II.D. There is exactly one rref matrix row-equivalent to any \( A \)

Recall that “\( A \) is row-equivalent to \( B \)” means \( A = E_M \cdots E_1 B \), where \( E_i \) are (invertible) elementary matrices, which we think of as row-operations. In §I.D, to each matrix \( A \) we associated a (well-defined) row-equivalent matrix \( \text{rref}(A) \) in reduced-row echelon form, by means of an algorithmic sequence of row-operations \( E_1, \ldots, E_N \): writing \( \mathcal{E} = E_N \cdots E_1 \) for their product, we have

\[
R := \text{rref}(A) := \mathcal{E} \cdot A.
\]

What if a different series of invertible row operations with product \( \mathcal{E} \) yields an rref matrix \( \mathcal{E} \cdot A \)? Consider this mysterious example:

\[
(A : I_3) \quad \left( \begin{array}{ccc|ccc}
1 & 2 & 4 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 \\
0 & -6 & -8 & 0 & 0 & 1 \\
\end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc}
1 & 2 & 4 & 1 & 0 & 0 \\
0 & -3 & -4 & -1 & 1 & 0 \\
0 & -6 & -8 & 0 & 0 & 1 \\
\end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc}
1 & 2 & 4 & 1 & 0 & 0 \\
0 & -3 & -4 & -1 & 1 & 0 \\
0 & 0 & 0 & 2 & -2 & 1 \\
\end{array} \right)
\]

\[
(R : \mathcal{E}) \quad \left( \begin{array}{ccc|ccc}
1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & 2 & -2 & 1 \\
\end{array} \right)
\]

\[
('R : \mathcal{E}) \quad \left( \begin{array}{ccc|ccc}
1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & 1 & -1 & \frac{1}{2} \\
\end{array} \right)
\]

The top row is our usual route, while the vertical arrow deviates from it: \( \mathcal{E} \neq \mathcal{E} \), and yet \( 'R = R \). How can this be? If \( 'R = R \) then \( \mathcal{E} \cdot A = \mathcal{E} \cdot A \), or \( (\mathcal{E}^{-1} \cdot \mathcal{E}) A = A \), and so \( \mathcal{E}^{-1} \cdot \mathcal{E} = \text{identity} \), right? If \( A \) is invertible, right; otherwise, watch out! For example,

\[
\left( \begin{array}{cc}
1 & a \\
0 & b \\
0 & c \\
\end{array} \right) \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{array} \right) = \left( \begin{array}{cc}
1 & 0 \\
0 & 1 \\
0 & 0 \\
\end{array} \right).
\]

In fact, no matter what procedure you use, you’ll get the same \( R \), as we now prove.

II.D.1. DEFINITION. If \( \{\vec{r}_j\}_{j=1}^m \) are the rows of an \( m \times n \) matrix \( A \) then the row space is

\[
V_{\text{row}}(A) := \text{span}\{\vec{r}_1, \ldots, \vec{r}_m\}.
\]
For $P$ a $k \times m$ matrix, the rows of $PA$ are
\[ \left\{ \sum_{j=1}^{m} P_{ij}r_j \right\}_{i=1}^{k}, \]
which are linear combinations of the rows of $A$ (and thus in their
span), so
\[ V_{\text{row}}(P \cdot A) \subseteq V_{\text{row}}(A). \]
If $P$ is invertible then
\[ V_{\text{row}}(PA) \supseteq V_{\text{row}}(P^{-1} \cdot PA) = V_{\text{row}}(A). \]
Since elementary row-operations are invertible, \textit{row-equivalent matrices}
\textit{have the same row space}. Now take $R = E \cdot A$ and \( 'R = 'E \cdot A \) any
two \textit{rref} matrices row-equivalent to $A$. Then
\[ V_{\text{row}}(R) = V_{\text{row}}(A) = V_{\text{row}}('R), \]
and the problem boils down to showing
\[ (\text{II.D.2}) \quad V_{\text{row}}(R) = V_{\text{row}}('R) \implies R = 'R \]
for any two \textit{rref} matrices, period.

