Before formalizing Gauss-Jordan in terms of a fixed procedure for row-reducing $A$, we briefly review some properties of matrix multiplication.

Let $m\{ [A_{ij}] \}$, $n\{ [B_{jk}] \}$, $p\{ [C_{kl}] \}$ be matrices, with entries in (say) $\mathbb{R}$, or more generally any “field” (cf. §II.A). Recall that the transpose $^tA$ is the $n \times m$ matrix with entries

$$(^tA)_{ij} = A_{ji}.$$ 

I write the superscript “$t$” on the left so that later we can talk about the transpose inverse $^tA^{-1}$ without parentheses. The $m \times m$ “identity” matrix – with entries 1 if $i = j$ and 0 otherwise – will be denoted by $I_m$ (or just $I$); and we will write $E^m_{ij}$ (or just $E_{ij}$) for the matrix with $(i, j)^{th}$ entry 1 and all other entries zero.

**Multiplication:** At any rate, we define the matrix product $AB$ to be the $m \times p$ matrix with entries

$$(AB)_{ik} := \sum_{j=1}^{n} A_{ij}B_{jk}.$$ 

Associativity of this product follows from associativity of the ground field:

$$(AB)C_{il} := \sum_k (AB)_{ik}C_{kl} = \sum_k (\sum_j A_{ij}B_{jk})C_{kl} = \sum_{j,k} (A_{ij}B_{jk})C_{kl}$$

$$= \sum_{j,k} A_{ij}(B_{jk}C_{kl}) = \ldots = (A(BC))_{il}.$$
Commutativity fails: $BA$ is not even defined unless $p = m$, in which case the closest one has is $BA = \, ^tA^tB$ . An example where $A$ and $B$ are actually symmetric:
\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
0 & 0 \\
\end{bmatrix}
\neq \begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
\end{bmatrix}.
\]

For a physicist, noncommutativity is essential, since it’s the entire point of the Heisenberg uncertainty principle that the position and momentum operators don’t commute! Or as the Mad Hatter says, “seeing what you eat” and “eating what you see” are not at all the same thing.\(^1\)

\textbf{Inverses:} if $A$ is $m \times n$ for $m < n$, it cannot have a left inverse ($L$ such that $LA = \mathbb{I}_n$) but may have many right inverses ($R$ such that $AR = \mathbb{I}_m$). An example, where $a, b$ can be any real numbers:
\[
A
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
t & 1 & \cdots & 0 \\
0 & t & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 - at & -bt \\
a & b \\
0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}.
\]

If $m > n$ then the situation is just reversed. For square matrices ($m = n$) we will prove in §I.D that $\exists$ of a left inverse$\iff \exists$ of a right inverse]. But if \textit{both} exist for a matrix $A$, they must be the same: $BA = \mathbb{I}$, $AC = \mathbb{I} \implies B = BI = B(AC) = (BA)C = IC = C$. In this situation we say $A$ is invertible, denoting the (left and right) inverse matrix by $A^{-1}$. Products and inverses of invertible matrices are invertible; e.g. for products $AB$, $B^{-1}A^{-1}$ furnishes a 2-sided inverse.

\textbf{Example 1.} If $\alpha \delta - \beta \gamma \neq 0$ (for $\alpha, \beta, \gamma, \delta$ in your favorite field), we have
\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{bmatrix}^{-1} = \frac{1}{\alpha \delta - \beta \gamma} \begin{bmatrix}
\delta & -\beta \\
-\gamma & \alpha \\
\end{bmatrix}.
\]

\(^1\)though putting these words in the Mad Hatter’s mouth may have been a polemic on Lewis Carroll’s part against the quaternions.
Vectors and matrix multiplication: For \( \vec{x} \in \mathbb{R}^n \), here are some characterizations of the matrix-vector product \( A\vec{x} \) in terms of rows and columns of \( A \):

\[
A\vec{x} = \left( \begin{array}{c} \uparrow \\ \vec{c}_1 \\ \downarrow \\
\vdots \\ \vec{c}_n \\
\downarrow \end{array} \right) \left( \begin{array}{c} \uparrow \\ \vec{x} \\ \downarrow \end{array} \right) = \left( \begin{array}{c} \vec{r}_1 \cdot \vec{x} \\
\vdots \\
\vec{r}_m \cdot \vec{x} \end{array} \right)
\]

\[
= \left( \begin{array}{c} \uparrow \\ \vec{c}_1 \\ \downarrow \\
\vdots \\ \vec{c}_n \\
\downarrow \end{array} \right) \left( \begin{array}{c} x_1 \\
\vdots \\
x_n \end{array} \right) = x_1\vec{c}_1 + \ldots + x_n\vec{c}_n = \sum_{i=1}^{n} x_i\vec{c}_i.
\]

