Lecture 37: Quadratic Forms

<u>Definition</u>: A quadratic form on $\mathbb{R}^n$ is a function $f: \mathbb{R}^n \to \mathbb{R}$ of the form $f(\vec{x}) = \vec{x}^T A \vec{x}$ for some symmetric (real, $n \times n$) matrix A.

called the matrix of the form f

Ex 1/ Compute $\vec{x}^T A \vec{x}$, for $A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

$$(x_1\ x_2\ x_3)\begin{pmatrix} 4 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_1\ x_2\ x_3)\begin{pmatrix} 4x_1+3x_2 \\ 3x_1+2x_2+x_3 \\ x_2+x_3 \end{pmatrix} = 4x_1^2+2x_2^2+x_3^2 + 3x_1x_2+3x_2x_1 + x_2x_3+x_3x_2$$

$3x_1x_2+3x_2x_1 = 6x_1x_2$, $\quad x_2x_3+x_3x_2 = 2x_2x_3$ //

In general,

$$(x_1 \cdots x_n)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1 \cdots x_n)\begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{pmatrix} = \sum_{i,j=1}^n x_i a_{ij} x_j$$

$$= \sum_{i=1}^n a_{ii}x_i^2 + \sum_{i<j} a_{ij}x_ix_j + \sum_{i>j} a_{ij}x_ix_j$$

$$\sum_{i>j} a_{ij}x_ix_j \underset{\binom{\text{swap}}{i \,\&\, j}}{=} \sum_{i<j} a_{ji}x_jx_i \underset{(A=A^T)}{=} \sum_{i<j} a_{ij}x_ix_j$$

$$= \sum_{i=1}^n a_{ii}x_i^2 + \sum_{i<j}(2a_{ij})x_ix_j .$$

Ex 2/ Find the matrix of each form:

- (on $\mathbb{R}^2$) $x_1^2+x_2^2 = (x_1\ x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}^T I_2 \vec{x} \Rightarrow A = I_2$
- x_1x_2 $\left[= (x_1\ x_2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right.$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ain't symmetric.$\left.\right]$
  $= \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_1 = (x_1\ x_2)\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.
- $10x_1^2 - 6x_1x_2 - 3x_2^2 = (x_1\ x_2)\begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 10 & -3 \\ -3 & -3 \end{pmatrix}$

(on $\mathbb{R}^3$) • $8x_1^2 + 7x_2^2 - 3x_3^2 \underbrace{- 6x_1x_2 + 4x_1x_3 - 2x_2x_3}_{\substack{\text{think: } -3x_1x_2 - 3x_2x_1 + 2x_1x_3 + 2x_3x_1 \\ -x_2x_3 - x_3x_2}} \Rightarrow A = \begin{pmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{pmatrix}$ //

Level sets of quadratic forms

What does the solution set of

$$8x_1^2 - 4x_1x_2 + 5x_2^2 = 1$$

look like? It's an ellipse, but for that to become clear you'll need to perform a rotation of coordinates. Begin by recognizing the left-hand side as a quadratic form

$$(x_1 \ \ x_2)\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}^T A \vec{x} = Q(\vec{x}),$$

where A is symmetric.

But let's step back for a moment and look more generally at the equation

$$(*) \qquad \vec{x}^T A \vec{x} = 1 \leftarrow \text{or any positive number}$$

with A symmetric $n \times n$. By the Spectral Theorem (Lecture 36), there is an orthonormal eigenbasis $\mathcal{B} = \{\hat{u}_1, \ldots, \hat{u}_n\}$ for A, so that $P_{\mathcal{B}}^{-1} = P_{\mathcal{B}}^T$ and

$$A = P_{\mathcal{B}} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{D} P_{\mathcal{B}}^T$$

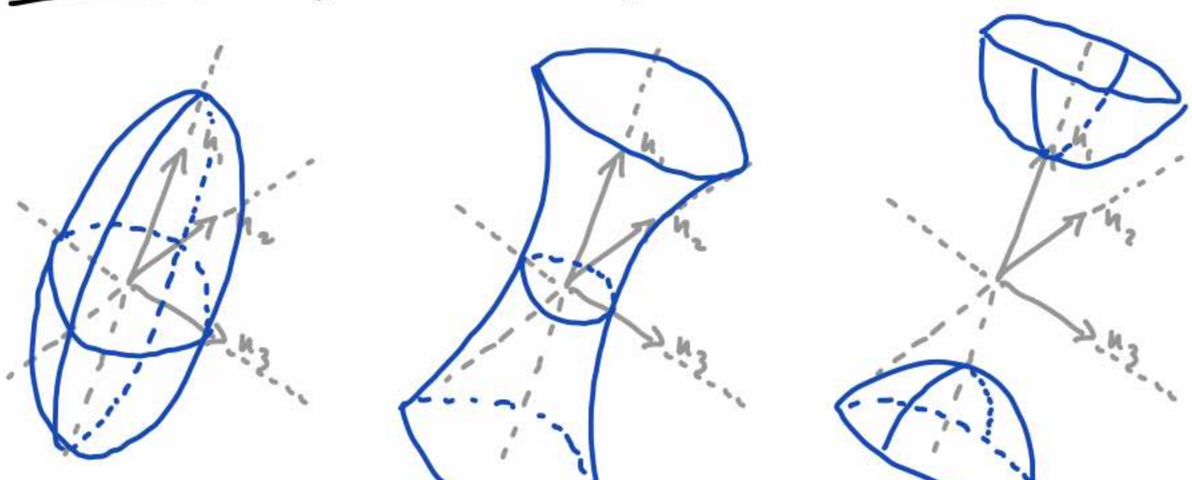
Writing $\vec{y} = [\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x} = P_{\mathcal{B}}^T \vec{x}$ for the "eigencoordinates", we have

$$\vec{x}^T A \vec{x} = \vec{x}^T P_B D P_B^T \vec{x} = \left(P_B^T \vec{x}\right)^T D \left(P_B^T \vec{x}\right) = \vec{y}^T D \vec{y},$$

and our equation (*) becomes

$$(**) \qquad \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 = 1.$$

To interpret this geometrically for $n=3$, assume that $\lambda_1 \le \lambda_2 \le \lambda_3$. <u>If all 3 eigenvalues are positive</u>, then (**) will define an <u>ellipsoid</u> with principal axes (in the directions of $\vec{u}_1, \vec{u}_2, \vec{u}_3$) of lengths $\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \frac{1}{\sqrt{\lambda_3}}$. <u>If only $\lambda_2, \lambda_3 > 0$</u>, then (**) is an <u>elliptic hyperboloid</u>; and <u>if only $\lambda_3 > 0$</u>, (**) is a <u>hyperboloid of 2 sheets</u>.



ELLIPSOID   ELLIPTIC HYPERBOLOID   HYPERBOLOID OF TWO SHEETS

Back to $n=2$, and our original equation: A orthogonally diagonalizes

$$\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

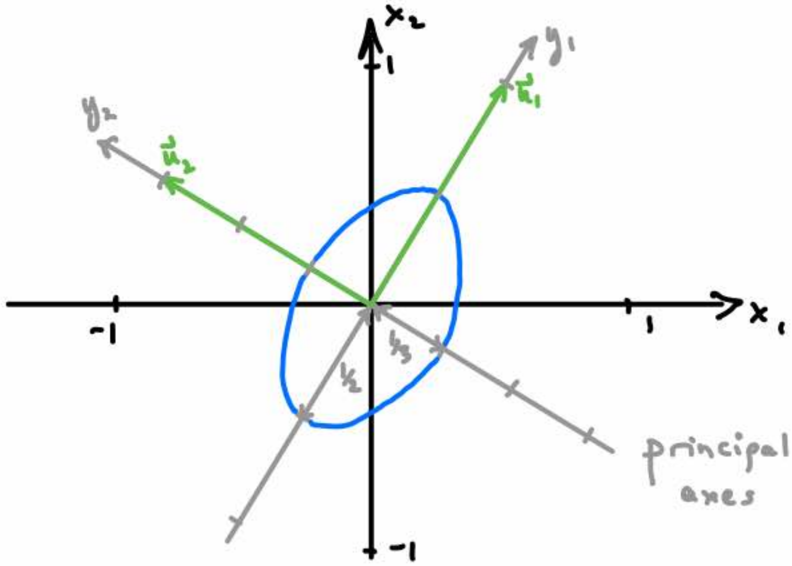
and so if (y_1, y_2) are coordinates along the axes defined by the (unit) eigenvectors

