(II.B) Basis and dimension

How would you explain that a plane has two dimensions? Well, you can go in two independent directions, and no more. To make this idea precise, we formulate the

**DEFINITION 1.** A (finite) subset \( \{ \vec{v}_1, \ldots, \vec{v}_m \} \) of a vector space \( V \) (over a field \( F \)) is a *basis* of \( V \) if (a) they are linearly independent (over \( F \)), and (b) they span \( V \) (over \( F \)).

The “over \( F \)” means that the linear combinations are understood to be of the form \( \sum \alpha_i \vec{v}_i \), with \( \alpha_i \in F \). As usual, our “default” will be \( F = \mathbb{R} \).

**EXAMPLE 2.** A basis of \( \mathbb{R}^m \) is given by the standard basis vectors \( \{ \hat{e}_i \}_{i=1}^m \).

Of course, one could say the same for any field: the standard basis vectors are also basis for \( \mathbb{C}^m \), viewed as a vector space over \( \mathbb{C} \). But what if we view \( \mathbb{C}^m \) as a vector space over \( \mathbb{R} \)? Then you need the \( 2m \) vectors \( \hat{e}_1, \ldots, \hat{e}_m \) and \( i\hat{e}_1, \ldots, i\hat{e}_m \) (Why?)

**EXAMPLE 3.** Let \( P_2(x,y) \) denote the vector space of quadratic polynomials in two variables \( x \) and \( y \), with real coefficients. This has the “monomial basis” \( \{ \rho^2_i(x,y) \}_{i=1}^6 = \{ 1, x, y, xy, x^2, y^2 \} \). Similarly, \( \{ \rho^3_i(x,y) \}_{i=1}^{10} = \{ 1, x, y, xy, x^2, y^2, x^2y, y^2x, x^3, y^3 \} \) gives a basis for the space \( P_3(x,y) \) of cubic polynomials.

We’ll give a geometric application of this example below.

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\(^1\)I am not saying that you can’t have infinite bases; indeed, non-finitely-generated vector spaces (like \( P_\infty(x) := \text{all polynomials} \)) may have such a basis. I’m just not defining or discussing them at this stage.
EXAMPLE 4. Let $A$ be an $m \times n$ matrix, and $V$ be the vector subspace of $\mathbb{R}^n$ consisting of solutions to $A\vec{x} = 0$. How do you compute a basis of $V$?

By making use of the row-reduction algorithm, of course! For instance, we might have

$$A = \begin{pmatrix}
0 & 0 & 1 & -1 & -1 \\
2 & 4 & 2 & 4 & 2 \\
2 & 4 & 3 & 3 & 3 \\
3 & 6 & 6 & 3 & 6
\end{pmatrix} \rightarrow \text{rref}(A) = \begin{pmatrix}
1 & 2 & 0 & 3 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$ 

The free variables are $x_2$ and $x_4$, and an arbitrary solution is given by plugging arbitrary values in for these and solving for the pivot variables $x_1, x_3, x_5$. To get a basis, you plug in $(x_2 = 1, x_4 = 0)$ and $(x_2 = 0, x_4 = 1)$ to obtain

$$\begin{pmatrix}
-2 \\
1 \\
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
-3 \\
0 \\
1 \\
1 \\
0
\end{pmatrix}.$$ 

More generally, the $i^{th}$ basis element is given by setting the $i^{th}$ free variable to 1 and the rest to zero.

EXAMPLE 5. Suppose $A$ is an invertible $m \times m$ matrix. I claim that its columns $\{\vec{c}_i\}_{i=1}^m$ are a basis for $\mathbb{R}^m$.

To establish this, check that:

- $\{\vec{c}_i\}$ are linearly independent: $\gamma_1 \vec{c}_1 + \ldots + \gamma_m \vec{c}_m = 0 \implies A\vec{\gamma} = 0 \implies \vec{\gamma} = 0$ (since $A$ is invertible).

- $\{\vec{c}_i\}$ span $\mathbb{R}^m$: show every $\vec{y} \in \mathbb{R}^m$ can be written $\vec{y} = \gamma_1 \vec{c}_1 + \ldots + \gamma_m \vec{c}_m (= A\vec{\gamma})$. This is easy: just put $\vec{\gamma} = A^{-1}\vec{y}$. □

We say that $V$ is finitely generated if some finite subset $\{\vec{v}_1, \ldots, \vec{v}_m\}$ spans $V$. Here is the precise reason why dimension makes sense:

PROPOSITION 6. If $V$ is finitely generated then (i) it has a (finite) basis, and (ii) any 2 bases of $V$ have the same number of elements.
**Definition 7.** We define the *dimension* \( \dim_F(V) \) of \( V \) over \( F \) to be the number of elements in (ii).\(^2\)

We get (i) by going through the generating subset \( \{ \vec{v}_1, \ldots, \vec{v}_m \} \) and throwing out any \( \vec{v}_i \) that is a linear combination of the earlier vectors. What remains is a basis.