\textbf{II.D.3. DEFINITION.} Given any \textit{rref} matrix $R$, define
\[ \mathcal{K}(R) := \left\{ k \in \mathbb{N} \mid R \text{ has a leading 1 in the } k^{\text{th}} \text{ column} \right\}. \]
Clearly these are just the $k_1 < \cdots < k_r$ indexing the pivot columns,
with $r$ the rank of $R$. Denote the rows of $R$ by $\rho^1(R), \ldots, \rho^m(R)$ (only
the first $r$ are nonzero). Then we have the following restatement of
Theorem II.C.4:

\textbf{II.D.4. THEOREM.} Let $R$ be any $m \times n$ \textit{rref} matrix, and $W = V_{\text{row}}(R)$.
Then $\mathcal{K}(R) = \mathcal{L}(W)$ and \( \{\rho^i(R)\}_{i=1}^{r} = \{\Psi^i(W)\}_{i=1}^{r} \). This says the rows
of an \textit{rref} matrix are the standard basis of its row space, and its rank is the
dimension of its row space.

\textbf{II.D.5. REMARK.} An easy corollary of the last statement is that
the nonzero rows of an \textit{rref} are always linearly independent (this is
not too hard to show by other means).
II.D. THERE IS EXACTLY ONE RREF MATRIX ROW-EQUIVALENT TO ANY $A$

PROOF OF THEOREM II.D.4. First observe that every row $\bar{\rho}$ of $R$ is in $W$ (= row space of $R$). If $k \in \mathcal{K}(R)$ then the $k^\text{th}$ column of $R$ has a “leading 1”. Let $\bar{\rho}$ be the row of $R$ to which this leading 1 belongs, so that $\rho_k = 1$ is its first nonzero entry ($\rho_1 = \cdots = \rho_{k-1} = 0$). Since also $\bar{\rho} \in W$, $k$ must be in the leading set of $W$, i.e. $k \in \mathcal{L}(W)$. We have just shown $\mathcal{K}(R) \subseteq \mathcal{L}(W)$.

Write $r$ for $\text{rank}(R) = \# \text{ of leading 1s} = \# \{ \mathcal{K}(R) \}$, and recall that also $r = \# \text{ of nonzero rows of } R$. Since $r$ vectors span a space of dimension $\leq r$,

$$\dim(V_{\text{row}}(R)) = \dim(\text{span}\{\bar{\rho}_1, \ldots, \bar{\rho}_r\}) \leq r$$

or $$\dim(W) \leq \# \{ \mathcal{K}(R) \}.$$ 

Now $\# \{ \mathcal{L}(W) \}$ is the number of elements in a basis (the standard basis!) for $W$, so

$$\dim(W) = \# \{ \mathcal{L}(W) \}, \quad \text{and}$$

$$\# \{ \mathcal{K}(R) \} \geq \# \{ \mathcal{L}(W) \}.$$ 

On the other hand, as sets we showed $\mathcal{K}(R) \subseteq \mathcal{L}(W)$ above, which implies

$$\# \{ \mathcal{K}(R) \} \leq \# \{ \mathcal{L}(W) \}.$$ 

So $\# \{ \mathcal{K}(R) \} = \# \{ \mathcal{L}(W) \}$, and the inclusion $\mathcal{K}(R) \subseteq \mathcal{L}(W)$ must be an equality (of sets). Therefore $\{ k_1 < \cdots < k_r \} = \{ \ell_1 < \cdots < \ell_\gamma \}$ (and $r = \gamma$), or simply $k_i = \ell_i$ (for all $i$).

Now $\bar{\Psi}_i(W)$ is the unique vector in the row space ($W = V_{\text{row}}(R)$) of $R$ with 1 in the $\ell_i (= k_i)$th place, and a 0 in all $\ell_j (= k_j)$th places, for $j \neq i$. But $\bar{\rho}^i(R)$ gives such a vector, since it has a leading 1 in the $k_i$th place, and 0’s in the places where the other leading 1’s occur (namely, the $k_j$ places, $j \neq i$). These are the same conditions. By uniqueness of the vector satisfying them $\bar{\Psi}_i = \bar{\rho}^i$. This concludes the proof of the Theorem. \qed
So if
\[ V_{row}(R) = W = V_{row}(′R), \]
we apply the Theorem to each equality to get
\[ \mathcal{K}(R) = \mathcal{L}(W) = \mathcal{K}(′R), \]
\[ \{\tilde{\rho}^i(R)\} = \{\tilde{\Psi}^i(W)\} = \{\tilde{\rho}^i(R)\}. \]
But if the rows \( \tilde{\rho}^i(R) \) and \( \tilde{\rho}^i(′R) \) of the two \( \text{ref} \) matrices are the same, they’re the same matrix: \( R = ′ R. \) This concludes the proof of (II.D.2), . . . and of the title of this section! \( \square \)

