Lecture 8 : Matrix of a linear transformation

Consider the examples

① $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} y \\ x \end{pmatrix}$ "flip"

and

② $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$ "projection"

of linear transformations. These are given by matrices

$$(A =) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For both of these, we can ask two questions:

Q1 ("Existence"): Is every $\vec{b} \in \mathbb{R}^{m\,(=2 \text{ resp. } 3)}$ in the image of T? Recall that this is the same thing as existence of $\vec{x}$ such that $T(\vec{x}) (= A\vec{x}) = \vec{b}$. So, YES for ①, NO for ②.

Q2 ("Uniqueness"): Given a $\vec{b}$ for which $T(\vec{x}) = \vec{b}$ has a solution, is this solution unique?

Again: YES for ①, NO for ②.

We'll develop a systematic way to check this shortly.

First, we ask whether every linear transformation is a matrix transformation? If we write

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n,$$

then using linearity of $T : \mathbb{R}^n \to \mathbb{R}^m$ gives

$$T(\vec{x}) = T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) = x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n)$$

$$= \underbrace{\begin{pmatrix} \uparrow & & \uparrow \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ \downarrow & & \downarrow \end{pmatrix}}_{=:\, A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

"[Standard] matrix of T"

$$= A\vec{x}.$$

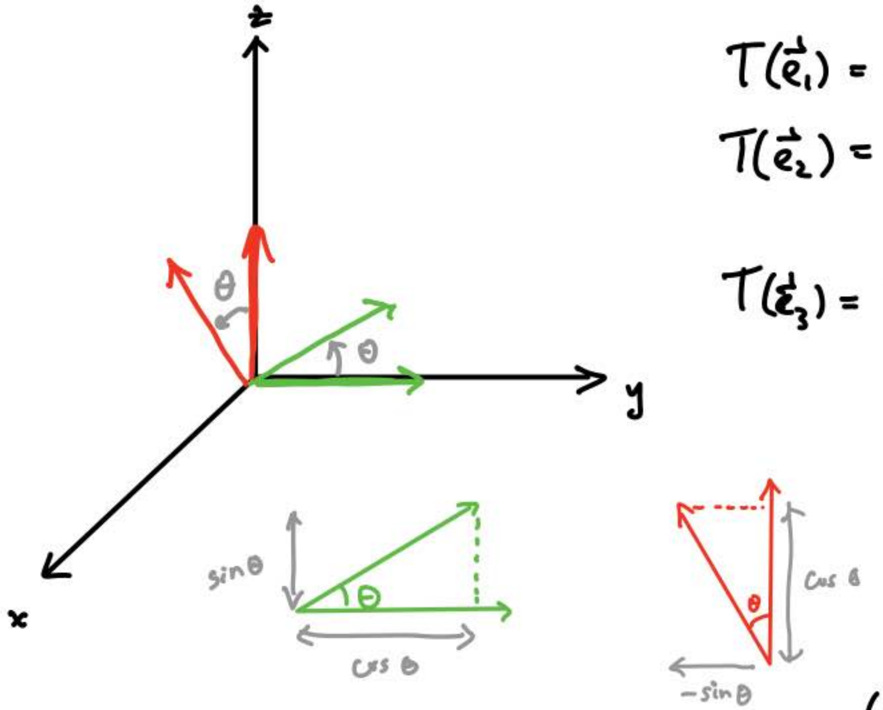
So we not only see that the answer is YES — we get a formula!

Ex/ Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ has $T(\vec{e}_1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $T(\vec{e}_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and $T(\vec{e}_3) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What is its matrix?

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

//

Ex/ Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation by angle θ about the x-axis. Find A.



