Lecture 4: Matrix Equations

Multiplying a matrix by a (column) vector: 2 approaches

① Row-by-column (more computationally efficient)

row vectors

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \underset{\text{definition}}{:=} \begin{pmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \end{pmatrix}.$$

take "dot products"

Notice that this $= \begin{pmatrix} a_1x \\ a_2x \end{pmatrix} + \begin{pmatrix} b_1y \\ b_2y \end{pmatrix} + \begin{pmatrix} c_1z \\ c_2z \end{pmatrix}$

$$= x\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + y\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + z\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

which leads to ...

② Linear combinations (more conceptual)

Writing $A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

column vectors

(*) $$\boxed{A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n}$$

As an example of the second approach's utility, we can use it to check the

<u>Linearity property</u>: $A(c\vec{u} + d\vec{w}) = c A\vec{u} + d A\vec{w}$

$$\underbrace{A(c\vec{u} + d\vec{w})}_{\begin{pmatrix} cu_1+dw_1 \\ \vdots \\ cu_n+dw_n \end{pmatrix}} = c A\underbrace{\vec{u}}_{\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}} + d A\underbrace{\vec{w}}_{\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}}$$

By (*),

$$\begin{aligned} \text{LHS} &= (cu_1 + dw_1)\vec{v}_1 + \cdots + (cu_n + dw_n)\vec{v}_n \\ &= c(u_1\vec{v}_1 + \cdots + u_n\vec{v}_n) + d(w_1\vec{v}_1 + \cdots + w_n\vec{v}_n) \\ &= \text{RHS}. \end{aligned}$$

More importantly, we see that

<u>Statement 1</u>: $A\vec{x} = \vec{b}$ has a solution in $\vec{x}$ (i.e. is consistent)

$\Updownarrow$ is equivalent to

<u>Statement 2</u>: $\vec{b} \in \operatorname{span}\{\text{columns of } A\}$

The link is (*): $\vec{b} = A\vec{x} \overset{(*)}{=} x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$ says "you can choose $x_1, \ldots, x_n$ so that $\vec{b}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$".

So to check <u>Statement 2</u>, you just row-reduce $[A \mid \vec{b}]$.

there is another set of equivalent statements:

<u>Assertion (A)</u>: Span{columns of A} = (all of) $\mathbb{R}^m$

$\Updownarrow$ (clear from above)

<u>Assertion (B)</u>: $A\vec{x} = \vec{b}$ is consistent for any $\vec{b}$

$\Updownarrow$ why?

<u>Assertion (C)</u>: rref(A) has no rows of all zeroes.

First,

- $[A \mid \vec{b}] \underset{\text{row-equiv.}}{\sim} [\text{rref}(A) \mid \vec{c}]$ for some vector $\vec{c}$

  (apply the sequence of row operations that puts A in RREF, to the <u>augmented</u> matrix)

- we can choose $\vec{b}$ so that $\vec{c}$ is any vector

  (because row operations are reversible)

<u>(C) $\Rightarrow$ (B)</u>: If (C) holds, then $\underline{[\text{rref}(A) \mid \vec{c}]}$ has a leading '1' in every row in the "rref(A)" part, hence is in RREF itself and $\underline{= \text{rref}[A \mid \vec{b}]}$. Since the leading '1's occur to the left of $\vec{c}$, $\vec{c}$ is not a pivot column and the system is consistent (regardless of $\vec{b}$). So (B) holds.

$\underline{(B) \Rightarrow (C)}$: If (C) fails, choose $\vec{b}$ so that $\vec{c}$ has a nonzero last entry. Since the last row of rref(A) is all '0's, $\vec{c}$ is a pivot column (for this choice of $\vec{b}$), and so (B) fails.

Ex 1/ For which $\vec{b}$ is $\begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix} \vec{x} = \vec{b}$ solvable?

(Equivalent question: determine $\mathrm{span}\left\{\begin{pmatrix} 3 \\ -9 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\right\}$.)

$$\left[\begin{array}{cc|c} 3 & -1 & b_1 \\ -9 & 3 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & b_1/3 \\ -9 & 3 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & b_1/3 \\ 0 & 0 & b_2+3b_1 \end{array}\right]$$

failure of (C)

so we must have $b_2 = -3b_1$, i.e. $\vec{b}$ must be a scalar multiple of the vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$. //

failure of (A)

Ex 2/ Do the columns of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ span $\mathbb{R}^3$?

(Equivalent question: does rref(A) have no rows of "all 0"?)

Row-reduce:

$$A \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -24 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A)$$ //

Answer: <u>NO</u>.

Ex 3/ Suppose $A = 4\times 4$ matrix, and $\vec{b} \in \mathbb{R}^4$ are such that $A\vec{x} = \vec{b}$ has a unique solution. Must columns of A span $\mathbb{R}^4$?

Consider the augmented matrix $[A \mid \vec{b}]$ and its rref, which must have

- $\vec{b}$ non-pivot column (for existence of a solution)
- no $\underbrace{\text{non-pivot columns}}_{=\text{ free variables}}$ in the "[⋯ |" part of the augmented matrix (for uniqueness)

So all columns of A are pivot AND A 4×4 $\Rightarrow$ all rows of rref have a leading '1'

$\Rightarrow$ no rows of zeroes

$\Rightarrow$ columns span $\mathbb{R}^4$.

Notice that if A was instead 5×4, this argument breaks down and columns need not span $\mathbb{R}^5$ (in fact, they can't).

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