Now I give you a handful of vectors, say
\[ \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 9 \\ 10 \\ 11 \\ 12 \end{pmatrix}, \]
and ask you for the dimension of \( W = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}, \) and a basis to back it up, fast! Then you perform the following magic trick: write
\[ A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \rightarrow \text{ref}(A) = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \end{pmatrix}, \]
and, voilà, your basis is
\[ \tilde{\rho}^1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \quad \tilde{\rho}^2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \]
so that \( \text{dim}(W) = 2. \) Why does this work? Since \( A \) and \( \text{ref}(A) \) are row-equivalent,
\[ \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = V_{row}(A) = V_{row}(\text{ref}(A)), \]
and by our theorem the (nonzero) entries of \( \text{ref}(A) \) give the standard basis for \( V_{row}(\text{ref}(A)). \)
II.D. There is exactly one RREF matrix row-equivalent to any $A$ 43

Now recall from §II.C that for $W \subseteq \mathbb{R}^n$ with standard basis $\{\vec{v}^i\}$ and leading set $\mathcal{L}(W)$, any $\vec{\beta} \in W$ is of the form

$$\vec{\beta} = \sum_{i=1}^{\dim(W)} \beta_{\ell_i} \vec{v}^i.$$  

If $W = V_{row}(A)$ for some $m \times n$ matrix $A$, and $\vec{\rho}^i$ are the (nonzero) rows of $rref(A)$, then $W = V_{row}(rref(A))$ and our Theorem gives

$$\vec{\beta} = \sum_{i=1}^r \beta_{k_i} \vec{\rho}^i. \tag{II.D.6}$$

Writing

$$A = \begin{pmatrix} \leftarrow & \vec{\nu}_1 & \rightarrow \\ \vdots \\ \leftarrow & \vec{\nu}_m & \rightarrow \end{pmatrix}$$

this makes it easy (by taking $rref(A)$ and finding $\{\vec{\rho}^i\}$) to determine whether a given $\vec{\beta}$ is in $span\{\vec{v}_1, \ldots, \vec{v}_m\}$. (Does $\vec{\beta}$ satisfy (II.D.6)? Then it is in the span.) If it is, then it’s easy to write $\vec{\beta}$ as a linear combination of the $\vec{\rho}^i$’s: you know the $k_i$ (location of $i^{th}$ pivot column in $rref(A)$); and since you know $\vec{\beta}$, you know its $k_i^{th}$ entries $\beta_{k_i}$.

On the other hand, how do you write $\vec{\beta}$ as a linear combination of the original $\{\vec{v}_i\}$? One way is to solve the inhomogeneous system

$$^tA \cdot \vec{\alpha} = \vec{\beta}, \quad \text{i.e.}$$

$$\alpha_1 \vec{v}_1 + \ldots + \alpha_m \vec{v}_m = \begin{pmatrix} \uparrow & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_m \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \vec{\beta},$$

Alternatively, we may write

$$^t\vec{\beta} = \sum \beta_{k_i} ^t \vec{\rho}^i = \begin{bmatrix} \beta_{k_1} & \cdots & \beta_{k_r} & 0 & \cdots & 0 \end{bmatrix}_m \begin{pmatrix} \leftarrow & \vec{\rho}^1 & \rightarrow \\ \vdots \\ \leftarrow & \vec{\rho}^r & \rightarrow \\ \leftarrow & 0 & \rightarrow \end{pmatrix}$$
II. VECTOR SPACES

\[
\begin{pmatrix}
\beta_{k_1} & \cdots & \beta_{k_r} & 0 & \cdots & 0
\end{pmatrix} \cdot \text{rref}(A)
\]