$$\vec{u}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Our equation becomes

$$1 = 4y_1^2 + 9y_2^2 = \frac{y_1^2}{(1/2)^2} + \frac{y_2^2}{(1/3)^2}.$$

Now it is really easy to sketch the solution:



<u>Q</u>: What sorts of figures do you get if

$(n=2)$ $\quad \lambda_2 > 0$ and $\lambda_1 = 0$

$(n=3)$ $\quad \lambda_3 > 0, \ \lambda_2 = 0, \ \lambda_1 < 0$?

Quadratic forms as functions

Suppose $A = PDP^T$ is an orthogonal diagonalization. Then

$$(\dagger) \quad Q(\vec{x}) = \vec{x}^T A \vec{x} = \vec{x}^T P D P^T \vec{x} = (P^T\vec{x}) D (P^T\vec{x})$$
$$= \vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

as before. So it's pretty clear that Q is "controlled by the eigenvalues λ_i" once you do the orthogonal change of coordinates.

<u>Definition</u>: (i) Q is <u>positive definite</u> $\Longleftrightarrow$ $Q(\vec{x}) > 0 \quad \forall \vec{x} \neq \vec{0}$

<u>positive semidefinite</u> $\Longleftrightarrow$ $Q(\vec{x}) \geq 0 \quad \forall \vec{x}$

<u>indefinite</u> $\Longleftrightarrow$ Q assumes both positive & negative values on nonzero $\vec{x}$'s

<u>negative semidefinite</u> $\Longleftrightarrow$ $Q(\vec{x}) \leq 0 \quad \forall \vec{x}$

<u>negative definite</u> $\Longleftrightarrow$ $Q(\vec{x}) < 0 \quad \forall \vec{x} \neq \vec{0}$

(ii) A is <u>positive definite</u> $\Longleftrightarrow$ all $\lambda_i > 0$

<u>positive semidefinite</u> $\Longleftrightarrow$ all $\lambda_i \geq 0$

<u>indefinite</u> $\Longleftrightarrow$ A has both positive & negative eigenvalues

<u>negative semidefinite</u> $\Longleftrightarrow$ all $\lambda_i \leq 0$

<u>negative definite</u> $\Longleftrightarrow$ all $\lambda_i < 0$.

<u>Theorem</u>: Q is $\boxed{}$ $\Longleftrightarrow$ A is $\boxed{}$

(Insert same term in both places.)

<u>Proof</u>: Clear from $(\dagger)$. e.g., if λ_i's are > 0, then

$$Q(\vec{x}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 \geq 0, \text{ and } = 0 \Longleftrightarrow \vec{y} = \vec{0} \Longleftrightarrow \vec{x} = \vec{0}. \quad \square$$

Quadratic forms and inner products

Let A be an $n \times n$ symmetric matrix.

Corollary (to the Theorem): $\langle \vec{x}, \vec{y} \rangle := \vec{x}^T A \vec{y}$ is an inner product on $\mathbb{R}^n$ $\iff$ A is positive definite.

Proof: A is positive-definite $\underset{\text{Thm.}}{\iff}$ $Q(\vec{x}) := \langle \vec{x}, \vec{x} \rangle$ is positive-definite

(eigenvalues > 0) ($\langle \vec{x}, \vec{x} \rangle \geq 0$ with equality $\iff \vec{x} = \vec{0}$)

$\iff$ the symm. bilinear form $\langle \cdot , \cdot \rangle$ satisfies the positive-definiteness property

$\iff$ $\langle \cdot , \cdot \rangle$ is an inner product. $\square$

More geometrically,

$\vec{x}^T A \vec{y}$ gives an inner product

$$\iff$$

the set $\|\vec{x}\| = 1$ (comprising elements of "length" 1) is an ellipsoid.