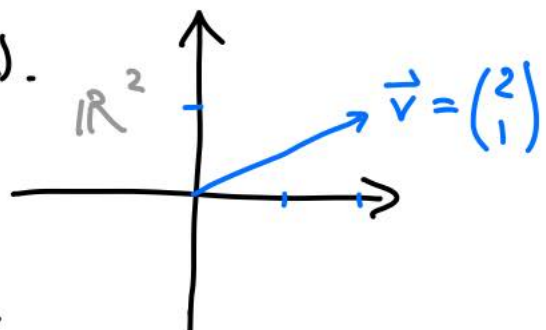


Lecture 3: Linear Combinations

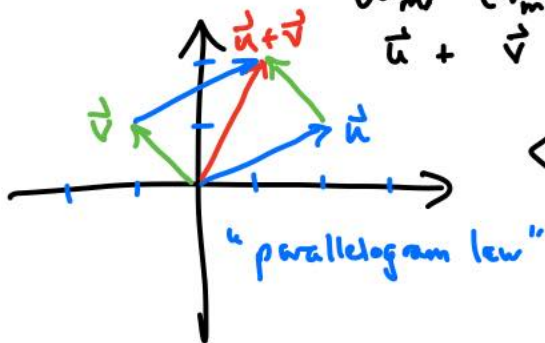
A vector ^{until we abstractify further} is an m -tuple of real numbers (for some m), which we shall write sometimes in a column ($= m \times 1$ matrix) — the default — and sometimes as a row ($= 1 \times m$ matrix). Real m -space $\mathbb{R}^m$ consists of m -tuples ^{think: coordinates of a point} of real numbers, with the "origin" at $(0, \dots, 0)$. But when we say the word "vector", we are thinking more of a directed line segment ($=$ arrow) — a "magnitude + direction" — than a point. More precisely, the vector $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$  has direction & length such

that if you place its tail at the origin in $\mathbb{R}^m$, its head lies at the point $(a_1, \dots, a_m)$.

Having said all this, we will write $\vec{v} \in \mathbb{R}^m$ when we want to consider a vector in m -space.



Adding vectors: $\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_m + v_m \end{pmatrix}$



geometric interpretation:
place vectors head to tail

Scalar multiplication: $c \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} cu_1 \\ \vdots \\ cu_m \end{pmatrix}$, for $c \in \mathbb{R}$:
 $c \cdot \vec{u}$

Corresponds to shrinking or stretching the directed segment (and reversing its direction if $c < 0$).

Definition 1: A linear combination of vectors

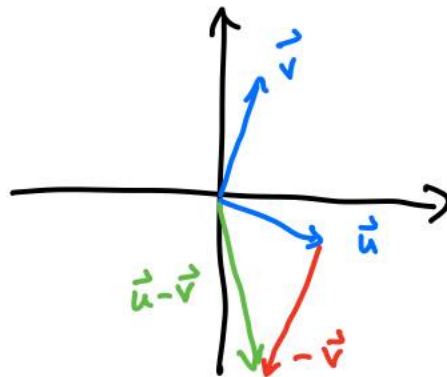
$\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$ is a vector $\vec{v} \in \mathbb{R}^m$ of the

form $(\vec{v} =) c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_n \vec{u}_n$, where

$c_1, \dots, c_n$ are real numbers.

Ex 1/

(in $\mathbb{R}^2$)



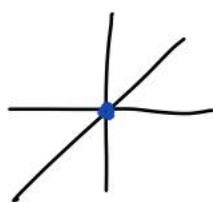
//

Definition 2: The span of $\vec{u}_1, \dots, \vec{u}_n \in \mathbb{R}^m$ is the set (in $\mathbb{R}^m$) of all linear combinations of them:

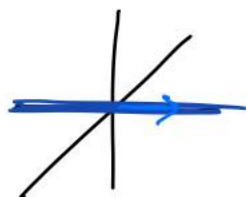
$$\text{span}\{\vec{u}_1, \dots, \vec{u}_n\} := \{c_1\vec{u}_1 + \dots + c_n\vec{u}_n \mid c_1, \dots, c_n \in \mathbb{R}\}.$$

When drawing a span, we draw the set of points — can't draw all those vectors at once.

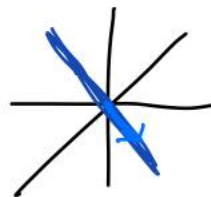
Ex 2 / Notation: $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
(in $\mathbb{R}^3$)



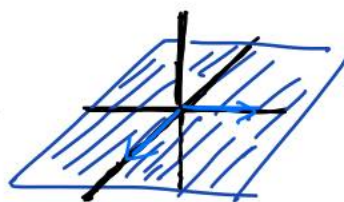
$\text{span}\{\vec{0}\}$



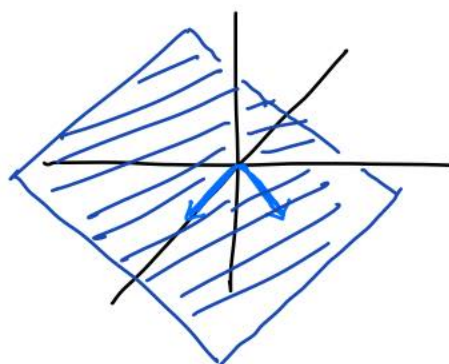
$\text{span}\{\vec{e}_1\}$



$\text{span}\{\vec{e}_1 + \vec{e}_2\}$



$\text{span}\{\vec{e}_1, \vec{e}_2\}$



$\text{span}\{\vec{e}_1, \vec{e}_2 - \vec{e}_3\}$

Going back to linear systems, consider the following three statements:

Statement 1: There exists a solution of the linear system

$$\left. \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{array} \right\} \begin{array}{l} m \text{ equations} \\ \text{in } n \text{ variables} \end{array}$$

Statement 2: There exist $x_1, \dots, x_n \in \mathbb{R}$ such that

$$x_1 \underbrace{\begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}}_{\vec{u}_1} + \dots + x_n \underbrace{\begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{\vec{u}_n} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{\vec{b}}$$

Statement 3: $\vec{b}$ is a linear combination of $\vec{u}_1, \dots, \vec{u}_n$;
that is, $\vec{b} \in \text{span}\{\vec{u}_1, \dots, \vec{u}_n\}$.