\[
= \begin{pmatrix}
\beta_{k_1} & \cdots & \beta_{k_r} & 0 & \cdots & 0
\end{pmatrix} \cdot E(A) \cdot A
\]

\[
= \left\{ \begin{pmatrix}
\beta_{k_1} & \cdots & \beta_{k_r} & 0 & \cdots & 0
\end{pmatrix} : \begin{pmatrix}
\leftarrow \bar{v}_1 \rightarrow \\
\vdots \\
\leftarrow \bar{v}_m \rightarrow
\end{pmatrix} \right\}
\]

setting

\[
\begin{pmatrix}
\alpha_1 & \cdots & \alpha_m
\end{pmatrix} := \begin{pmatrix}
\beta_{k_1} & \cdots & \beta_{k_r} & 0 & \cdots & 0
\end{pmatrix} \cdot E(A)
\]

then yields

\[
\bar{\beta} = \sum_{i=1}^{m} \alpha_i \bar{v}_i.
\]

Since this way of writing \(\bar{\beta}\) as a linear combination of \(\bar{v}_i\)'s (as opposed to the first option, the inhomogeneous system) involves finding\(^{11}\) \(E(A)\), it’s useful primarily when you have more than one \(\bar{\beta}\) and need a systematic way of dealing with them so that you don’t row-reduce for each one.

**Grassmanians.** We conclude with a geometric application (in the same spirit as Pascal’s hexagon). Define

\[
G_m(\mathbb{R}^n) = \text{Grassmanian of } m\text{-planes in } n\text{-space}
\]

to be the set of \(m\)-dimensional subspaces \(W \subseteq \mathbb{R}^n\) thought of as a “topological space”. That is, each “\(m\)-plane” (thru the origin) is considered a point of \(G_m(\mathbb{R}^n)\); alternately, you can think of \(G_m(\mathbb{R}^n)\) as a smooth manifold whose points are in 1-to-1 correspondence with \(m\)-planes.

We can use Theorem II.D.4 to begin to see what these look like, in particular to get a coordinate system on (a big open set of) them. Since \(W\) is the row-space of a unique \(m \times n\) \text{rref} matrix, there is a 1-to-1 correspondence

\[
\{\text{points of } G_m(\mathbb{R}^n)\} \leftrightarrow \{m \times n\ \text{rref matrices of rank } m\}
\]

\(^{11}\)It also depends on your choice of \(E(A)\), if \(A\) isn’t invertible.
II.D. THERE IS EXACTLY ONE RREF MATRIX ROW-EQUIVALENT TO ANY \( A \)

\[ W \leftrightarrow R. \]

If we write \( \mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \ldots \subseteq \mathbb{R}^n \) where
\[
\mathbb{R}^1 = \text{span}\{\hat{e}_n\}, \quad \mathbb{R}^2 = \text{span}\{\hat{e}_{n-1}, \hat{e}_n\}, \quad \mathbb{R}^3 = \text{span}\{\hat{e}_{n-2}, \hat{e}_{n-1}, \hat{e}_n\},
\]
then the jumps (for a given \( W \)) in the sequence
\[
0 \leq \dim(W \cap \mathbb{R}^1) \leq \dim(W \cap \mathbb{R}^2) \leq \ldots \leq \dim(W \cap \mathbb{R}^n) = m
\]
correspond to the integers \( k_i \) (= columns in which leading 1s occur, in \( R \)) via
\[
\dim(W \cap \mathbb{R}^{n-k_i+1}) = 1 + \dim(W \cap \mathbb{R}^{n-k_i}).
\]

So for the Grassmanian of 2-planes in 4-space the “points” correspond to (row spaces of) matrices of the form:
\[
\begin{pmatrix}
1 & 0 & * & * \\
0 & 1 & * & *
\end{pmatrix},
\]
\[
\begin{pmatrix}
1 & * & 0 & * \\
0 & 0 & 1 & *
\end{pmatrix},
\]
\[
\begin{pmatrix}
0 & 1 & * & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
\[
\begin{pmatrix}
0 & 1 & * & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
where the *’s are the coordinates. Since the first type of point has the most degrees of freedom, it corresponds to a big open set of the same dimension as \( G_2(\mathbb{R}^4) \), which in this case is 4. (The remaining \( rref \)'s correspond to [points in] subsets of \( G_2(\mathbb{R}^4) \) of lesser dimension.) More generally, \( \dim(G_m(\mathbb{R}^n)) = (n-m)m \) as we can see from the \( m \times n \text{ rref} \) matrix
\[
\begin{pmatrix}
1 & 0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & * & \cdots & *
\end{pmatrix}.
\]

