

Lecture 3D: The dot product

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. Define the dot product by

$$\vec{u} \cdot \vec{v} := \vec{u}^T \vec{v} = (u_1 \dots u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1 v_1 + \dots + u_n v_n.$$

Properties of "·":

(a) [commutativity] $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) [bilinearity] $\vec{u} \cdot (a\vec{v} + b\vec{w}) = a\vec{u} \cdot \vec{v} + b\vec{u} \cdot \vec{w}$

(c) [positive definite] $\vec{u} \cdot \vec{u} \geq 0$, with equality $\Leftrightarrow \vec{u} = \vec{0}$

Ex 1 / $\vec{u} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -7 \\ -4 \\ 6 \end{pmatrix}$

$$\vec{u} \cdot \vec{u} = 30, \quad \vec{v} \cdot \vec{v} = 101, \quad \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} = 0 \quad //$$

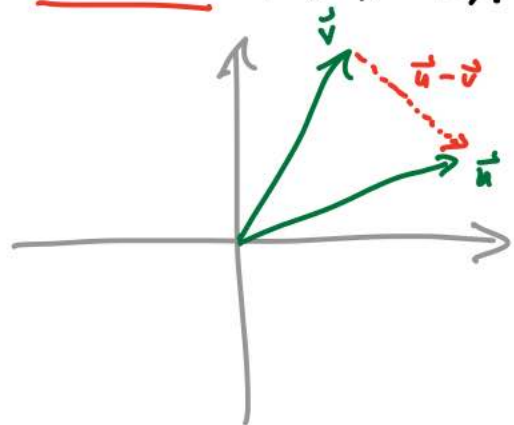
Next, we use (c) to define a notion of length (or norm):

$$\|\vec{u}\| := \sqrt{\vec{u} \cdot \vec{u}} \quad (\vec{u} \text{ is a } \underline{\text{unit vector}} \text{ if } \|\vec{u}\| = 1).$$

This gives a notion of distance

as well:

$$\text{dist}(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|.$$



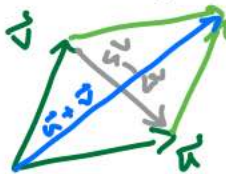
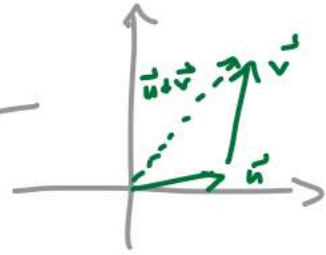
Properties of $\| \cdot \|$:

(a) $\|c\vec{v}\| = |c| \|\vec{v}\|$

(b) [triangle inequality] $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

(c) [Cauchy-Schwarz inequality] $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

(d) (parallelogram law) $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$



Let's check these:

(a) $\|c\vec{v}\|^2 = c\vec{v} \cdot c\vec{v} = c^2 \vec{v} \cdot \vec{v} = c^2 \|\vec{v}\|^2$. Take $\sqrt{\quad}$.

(d) $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) + (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$
 $= 2\vec{u} \cdot \vec{u} + 2\vec{v} \cdot \vec{v} + \cancel{2\vec{u} \cdot \vec{v}} - \cancel{2\vec{u} \cdot \vec{v}}$
 $= 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$

(c) $0 \leq \left\| \|\vec{v}\| \frac{\vec{u}}{\|\vec{u}\|} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|} \right\|^2 = \left(\|\vec{v}\| \frac{\vec{u}}{\|\vec{u}\|} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|} \right) \cdot \left(\|\vec{v}\| \frac{\vec{u}}{\|\vec{u}\|} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|} \right)$

\uparrow we can assume $\vec{v} \neq 0$

$= \|\vec{v}\|^2 \frac{\vec{u} \cdot \vec{u}}{\|\vec{u}\|^2} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} + \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} \frac{\vec{v} \cdot \vec{v}}{\|\vec{v}\|^2}$
 $= \|\vec{v}\|^2 \frac{\|\vec{u}\|^2}{\|\vec{u}\|^2} - 2 \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} + \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2}$
 $= \|\vec{v}\|^2 \|\vec{u}\|^2 - (\vec{u} \cdot \vec{v})^2$

So $(\vec{u} \cdot \vec{v})^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$. Take $\sqrt{\quad}$'s.

(b) $\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2\vec{u} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v}$

$\stackrel{(c)}{\leq} \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\| \|\vec{v}\| = (\|\vec{u}\| + \|\vec{v}\|)^2$. Take $\sqrt{\quad}$'s.

