Lecture 9: Algebra with Matrices

Adding Matrices

- must be of same "dimensions" (both $m \times n$)
- add entry by entry

Ex/ $\begin{pmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 5 \end{pmatrix}$ //

- commutative : $A+B = B+A$
- associative : $(A+B)+C = A+(B+C)$
- additive identity : $A+0=A$, where 0 = zero matrix
- additive inverse : $A+(-A)=0$
  negate all entries
- subtraction : $A-B := A+(-B)$
- cancellation : $A+B = C+B \Rightarrow A=C$

Scalar multiplication

- multiply each matrix entry by the scalar

Ex/ $3\begin{pmatrix} 0 & 1 & -1 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -3 \\ 9 & 0 & 6 \end{pmatrix}$ //

- distributivity : $c(A+B) = cA+cB$
  $(c+d)A = cA + dA$
- $(cd)A = c(dA)$

Multiplying matrices

- A $m\times n$, B $k\times l$: AB is defined when $k=n$ (and is then an $m\times l$ matrix)
  - if $B = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{w}_1 & \cdots & \vec{w}_l \\ \downarrow & & \downarrow \end{pmatrix}$, then $AB := \begin{pmatrix} \uparrow & & \uparrow \\ A\vec{w}_1 & \cdots & A\vec{w}_l \\ \downarrow & & \downarrow \end{pmatrix}$.

- So the q^{th} column of AB is

$$A\vec{w}_q = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} b_{1q} \\ \vdots \\ b_{nq} \end{pmatrix} = b_{1q}\vec{v}_1 + \cdots + b_{nq}\vec{v}_n$$

(column-vector interpretation)

$$= \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{1q} \\ \vdots \\ b_{nq} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} b_{jq} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} b_{jq} \end{pmatrix},$$

from which we see that the $(p,q)^{\text{th}}$ entry of AB is

$$[AB]_{\underline{pq}} = \sum_{j=1}^{n} a_{\underline{p}j} b_{j\underline{q}}$$

- powers of a matrix: A^m means $\underbrace{A\cdot A\cdot\cdots\cdot A}_{m \text{ times}}$

- identity matrix:

$$\mathbb{I}_n := \underset{n\times n}{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{e}_1 & \cdots & \vec{e}_n \\ \downarrow & & \downarrow \end{pmatrix}$$

$$A I_n = \begin{pmatrix} \uparrow & & \uparrow \\ A\vec{e}_1 & \cdots & A\vec{e}_n \\ \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix} = A = I_m A$$

- distributivity : $A(\beta B + \gamma C) = \beta(AB) + \gamma(AC)$

  $(\beta B + \gamma C) A = \beta (BA) + \gamma (CA)$

- NOT commutative: $AB \neq BA$ in general

Ex/ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ //

- Multiplicative inverse? In general, NO.

  - If $m > n$, there can't be an $n \times m$ B with $AB = I_m$.
    [Otherwise, for any $\vec{b} \in \mathbb{R}^m$ we'd have $A(B\vec{b}) = I_m \vec{b} = \vec{b}$, which contradicts the fact that rref(A) has rows of 0's at the bottom.] On the other hand, there could be lots of matrices B with $BA = I_n$.

  - For square matrices, there can be both left and right inverses. [See the next lecture.] But there also might not be: the projection matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ can't have a left-inverse, since BA has first column $\vec{0}$ for any B.

  - If there aren't inverses, you can't "cancel":

Ex/ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$B \cdot A = 0 = C \cdot A$

but clearly $B \neq C$. //

- Associativity holds: $(AB)C = A(BC)$.

If A $m\times n$, B $n\times r$, C $r\times s$, then

$$[(AB)C]_{i\ell} = \sum_k [AB]_{ik} C_{k\ell} = \sum_k \left(\sum_j A_{ij} B_{jk}\right) C_{k\ell}$$

$$= \sum_{j,k} (A_{ij} B_{jk}) C_{k\ell} = \sum_{j,k} A_{ij} (B_{jk} C_{k\ell})$$

since multiplication of real numbers is associative

$$= \sum_j A_{ij} \left(\sum_k B_{jk} C_{k\ell}\right) = \sum_j A_{ij} [BC]_{j\ell}$$

$$= [A(BC)]_{i\ell}.$$

<u>Theorem</u>: Let $\begin{cases} S: \mathbb{R}^n \to \mathbb{R}^m \\ T: \mathbb{R}^r \to \mathbb{R}^n \end{cases}$ be linear transformations

with matrices $\begin{cases} A & m\times n \\ B & n\times r. \end{cases}$ Then $S\circ T: \mathbb{R}^r \to \mathbb{R}^m$ has matrix AB.

<u>Proof</u>: $(S\circ T)\vec{x} = S(T(\vec{x})) = S(B\cdot\vec{x}) = A\cdot(B\cdot\vec{x})$

$$= (A\cdot B)\cdot\vec{x}.$$

by associativity

$\square$

Ex / Suppose $B = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$ with $\vec{w}_3 = \vec{w}_1 + \vec{w}_2$.

What can you say about the 3rd column of AB?

$AB = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ A\vec{w}_1 & A\vec{w}_2 & A\vec{w}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$ and $A\vec{w}_3 = A(\vec{w}_1 + \vec{w}_2) = A\vec{w}_1 + A\vec{w}_2$. //

Ex/ Suppose the last column of AB is $\vec{0}$, but B has no $\vec{0}$ column. What can you say about A's columns?

If $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ = last column of B, and $A = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$, then the last column of AB is $A\vec{w} = w_1\vec{v}_1 + \cdots + w_n\vec{v}_n$. Since this is $\vec{0}$ and $\vec{w} \neq \vec{0}$, $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent. //

- Transpose of a matrix: A $m \times n$ $\rightsquigarrow$ A^T $n \times m$, given by $[A^T]_{ij} := [A]_{ji}$. ($A = A^T \iff A$ is <u>symmetric</u>.)

Ex/ $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ //

- $(AB)^T = B^T A^T$:

$$\begin{aligned} [(AB)^T]_{ij} &= [AB]_{ji} = \sum_k a_{jk} b_{ki} \\ &= \sum_k b_{ki} a_{jk} = \sum_k [B^T]_{ik} [A^T]_{kj} \\ &= [B^T A^T]_{ij}. \end{aligned}$$