The most widely used of these is the case of “projective \( n \)-space”
\[
\mathbb{R}\mathbb{P}^n := G_1(\mathbb{R}^{n+1}),
\]
consisting of all lines (considered as points) in \( n + 1 \) space. Its importance is that it can be thought of as \( \mathbb{R}^n + \text{“directions at } \infty \text{”} \); the
points at $\infty$ are important for completing solution sets (of polynomial equations) to compact sets. In any case, the $\mathbb{R}^n$ part comprises $rref$’s of the form
\[ \left( 1 \ * \ \cdots \ * \right), \]
and the stuff “at $\infty$” is parametrized by all the other $rref$’s:
\[ \left( 0 \ 1 \ * \ \cdots \ * \right), \ldots, \left( 0 \ \ldots \ 0 \ 1 \right). \]

In particular, if $n = 2$ then the addition of a “$\mathbb{RP}^1$ at infinity” to $\mathbb{R}^2$ gives parallel lines a place to intersect!\(^{12}\) More generally, Bézout’s Theorem says that if $f(x,y)$ and $g(x,y)$ are polynomials of degrees $d$ and $e$ respectively, then the “curves” defined by $f = 0$ and $g = 0$ intersect in exactly $de$ points provided one works in the complex projective plane $\mathbb{CP}^2$ and counts an “order $m$ intersection point” $m$ times (e.g. $y = 0$ and $y = x^2$ meet “twice” at $(0,0)$ because they are tangent there).

An even more fundamental object in mathematics is the complex projective line, $\mathbb{CP}^1$. Topologically, it looks like a sphere; if you remove the North pole you are left with the complex plane.\(^ {13}\) Its points are identified with lines through the origin in $\mathbb{C}^2$; we can keep track of these in two ways. The first is by $rref$ $1 \times 2$ matrices: namely, $(1 \ z)$ (with $z \in \mathbb{C}$) and $(0 \ 1)$ (the “point at $\infty$”/North pole). Alternatively, we may use ordered pairs $[Z_0 : Z_1]$ in $\mathbb{C}^2 \setminus \{(0,0)\}$ representing (a nonzero point on) a line through the origin; given $\lambda \neq 0$, $[\lambda Z_0 : \lambda Z_1]$ is on the same line, so determines the same “point” in $\mathbb{CP}^1$. (These ordered pairs modulo scalings are called projective coordinates and generalize naturally to projective $n$-space over any field.) The coordinate $z$ above is recovered by $Z_1/Z_0$.

Going back to “reality”, here’s another way of thinking of $\mathbb{RP}^n$: take the $n$-sphere (solutions of $x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1$ in $\mathbb{R}^{n+1}$), and identify all opposite points. This makes $\mathbb{RP}^1$ just a circle (wound up

\(^{12}\)You may have noticed that a fix like this is needed for the Pascal construction to work when some pairs of lines are parallel.

\(^ {13}\) $\mathbb{CP}^2$ has real dimension 4 so is harder to visualize . . .
twice), but \( \mathbb{R}P^2 \) already hard to “see”. This is because it can’t fit in \( \mathbb{R}^3 \) without crossing itself! (It can fit in \( \mathbb{R}^4 \). More generally, for \( n \) a power of 2, \( \mathbb{R}P^n \) can’t fit inside \( \mathbb{R}^{2n-1} \).)

**Exercises**

(1) Let \( V \subset \mathbb{R}^5 \) be the vector space spanned by the rows of the matrix

\[
A = \begin{pmatrix}
3 & 21 & 0 & 9 & 0 \\
1 & 7 & -1 & -2 & -1 \\
2 & 14 & 0 & 6 & 1 \\
6 & 42 & -1 & 13 & 0
\end{pmatrix}.
\]

(a) Find the “standard basis” \( \{ \overline{\Psi}^i \} \) of \( V \).