Writing \( \hat{e}_i \) for the coordinate vectors of \( \mathbb{R}^n \), we see that \( A\hat{e}_i = \vec{c}_i \) and so

\[
A = \left( \begin{array}{c} \uparrow \\ A\hat{e}_1 \\ \downarrow \\
\vdots \\ A\hat{e}_n \\
\downarrow \end{array} \right).
\]

(With this understood, you should now be able to easily convince yourself that the columns of a matrix product \( AB \) are linear combinations of the columns of \( A \)!) 

There are two different ways to multiply vectors as matrices:

\[
\begin{bmatrix} 2 & -1 & 1 \\ \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = 4 = \text{dot (interior) product},
\]

\[
\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -2 \\ -3 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix} = \text{exterior product}.
\]

In particular, if \( \vec{x} \) and \( \vec{y} \) are two column vectors, then the dot product \( \vec{x} \cdot \vec{y} \) in terms of matrix multiplication is \( \vec{x}^T \cdot \vec{y} \) (where \( X \) and \( Y \) are the corresponding \( m \times 1 \) matrices).
Elementary matrices: These are \( m \times m \) (square) matrices of one of the following three types:

\[
S^m_{ij} := \mathbb{I}_m - E^m_{ii} - E^m_{jj} + E^m_{ij} + E^m_{ji} = \\
\begin{pmatrix}
1 & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & 1
\end{pmatrix},
\]

\[
S^m_i(\frac{1}{a}) := \mathbb{I}_m + (\frac{1}{a} - 1)E_{ii} = \\
\begin{pmatrix}
1 & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & \frac{1}{a}
\end{pmatrix},
\]

\[
and \ R^m_{ji}(-b) := \mathbb{I}_m - bE^m_{ji} = \\
\begin{pmatrix}
1 & & & & \\
& \ddots & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & 1
\end{pmatrix}.
\]

I’ll drop the superscript \( m \) in the sequel sometimes. The elementary row operations of §1.B may be interpreted as left-multiplying the augmented matrix (representing our linear system) by one of these elementary matrices: the first exchanges the \( i \)th and \( j \)th rows of the matrix it operates on; the second divides the \( i \)th row by \( a \); and the third subtracts \( b \times (i^{th} \text{ row}) \) from the \( j^{th} \) row. All three types of matrices are clearly invertible, with

\[
S_{ij}^{-1} = S_{ij}, \quad S_i(\frac{1}{a})^{-1} = S_i(a), \quad \text{and} \quad R_{ji}(-b)^{-1} = R_{ji}(b).
\]

Exercises

1. Find two different \( 2 \times 2 \) matrices \( A \) such that \( A^2 = 0 \) but \( A \neq 0 \).

2. By carrying out Gauss-Jordan and keeping track of your steps, find elementary matrices \( E_1, \ldots, E_k \) such that \( E_k \cdots E_2E_1A = \mathbb{I} \),
where
\[
A := \begin{pmatrix}
1 & -1 & 1 \\
2 & 0 & 1 \\
3 & 0 & 1
\end{pmatrix}.
\]

(3) Let \(A\) be an upper triangular \(m \times m\) matrix. (That is, \(A_{ij} = 0\) for \(i > j\).) Show that \(A\) is invertible if and only if all the diagonal entries \(A_{ii}\) are nonzero.

(4) Consider the set \(H \subset M_2(\mathbb{C})\) (of \(2 \times 2\) matrices with complex entries) of the form
\[
x = \begin{pmatrix}
\alpha & \beta \\
-\bar{\beta} & \bar{\alpha}
\end{pmatrix} = \begin{pmatrix}
a_0 + a_1 \sqrt{-1} & a_2 + a_3 \sqrt{-1} \\
-a_2 + a_3 \sqrt{-1} & a_0 - a_1 \sqrt{-1}
\end{pmatrix}, \quad a_i \in \mathbb{R}.
\]
Show that \(H\) is closed under addition and multiplication, and that every nonzero element is invertible. Give an example to show that multiplication is not commutative.