

Lecture 5: Parametrizing solutions

Review of "span"

Recall that the span of a collection of vectors is the set of all linear combinations of them:

$$\begin{aligned} \text{Ex 1/ } \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -1 \\ 0 \end{pmatrix} \right\} &= \left\{ s \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 3 \\ -1 \\ 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} s \\ 0 \\ -2s \\ -s \end{pmatrix} + \begin{pmatrix} -t \\ 3t \\ -t \\ 0 \end{pmatrix} \mid s, t \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} s-t \\ 3t \\ -2s+3t \\ -s \end{pmatrix} \mid s, t \in \mathbb{R} \right\} // \end{aligned}$$

So if you are asked

"Taking $W =$ set of all vectors of the form $\begin{pmatrix} 3s-4t \\ -4s+4t \\ -4s-4t \\ -3s+5t \end{pmatrix}$,
find $\vec{u}$ & $\vec{v}$ such that $W = \text{span}\{\vec{u}, \vec{v}\}$ "

you'll just reverse the process, obtaining

$$\vec{u} = \begin{pmatrix} 3 \\ -4 \\ -4 \\ -3 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} -4 \\ 4 \\ -4 \\ 5 \end{pmatrix}.$$

Homogeneous linear systems

These are systems of the form

$$A\vec{x} = \vec{0}.$$

They always have the "trivial solution" $\vec{x} = \vec{0}$, and the solution set is always the span of a set of vectors (as we now demonstrate).

Ex 2 / Determine all solutions of
$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 0 \\ -4x_1 - 4x_2 - 8x_3 = 0 \\ -3x_2 - 3x_3 = 0. \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \end{array} \right]$$

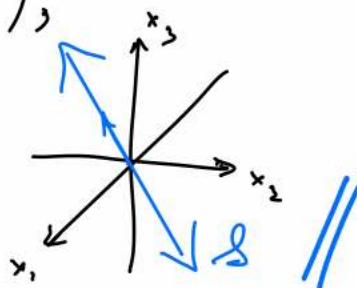
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] = \text{rref}$$

x_3 free

[N.B.: You may drop the right-hand column of 0's if you are comfortable doing that.]

$$\text{So } \begin{cases} x_1 = -x_3 \\ x_2 = -x_3 \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

and the solution set is $\mathcal{S} = \text{span} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$



In homogeneous linear systems

These are the systems $A\vec{x} = \vec{b}$ with $\vec{b} \neq \vec{0}$.

In this case, $\vec{x} = \vec{0}$ is NEVER a solution. So the solution set $\mathcal{S}$ can't be a span: if nonempty, it will be a "parallel translate" of a span.

Ex 3/ Determine all solutions of
$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 8 \\ -4x_1 - 4x_2 - 8x_3 = -16 \\ -3x_2 - 3x_3 = 12 \end{cases}$$

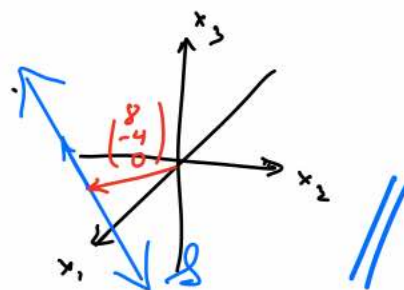
$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array} \right] = \text{rref}$$

x_3 free

$$\Rightarrow \begin{cases} x_1 = 8 - x_3 \\ x_2 = -4 - x_3 \end{cases} \Rightarrow \vec{x} = \begin{pmatrix} 8 - x_3 \\ -4 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{S} = \left\{ \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$$



Here is another homogeneous system problem, with the matrix already nearly row-reduced:

Ex 4/ If $A \sim \begin{pmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,
row-equiv.

describe the solution set to $A\vec{x} = \vec{0}$.

$$\rightarrow \begin{pmatrix} 1 & -4 & 0 & 0 & 3 & -7 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -4 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{ref}(A)$$

$\uparrow$ $\uparrow$ $\uparrow$
 x_2 x_4 x_6 free

Now mentally adjoining a column of zeroes to the right,

we get $\begin{cases} x_1 = 4x_2 - 5x_6 \\ x_3 = x_6 \\ x_5 = 4x_6 \end{cases} \Rightarrow$

$$\mathcal{S} = \left\{ \begin{pmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{pmatrix} \mid x_2, x_4, x_6 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \mid x_2, x_4, x_6 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}.$$

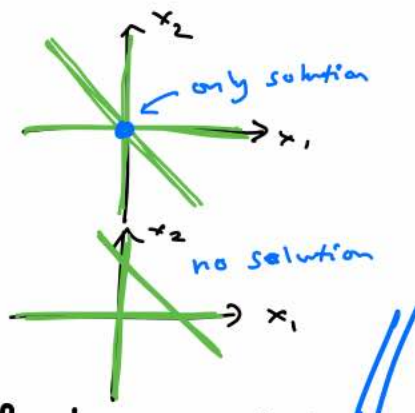
Relation between homogeneous & inhomogeneous systems

Consider $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b} (\neq \vec{0})$, with solution sets $\mathcal{S}_0$ and $\mathcal{S}_b$.

Case I: $\mathcal{S}_b$ empty

Ex 5 / $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Leftrightarrow$$



Case II: $\mathcal{S}_b$ nonempty. Let $\vec{p} \in \mathcal{S}_b$ be any solution (of the inhomogeneous system).

Then $\boxed{\mathcal{S}_b = \mathcal{S}_0 + \vec{p}} = \{ \vec{w} + \vec{p} \mid \vec{w} \in \mathcal{S}_0 \}$ is the translate of the homogeneous solution set by $\vec{p}$.

Why?

- If $\vec{w} \in \mathcal{S}_0$, then $A(\vec{w} + \vec{p}) = A\vec{w} + A\vec{p} = \vec{0} + \vec{b} = \vec{b}$.
So $\mathcal{S}_0 + \vec{p} \subseteq \mathcal{S}_b$.
- If $\vec{x} \in \mathcal{S}_b$, then write $\vec{x} = (\vec{x} - \vec{p}) + \vec{p}$. Since $A(\vec{x} - \vec{p}) = A\vec{x} - A\vec{p} = \vec{b} - \vec{b} = \vec{0}$, we see that $\vec{x} - \vec{p} \in \mathcal{S}_0$. So $\vec{x} \in \mathcal{S}_0 + \vec{p}$. This shows the reverse inclusion $\mathcal{S}_b \subseteq \mathcal{S}_0 + \vec{p}$.