(b) Given \( \overline{\beta} \in V \), write out \( \overline{\beta} = \sum_{i=1}^{\dim(V)} \beta_{\ell_i} \overline{\Psi}^i \) explicitly. (Express your answer in the form \( \overline{\beta} = (\ast, \ast, \ast, \ast) \), where some \( \ast \)'s are \( \beta_{\ell_i} \) and the others are linear functions of the \( \beta_{\ell_i} \).

(c) Which of the following are in the row space: \((-2, -14, 1, -1, 8)\), \((1, 7, -1, 2, 5)\), \((1, 7, -1, -2, -5)\), \((-2, -14, 1, 1, 8)\)? (You should be able to do this without any additional work.)

(2) For which \((y_1, y_2, y_3, y_4)\) does the “inhomogeneous linear system” \( A\vec{x} = \vec{y} \), where

\[
A = \begin{pmatrix}
3 & -6 & 2 & -1 \\
-2 & 4 & 1 & 3 \\
0 & 0 & 1 & 1 \\
1 & -2 & 1 & 0
\end{pmatrix},
\]

have a solution? Determine this in two different ways (which should give the same answer!):

(a) by applying the \( \text{rref} \) algorithm to the augmented matrix \((A \mid \vec{y})\); and

(b) by applying the \( \text{rref} \) algorithm to \( ^tA \) and then using criterion (II.D.6) for \( ^t\vec{y} \) to be in its row space.
(3) Fractional linear transformations are maps of the form
\[ f_M(z) = \frac{az + b}{cz + d}, \quad \text{where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}). \]

If we think of this as a map from \( \mathbb{C} \) to itself, it isn’t defined at \( z = -\frac{d}{c} \) and doesn’t hit the point \( z = \frac{a}{c} \). It’s better to think of it as a map from \( \mathbb{C}P^1 = \mathbb{C} \cup \{ \infty \} \) to itself. Show that:

(a) in projective coordinates \([Z_0 : Z_1]\), \( f_M \) is given by matrix-vector multiplication;

(b) composition of such maps is given by matrix multiplication.

(4) [highly optional but interesting] Let \( S_k = S_{k_1, \ldots, k_m} \) denote the set of \( \text{ref} \) \( m \times n \) matrices of rank \( m \) with pivot columns \( k_1, \ldots, k_m \). We may regard each such \( S_k \) as a subset of the Grassmannian \( G_m(\mathbb{R}^n) \), which is the (disjoint) union of all of them. These sets are indexed by the subsets of \( \{1, \ldots, n\} \) of cardinality \( m \), of which there are \( \binom{n}{m} \).

(a) Explain why \( S_{k'} \) is in the closure of \( S_k \) if and only if \( k'_i \geq k_i \) for each \( i \). (Consider limits of the row spaces of \( \text{ref} \) matrices as some of the “*” entries go to \( \infty \).) This yields a poset (i.e., partially ordered set) structure on these subsets of \( G_m(\mathbb{R}^n) \).

(b) Draw a directed graph (diagram with vertices and arrows between them) exhibiting this poset structure for some values of \( m \) and \( n \): an arrow points from \( k \) to \( k' \) iff \( k'_j > k_i \) for one \( i \) and \( k_j = k'_j \) for \( j \neq i \). (For instance, if \( (m, n) = (1, 2) \) then there are two dots with a single arrow between them.) When is this order linear? When can it be drawn in a plane without the arrows crossing?

(c) If we think of points of \( G_m(\mathbb{R}^n) \) as \( m \)-dimensional subspaces \( W \subset \mathbb{R}^n \) (instead of \( \text{ref} \) matrices), you can consider their image under multiplication by invertible upper-triangular \( n \times n \) matrices \( U \), by \( W \mapsto U(W) \subset \mathbb{R}^n \). Show that if \( W \in S_k \), then you get all other “points” in \( S_k \) (and nothing else) in this way. The \( \{ S_k \} \) are thus the orbits of the action on \( G_m(\mathbb{R}^n) \) by the group of such matrices.