In order to show (ii) we first prove the following

**Lemma 8.** If \( \vec{v}_1, \ldots, \vec{v}_m \) span \( V \) then any independent set \( \subseteq V \) has no more than \( m \) vectors.

**Proof.** Consider any set \( \vec{u}_1, \ldots, \vec{u}_n, n > m \) (of vectors in \( V \)). Since \( \vec{v}_1, \ldots, \vec{v}_m \) span \( V \), each \( \vec{u}_j \) is a linear combination of the \( \vec{v}_i \) :

\[
\vec{u}_j = \sum_{i=1}^{m} A_{ij} \vec{v}_i, \quad \text{for } A \, m \times n.
\]

Since \( m < n \), there is a nontrivial solution to \( A\vec{x} = \vec{0} \), some nonzero \( \vec{x} \in \mathbb{R}^n \). That is,

\[
\sum_{j=1}^{n} A_{ij} x_j = 0 \quad \text{for each } i,
\]

and so

\[
\sum_{j=1}^{n} x_j \vec{u}_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij} \vec{v}_i = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} x_j A_{ij} \right) \vec{v}_i = \sum_{i=1}^{m} 0 \vec{v}_i = \vec{0}.
\]

We conclude that \( \{ \vec{u}_j \}_{j=1}^{n} \) is *not* an independent set! \( \square \)

Now let \( \{ \vec{v}_1, \ldots, \vec{v}_m \} \) and \( \{ \vec{w}_1, \ldots, \vec{w}_n \} \) be two bases for \( V \). The \( \{ \vec{v}_i \} \) span \( V \) and the \( \{ \vec{w}_j \} \) are independent, so the Lemma \( \implies n \leq m \). The same argument with \( \{ \vec{v}_i \} \) and \( \{ \vec{w}_j \} \) swapped \( \implies n \geq m \), and so we’ve proved the Proposition. We can rephrase the Lemma as follows: if \( \dim(V) = m \) then (a) less than \( m \) vectors cannot span \( V \), while (b) more than \( m \) vectors are linearly dependent.

**A Geometric Application.** Given 5 points \( A, B, C, D, E \) “in general position” in the plane, there is a unique conic passing through

\(^2\)The \( F \) is typically omitted when there is no ambiguity. That said, in the first example we had \( \dim_{\mathbb{C}}(\mathbb{C}^n) = n \) while \( \dim_{\mathbb{R}}(\mathbb{C}^n) = 2n \).
them. Here’s why: the equation of a conic $Q$ has the form

$$0 = f_Q(x, y) = \sum_{i=1}^{6} a_i \rho_i^2(x, y) = a_1 + a_2 x + a_3 y + a_4 xy + a_5 x^2 + a_6 y^2,$$

and the statement that it passes through $A$ is

$$0 = f_Q(x_A, y_A) = \sum a_i \rho_i(x_A, y_A).$$

Continuing this for $B$ thru $E$ we get 5 equations in 6 unknowns $a_i$, a system which “generically” has a “line” of solutions: if $(a_1, \ldots, a_6)$ solves the system then so does $(c \cdot a_1, \ldots, c \cdot a_6)$. But this just gives multiples of the same $f_Q$, which won’t change the shape of the curve, and so there is really only one nontrivial solution: through five general points in the plane there exists a unique conic $Q$.

Similarly, given 8 points $A$, $B$, $C$, $D$, $E$, $P_1$, $P_2$, $P_3$ what can we say about the “vector space” of cubics passing through them (i.e. of cubic polynomials in $(x,y)$ vanishing at all 8 points)? First of all, the vector space of all cubics

$$\sum_{i=1}^{10} a_i \rho_i^3(x, y) = a_1 + a_2 x + a_3 y + \ldots + a_9 x^3 + a_{10} y^3$$

should be thought of in terms of the coefficient vector $\vec{a}$. The 8 constraints give a linear system

$$\begin{align*}
0 = \sum_{i=1}^{10} a_i \rho_i^3(x_A, y_A) \\
\vdots \\
0 = \sum_{i=1}^{10} a_i \rho_i^3(x_{P_3}, y_{P_3})
\end{align*}$$

whose space of solutions “in general” (i.e., under the assumption of maximal rank[=8]) is two-dimensional – there are 8 leading 1s in $\text{rref}$ of the $8 \times 10$ $[\rho]$ matrix, and therefore 2 parameters to choose freely.

So start with $A$, $B$, $C$, $D$, $E$ in “general position”; we’d like to construct points on the (unique) conic through them, using a straightedge. Begin by drawing the lines $AB$, $DE$, $CD$, and $BC$; label

more precisely, “generically” means here that the matrix $[\rho]$ (with rows indexed by $A$ thru $E$, columns by $i = 1$ thru 6) is of maximal rank ($= 5$).
$AB \cap DE$ by $P_1$. ($Q$ is drawn in a background color because we don’t know what it looks like yet.)