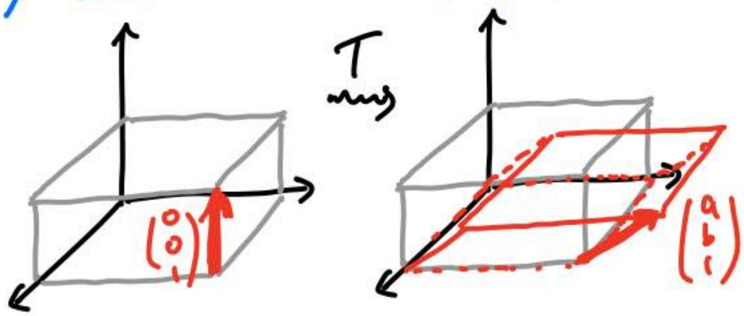
$$T(\vec{e}_1) = \vec{e}_1$$

$$T(\vec{e}_2) = \begin{pmatrix} 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

$$T(\vec{e}_3) = \begin{pmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{pmatrix}$$

So $$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$ //

Ex/ Find the matrix of the "shear" transformation depicted:



$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

(doesn't affect the xy-plane) //

"Onto" and "1-1" abbreviation for "one-to-one"

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, with matrix A.
domain codomain

Definition: (a) T is onto if the range (image) of T equals $\mathbb{R}^m$.

(b) T is 1-1 if no two distinct $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$ are sent (by T) to the same vector in $\mathbb{R}^m$.

Theorem 1: The following are equivalent:

$\boxed{S1}$ T onto (any $\vec{b} \in \mathbb{R}^m$ can be written $T(\vec{x})$)

$\boxed{S2}$ $A\vec{x} = \vec{b}$ is consistent for all $b \in \mathbb{R}^m$

$\boxed{S3}$ Columns of A span $\mathbb{R}^m$

$\boxed{S4}$ rref(A) has no rows of all zeroes.
(leading '1' in every row)

lecture 4: these are equivalent

Clearly this pertains to the "Existence" question above.

The uniqueness of a solution is determined by whether T is 1-1, about which we have:

<u>Theorem 2</u>: The following are equivalent:

$\boxed{S1'}$ T is 1-1

$\boxed{S2'}$ $A\vec{x} = \vec{0}$ has only the zero solution

$\boxed{S3'}$ The columns of A are linearly independent

$\boxed{S4'}$ rref(A) has a leading '1' in every <u>column</u>.

lecture 6: these are equivalent

Why are $S1'$ and $S2'$ equivalent? Clearly if T is 1-1, then $A\vec{x} = \vec{0}$ can't have more than the $\vec{0}$ solution.
If T is NOT 1-1, then there exist $\vec{v}_1, \vec{v}_2$ <u>distinct</u> with $T\vec{v}_1 = T\vec{v}_2$, which implies (by linearity of T) that $\vec{0} = T\vec{v}_1 - T\vec{v}_2 = T(\vec{v}_1 - \vec{v}_2)$, when $\vec{v}_1 - \vec{v}_2$ is different from $\vec{0}$.

Ex/ Consider the "flip" and "projection" examples on the first page. The flip had $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and is 1-1 and onto. The projection had $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and is neither 1-1 nor onto. But if we took its codomain to be instead $\mathbb{R}^2$, then $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and it is onto. (Just remember that "onto" means "onto the entire codomain".) //

Ex / Say $T: \mathbb{R}^4 \to \mathbb{R}^3$ has matrix $A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ -1 & 0 & -6 & -2 \end{pmatrix}$.

Is it 1-1? Onto?

Row-reduce: $A \to \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & -4 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 6 & 7 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 5 \end{pmatrix}$

$\to \begin{pmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. No rows of 0's $\Rightarrow$ T <u>onto</u>.

Has a non-pivot column $\Rightarrow$ T <u>not</u> 1-1. //

<u>QS</u>: Can a linear $T: \mathbb{R}^4 \to \mathbb{R}^3$ ever be 1-1?

Can a linear $T: \mathbb{R}^3 \to \mathbb{R}^4$ ever be onto?

<u>No</u> in both cases, using S4/S4′ in the two Theorems.