Ex 3/

$$\left. \begin{array}{l} 2x - 3y = 1 \\ -x + 6y = 2 \end{array} \right\} \Leftrightarrow x \begin{pmatrix} 2 \\ -1 \end{pmatrix} + y \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \text{span}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \end{pmatrix}\right\} //$$

These three statements are visibly equivalent. Thus:
to check whether a given vector is in the span of
some other vectors, amounts to solving a linear system!

Ex 4 / Is $\vec{b} = \begin{pmatrix} 3 \\ -7 \\ -3 \end{pmatrix}$ in the plane generated by (i.e., span of) $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix}$?

If this is "Statement 3", then "Statement 1" reads

$$\begin{cases} 1x_1 - 4x_2 = 3 \\ 0x_1 + 3x_2 = -7 \\ -2x_1 + 8x_2 = -3 \end{cases} \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 3 \\ 0 & 3 & -7 \\ -2 & 8 & -3 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 3 \\ 0 & 3 & -7 \\ 0 & 0 & 3 \end{array} \right]$$

inconsistent

Conclude that $\vec{b} \notin \text{span}\{\vec{u}_1, \vec{u}_2\}$.

Now suppose $\vec{b} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}$. Same question.

$$\left[\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 26/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{array} \right]$$

$$\Rightarrow (x_1, x_2) = \left(\frac{26}{3}, \frac{5}{3} \right). \quad \text{Check: } \frac{26}{3} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + \frac{5}{3} \begin{pmatrix} -4 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} //$$



In lecture 2, I claimed the

Theorem: Every matrix A is row-equivalent to a unique RREF matrix.

To show this, we need the following:

(*) If A is row-equivalent to B , then the rows of A are linear combinations of rows of B (and vice versa).

This is because

- the row operations — replace, swap, scale — simply replace a given row by a linear combination of rows.
- they are reversible.

To prove the Theorem, proceed in 3 steps:

① It is enough to show that

(**) 2 row-equivalent RREF matrices U and V must be the same.

Why? By the algorithm, $A \underset{\text{row-eq.}}{\sim} \text{rref}(A) =: U$.

If $A \underset{\text{row-eq.}}{\sim} V$ (also RREF), then $U \underset{\text{row-eq.}}{\sim} V$.

(So if (**) is known, we get $U = V$ as desired.)

For the remaining 2 steps, (which prove (**)) let U and V be 2 row-equivalent RREF matrices:

② The pivot columns of U & V are the same.

picture of U

$$\begin{bmatrix} 0 \dots 0 & 1 & * \dots * & 0 & * \dots * & 0 \dots \\ 0 \dots & & & 0 & 1 & * \dots * & 0 \dots \\ 0 \dots & & & & & 0 & 1 \dots \\ & & & \vdots & & & \end{bmatrix}$$

columns: $i_1 \quad \quad i_2 \quad \quad i_3 \dots$

picture of V

$$\begin{bmatrix} 0 \dots 0 & 1 & * \dots * & 0 & * \dots * & 0 \dots \\ 0 \dots & & & 0 & 1 & * \dots * & 0 \dots \\ 0 \dots & & & & & 0 & 1 \dots \\ & & & \vdots & & & \end{bmatrix}$$

columns: $j_1 \quad \quad j_2 \quad \quad j_3 \dots$

Write $\vec{u}_1, \vec{u}_2$, etc. for rows of U (viewed as vectors), same for V .

(A) By (x), $\vec{u}_1 \in \text{span}\{\text{rows of } V\}$. So:

$$\vec{u}_1 = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots \Rightarrow i_1 \geq j_1.$$

Reversing U & V gives $j_1 \geq i_1$. So $i_1 = j_1$ (and $a_1 = 1$).

(B) Next, $\vec{u}_2 = b_1 \vec{v}_1 + b_2 \vec{v}_2 + \dots \Rightarrow \vec{u}_2$ has b_1 in $(j_1 =) i_1^{\text{th}}$ coord.

$$\Rightarrow b_1 = 0$$

$$\Rightarrow \vec{u}_2 = b_2 \vec{v}_2 + b_3 \vec{v}_3 + \dots$$

(and vice versa).

(C) Striking out the first row of U and V , the remaining (still RREF!) matrices are row equivalent. Go back to (A), find $i_2 = j_2$, and repeat until there's nothing left.

③ The non-pivot columns of U & V are equal.

(D) We have $\vec{u}_i = \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_m \vec{v}_m$ (recall $a_1 = 1$ above)

$\Rightarrow \vec{u}_i$ has a_2 in the $(j_1 =) i_1^{\text{th}}$ coordinate

a_3 in the $(j_2 =) i_2^{\text{th}}$ coordinate, etc.

$$\Rightarrow 0 = a_2 = a_3 = \dots = a_m$$

So $\vec{u}_1 = \vec{v}_1$.

(E) Starting at the first row of $U \neq V$, go back to (D),
get $\vec{u}_2 = \vec{v}_2$, and so on. □