Now draw (almost) any line $\ell$ through $A$ – the choice is free except that $\ell$ shouldn’t go through $B$, $C$, $D$, or $E$. All the rest of the construction depends on this choice; in particular the final point $q$ depends on $\ell$, and so by varying $\ell$ we can vary $q$ and in this way (we hope) construct enough points on $Q$ to get an idea of what it looks like. Label $\ell \cap CD =: P_2$; then draw $P_1P_2$ and label $P_1P_2 \cap BC =: P_3$; finally set $q := EP_3 \cap \ell$. We must prove $q \in Q$. Adding dotted lines to our diagram for the lines depending on $\ell$, 
Now consider the following three cubic equations (where e.g. \( f_{AB}(x,y) = 0 \) is the linear equation of the line \( AB \), and so on):

\[
\begin{align*}
  f_1(x,y) &:= f_Q(x,y) \cdot f_{P_1P_2}(x,y) = 0, \\
  f_2(x,y) &:= f_{AB}(x,y) \cdot f_{CD}(x,y) \cdot f_{EP_3}(x,y) = 0, \\
  f_3(x,y) &:= f_1(x,y) \cdot f_{BC}(x,y) \cdot f_{DE}(x,y) = 0.
\end{align*}
\]

The graphical solutions of these equations are (respectively)

\[ Q \cup P_1P_2, \; AB \cup CD \cup EP_3, \; \ell \cup BC \cup DE. \]

All 3 of these unions contain the 8 points \( A, B, C, D, E, P_1, P_2, P_3 \); that is, the cubic polynomials \( f_1, f_2, \) and \( f_3 \) each vanish at all 8 points. Above we explained that the (vector) space of such cubic polynomials was \( 10 - 8 = 2 \) dimensional; it follows that 3 “vectors” \( f_1, f_2, f_3 \) cannot be linearly independent and there is a nontrivial relation \( \alpha f_1(x,y) + \beta f_2(x,y) + \gamma f_3(x,y) = 0 \). Now we extract the information we want (\( q \in Q \)) from this relation.

First of all \( \alpha \) cannot be zero. Otherwise \( \beta f_2 = -\gamma f_3 \) are multiples of each other, which means that \( AB \cup CD \cup EP_3 = \ell \cup BC \cup DE \). That would imply \( \ell \) was either \( AB, CD, \) or \( EP_3 \); and we said \( \ell \) cannot pass through \( B, C, D, \) or \( E \) so this gives a contradiction.

So we may divide by \( \alpha \) \( \implies f_1(x,y) = -\frac{\beta}{\alpha} f_2(x,y) - \frac{\gamma}{\alpha} f_3(x,y) \) . Now since \( q \) is contained in both \( EP_3 \) and \( \ell, f_2(x,y) \) and \( f_3(x,y) \) both vanish at \( q = (x_q, y_q) \). (This is because \( f_2 \) contains a factor of \( f_{EP} \) and \( f_3 \) contains a factor of \( f_\ell \).) According to the relation, therefore also \( f_1 = 0 \) at \( q \); this means that either \( f_Q \) or \( f_{P_1P_2} \) vanishes at \( q \), and so \( q \in Q \) or \( P_1P_2 \). Suppose the latter: if \( q \in P_1P_2 \) then the three lines \( P_1P_2, EP_3, \ell \) are the same, and so \( E \in \ell \). But once again, we disallowed such a choice of \( \ell \) at the outset. So \( q \in Q \), and we’re done.

Another way of thinking of this result is that intercepts of opposite edges of a hexagon inscribed in a conic lie on line. This is a theorem proved by Pascal in 1639 known as his “mystic hexagon”!
Exercises

(1) Let $V$ be a vector space over a subfield $F$ of the complex numbers. Suppose $\vec{u}, \vec{v}, \vec{w}$ are linearly independent vectors in $V$. Prove that $(\vec{u} + \vec{v}), (\vec{v} + \vec{w})$, and $(\vec{w} + \vec{u})$ are linearly independent.

(2) Let $V$ be the set of all $2 \times 2$ matrices $A$ with complex entries which satisfy $A_{11} + A_{22} = 0$.

(a) Show that $V$ is a vector space over the field of real numbers, with the usual operations of matrix addition and multiplication of a matrix by a scalar.

(b) Find a basis for this vector space.

(c) Let $W$ be the set of all matrices $A$ in $V$ such that $A_{21} = -\overline{A_{12}}$ (the bar denotes complex conjugation). Prove that $W$ is a subspace of $V$ and find a basis for $W$.

(3) Prove that the space of all $m \times n$ matrices over the field $F$ has dimension $mn$, by exhibiting a basis for this space.