Ex 2 / \vec{u} & \vec{v} as in Example 1.

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{30}, \quad \|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{101}$$

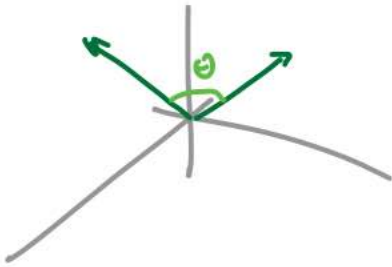
$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \left\| \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} - \begin{pmatrix} -7 \\ -4 \\ 6 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 9 \\ -1 \\ -7 \end{pmatrix} \right\| = \sqrt{131}$$

Unit vectors in the directions of \vec{u} & \vec{v} :

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}, \quad \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{101}} \begin{pmatrix} -7 \\ -4 \\ 6 \end{pmatrix} \quad (\text{works since } \frac{\vec{u}}{\|\vec{u}\|} \cdot \frac{\vec{u}}{\|\vec{u}\|} = \frac{\|\vec{u}\|^2}{\|\vec{u}\|^2} = 1)$$

We also can define the angle between 2 vectors by:

$$\Theta := \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right) \quad (\text{i.e. } \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \Theta).$$

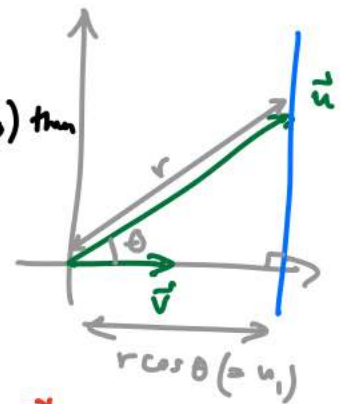


Here \vec{u} & \vec{v} must both be nonzero, and Θ belongs to $[0, \pi]$.

This makes sense b/c. Cauchy - Schwarz $\implies \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\| \|\vec{v}\|} \leq 1$
 $\implies \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \in [-1, 1] \implies \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$ is the cosine of some angle.

It is also consistent with angle in \mathbb{R}^2 : e.g. -

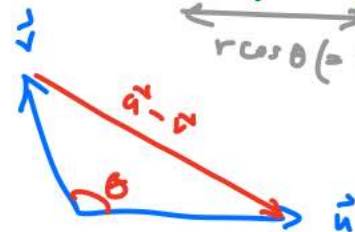
$$\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\sqrt{u_1^2 + u_2^2}} = \frac{u_1}{r} = \cos \Theta.$$



We have more generally the law of cosines

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \Theta$$

which follows at once from the definition (try it!).



Ex 3/ \vec{u}, \vec{v} as above. Then $\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = 0 \Rightarrow \theta = \frac{\pi}{2}$.

Next try $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. This gives

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{3}{\sqrt{6} \sqrt{2}} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}.$$

Finally we come to orthogonality:

• $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0$.
(\vec{u} is perpendicular to \vec{v})
orthogonal

• If $W \subseteq \mathbb{R}^n$ is a subspace, the orthogonal complement of W is the subspace

$$W^\perp := \{ \vec{u} \in \mathbb{R}^n \mid \vec{u} \perp \vec{w} \text{ for every } \vec{w} \in W \}.$$

Properties: (a) [Pythagorean Theorem]

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \iff \vec{u} \perp \vec{v}$$

(b) If A is an $m \times n$ matrix

(i) $(\text{Row } A)^\perp = \text{Nul } A$

(ii) $(\text{Col } A)^\perp = \text{Nul } A^T$

(c) For $W \subseteq \mathbb{R}^n$, $\dim W + \dim W^\perp = n$.

(a) is clear, from the law of cosines above.

(c) I'll address in a later lecture

Turning to (b) ... write $A = \begin{pmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_m \rightarrow \end{pmatrix}$:

(i) A vector \vec{v} is in $\text{Nul}(A)$

$$\Leftrightarrow \vec{r}_j \cdot \vec{v} = 0 \text{ for each } j \quad (\text{i.e. } \vec{v} \perp \vec{r}_j)$$

$$\Leftrightarrow \vec{v} \in \text{Row}(A)^\perp$$

$$(ii) (\text{Col } A)^\perp = (\text{Row } A^T)^\perp \stackrel{(i)}{=} \text{Nul } A^T.$$

In a picture:

$$\begin{array}{ccc} \vec{v} & \xrightarrow{\quad} & A\vec{v} \\ \mathbb{R}^m & \xrightarrow{\quad} & \mathbb{R}^n \end{array}$$

