Lecture 7: Linear Transformations

A transformation / function / map

$$T : \underset{\text{domain}}{\mathbb{R}^n} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^m}$$

is a rule assigning, to each vector in $\mathbb{R}^n$, a vector in $\mathbb{R}^m$:

$$\vec{x} \longmapsto T(\vec{x}).$$

The range / image of T is

$$T(\mathbb{R}^n) = \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n \} \subset \mathbb{R}^m,$$

the set of all vectors in the image of T.

Examples

1. $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T\begin{pmatrix} x \\ y \end{pmatrix} := x^2 + y^2$.

2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ " $T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x \\ -y \end{pmatrix}$

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ " $T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} y-1 \\ x+y \end{pmatrix}$

4. $T : \mathbb{R}^3 \to \mathbb{R}^3$ " $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$

(What are these geometrically?)

A transformation is linear if it satisfies

$$(*) \quad \boxed{\left.\begin{array}{l} T(c\vec{v}) = cT(\vec{v}) \\ T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \end{array}\right\} \text{ for all } c \in \mathbb{R} \text{ and } \vec{u},\vec{v} \in \mathbb{R}^n.}$$

(Equivalently, you can check that $T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$.)

An immediate consequence of $(*)$ is that $T(\vec{0}) = \vec{0}$:

$$T(\vec{0}) = T(0\cdot\vec{0}) = 0\cdot T(\vec{0}) = \vec{0}.$$

(take $c = 0$ in $(*)$)

Examples (cont'd.)

2 is linear since $T\left(c\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} cu_1+dv_1 \\ cu_2+dv_2 \end{pmatrix}\right)$

$$= \begin{pmatrix} cu_1+dv_1 \\ -(cu_2+dv_2) \end{pmatrix} = \begin{pmatrix} cu_1 \\ -cu_1 \end{pmatrix} + \begin{pmatrix} du_2 \\ -du_2 \end{pmatrix} = c\begin{pmatrix} u_1 \\ -u_1 \end{pmatrix} + d\begin{pmatrix} u_2 \\ -u_2 \end{pmatrix}$$

$$= cT\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + dT\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

4 is linear (similar reasoning)

3 is not because $T(\vec{0}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \neq \vec{0}$. This may seem strange: the basic point is that a linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is of the form $x \mapsto ax$, not $\underbrace{x \mapsto ax+b}_{\text{this is called affine rather than linear}}$.

1 is also non linear; while $T(\vec{0}) = 0$, $T\left(2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = T\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2^2+0^2 = 4 \neq 2 = 2T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Ex/ Which are linear/nonlinear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$:

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x \end{pmatrix}, \quad R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}.$$

Answer: the middle one is nonlinear:

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$ //

A matrix transformation is a transformation of the form

$$T(\vec{x}) := A\vec{x}.$$

In Lecture 4, we checked that

$$A(c\vec{u} + d\vec{v}) = cA\vec{u} + dA\vec{v};$$

that is, matrix transformations are linear.

Note that in the matrix-vector product $A\vec{x}$, an $m \times n$ matrix must be multiplied by an $n \times 1$ vector, yielding an $m \times 1$ vector. That is, the matrix transformation maps $\mathbb{R}^n \to \mathbb{R}^m$ if A is $m \times n$.

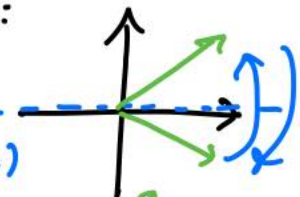
Ex/ Let $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$, and let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the corresponding transformation: $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$= \begin{pmatrix} 2x - 3y + z \\ x - z \end{pmatrix}.$$ //

Examples (2) and (4) I claim that we can write these as matrix transformations:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

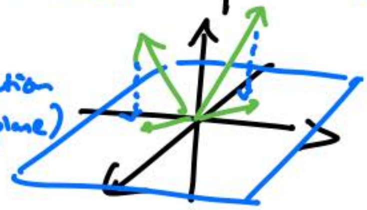
(reflection about x-axis)



and

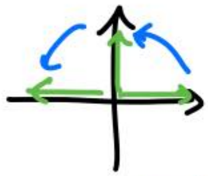
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

(projection to xy-plane)



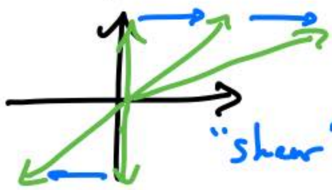
What about S & R from the example on the last page?

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = S\begin{pmatrix} x \\ y \end{pmatrix}$$



rotation?

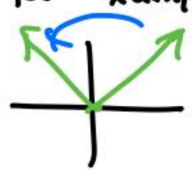
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix} = R\begin{pmatrix} x \\ y \end{pmatrix}$$



"shear"

(Is S a rotation? Sure looks like it: for example,

$$S\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



)

How does one go about finding these matrices for a linear transformation? Can one always do this?

Those are questions for the next lecture! But this would be useful, because then we could rephrase questions like "is $\vec{y}$ in the range of T" as "is $\vec{y} = A\vec{x}$ consistent" which we already know how to solve.

It's natural to ask what happens to linear (in)dependence under a linear transformation. If $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ is a dependence relation on $\{\vec{v}_1, \ldots, \vec{v}_n\}$,

$$\vec{0} = T(\vec{0}) = T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n)$$

is a dependence relation on $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$. So "T preserves dependency". Does it preserve independence?

To see that the answer is NO for an arbitrary transformation, just consider the projection above: