Let's take a closer look at the characteristic polynomial for a $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, namely

$$f_A(\lambda) = \det(\lambda I - A) = (\lambda - a)(\lambda - d) - bc$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc)$$

$$= \lambda^2 - (\text{tr} A)\lambda + (\det A).$$

More generally, for $A \in M_n(F)$ one can check by direct calculation that

$$f_A(\lambda) = \lambda^n - (\text{tr} A)\lambda^{n-1} + P_2(A)\lambda^{n-2} - \ldots + (-1)^n \det A$$

$$=: \sum_{k=0}^{n} (-1)^k P_k(A)\lambda^{n-k}$$

where $P_1(A) = \text{tr}(A)$, $P_n(A) = \det(A)$. The $P_k(A)$ so defined are “invariant” polynomials (of degree $k$) in the entries of $A$:

$$P_k(A) = P_k(S^{-1}AS),$$

as one can see by comparing coefficients of like powers of $\lambda$ in the end terms of

$$\sum (-1)^k P_k(A)\lambda^{n-k} = \det(\lambda I - A) = \det\{S^{-1}(\lambda I - A)S\}$$

$$= \det\{\lambda I - S^{-1}AS\} = \sum (-1)^k P_k(S^{-1}AS)\lambda^{n-k}.$$

If you didn’t know $\text{tr}(A) = \text{tr}(S^{-1}AS)$ for all invertible $S$ (say, by not doing the exercises), now you do.
In particular, if $A$ is diagonalizable ($= SDS^{-1}$) then
\[ P_k(A) = P_k(SDS^{-1}) = P_k(D). \]

Now by definition
\[
f_D(\lambda) = \det \begin{pmatrix}
\lambda - \lambda_1 \\
\vdots \\
\lambda - \lambda_n
\end{pmatrix} = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)
\]
\[ = \sum (-1)^k P_k(D) \lambda^{n-k}, \]
and so
\[
P_k(A) = \left\{ \text{coefficient of } \lambda^{n-k} \text{ in } (\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \right\} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}
\]
\[ =: "k^{\text{th}} \text{ elementary symmetric function}" \text{ in the } \{\lambda_i\}.\]

We caution that these are not formulas for the $P_k(A)$ any more than the special cases
\[
det(A) = P_n(A) = \lambda_1 \cdots \lambda_n, \quad tr(A) = P_1(A) = \lambda_1 + \ldots + \lambda_n
\]
are formulas for determinant and trace.

**Decoupling linear PDEs.** Frequently in physics and engineering one encounters systems of partial differential equations like
\[
\frac{\partial u_1}{\partial t} + a_{11} \frac{\partial u_1}{\partial x} + a_{12} \frac{\partial u_2}{\partial x} = 0,
\]
\[
\frac{\partial u_2}{\partial t} + a_{21} \frac{\partial u_1}{\partial x} + a_{22} \frac{\partial u_2}{\partial x} = 0
\]
which are “coupled” in the sense that the equation for the change in $u_1$ has a term involving $u_2$, and vice versa. We can decouple them by a linear change in coordinates. Assume
\[
A = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
\uparrow & \uparrow & \downarrow & \downarrow
\end{pmatrix} \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} \begin{pmatrix}
\uparrow & \uparrow & \downarrow & \downarrow
\end{pmatrix}^{-1}
\]
\[ = SDS^{-1} \]
is diagonalizable, with eigenbasis \( B = \{ \vec{v}_1, \vec{v}_2 \} \), and rewrite the system in matrix form
\[
\frac{\partial \vec{u}}{\partial t} + A \frac{\partial \vec{u}}{\partial x} = 0.
\]
Let \( \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{c} := S^{-1} \vec{u} \) be the coordinates of \( \vec{u} \) with respect to \( B \), and write
\[
\frac{\partial \vec{u}}{\partial t} + S D S^{-1} \frac{\partial \vec{u}}{\partial x} = 0 \quad \rightarrow \quad S^{-1} \frac{\partial \vec{u}}{\partial t} + D S^{-1} \frac{\partial \vec{u}}{\partial t} = 0.
\]
Since \( S^{-1} \) is a matrix of constants, the differential operators commute with it. The last equation becomes
\[
\frac{\partial \vec{c}}{\partial t} + D \frac{\partial \vec{c}}{\partial x} = 0
\]
which corresponds to the two PDEs
\[
\begin{align*}
\frac{\partial c_1}{\partial t} + \lambda_1 \frac{\partial c_1}{\partial x} &= 0, \\
\frac{\partial c_2}{\partial t} + \lambda_2 \frac{\partial c_2}{\partial x} &= 0.
\end{align*}
\]
These are not coupled and may be solved separately to get the “traveling waves”
\[
c_1(x, t) = \Phi_I(x - \lambda_1 t), \quad c_2(x, t) = \Phi_{II}(x - \lambda_2 t)
\]
where \( \Phi_I, \Phi_{II} \) are any functions on the real line. To get more specific we must specify the initial conditions
\[
u_1(x, 0) = g(x) \quad \text{and} \quad u_2(x, 0) = h(x)
\]
of the system.

**Example 1.** Let’s solve the system
\[
\begin{align*}
\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} + 4 \frac{\partial u_2}{\partial x} &= 0, \\
\frac{\partial u_2}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} &= 0
\end{align*}
\]
with initial conditions
\[
u_1(x, 0) = e^{-x}, \quad u_2(x, 0) = \cos x.
\]
The first step is to diagonalize
\[
A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{4} \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}
\]
\[
= S D S^{-1}.
\]
Then we solve for \(\Phi_I\) and \(\Phi_{II}\):
\[
\begin{pmatrix} \Phi_I(x) \\ \Phi_{II}(x) \end{pmatrix} = \begin{pmatrix} c_1(x, 0) \\ c_2(x, 0) \end{pmatrix} = S^{-1} \vec{u}(x, 0)
\]
\[
= \frac{1}{4} \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} e^{-x} \\ \cos x \end{pmatrix},
\]
i.e.
\[
\Phi_I(x) = \frac{1}{4} e^{-x} + \frac{1}{2} \cos x, \quad \Phi_{II}(x) = -\frac{1}{4} e^{-x} + \frac{1}{2} \cos x;
\]
and so (since \(\lambda_1 = 3\), \(\lambda_2 = -1\))
\[
\vec{c}(x, t) = \begin{pmatrix} \Phi_I(x - \lambda_1 t) \\ \Phi_{II}(x - \lambda_2 t) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} e^{-(x-3t)} + \frac{1}{2} \cos(x - 3t) \\ -\frac{1}{4} e^{-(x+t)} + \frac{1}{2} \cos(x + t) \end{pmatrix}.
\]
This is the solution in eigen-coordinates. In terms of the original variables \(u_1\) and \(u_2\) we have
\[
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \vec{u}(x, t) = S \vec{c}(x, t)
\]
\[
= \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} e^{-(x-3t)} + \frac{1}{2} \cos(x - 3t) \\ -\frac{1}{4} e^{-(x+t)} + \frac{1}{2} \cos(x + t) \end{pmatrix}
\]
\[
= \begin{pmatrix} \frac{1}{2} e^{-(x-3t)} + \cos(x - 3t) + \frac{1}{2} e^{-(x+t)} - \cos(x + t) \\ \frac{1}{4} e^{-(x-3t)} + \frac{1}{2} \cos(x - 3t) - \frac{1}{4} e^{-(x+t)} + \frac{1}{2} \cos(x + t) \end{pmatrix}.
\]
If you are interested in applications of this sort of thing, look in any book on applied PDEs (a beautiful subject in itself).

**Complex diagonalization of real matrices.** Let \(A \in M_n(\mathbb{R})\). (In this lecture \(A\) always has real entries.) If the characteristic polynomial \(f_A(\lambda)\) does not break completely into linear factors over \(\mathbb{R}\), then \(A\) does not have a complete set of eigenvalues in \(\mathbb{R}\). (The algebraic
multiplicities \( d_j \) of the real eigenvalues do not add to \( n \). In this case

\[
f_A(\lambda) = (\lambda - \sigma_1)^{d_1} \cdots (\lambda - \sigma_r)^{d_r} \cdot (\lambda^2 + \mu_1 \lambda + v_1)^{c_1} \cdots (\lambda^2 + \mu_s \lambda + v_s)^{c_s}
\]

irreducible quadratic factors: \( \mu_k^2 < 4v_k \)

where \( \sigma_i, \mu_j, v_j \) are all real and \( \sum d_j + 2 \sum c_k = n \).

Over \( \mathbb{C} \) this \( f_A(\lambda) \) factors completely: if we set \( \xi_k = -\mu_k^2, \eta_k = -\frac{1}{2} \sqrt{4v_k - \mu_k^2} \) (since \( \mu_k^2 < 4v_k \) by hypothesis, these will be real) then the quadratic formula shows

\[
(\lambda^2 + \mu_k \lambda + v_k) = \{\lambda - (\xi_k + i\eta_k)\} \cdot \{\lambda - (\xi_k - i\eta_k)\}.
\]

Therefore each irreducible quadratic factor gives rise to a conjugate pair of complex eigenvalues, and the algebraic multiplicities of

\[
\sigma_1, \ldots, \sigma_r; \xi_1 + i\eta_1, \xi_1 - i\eta_1, \ldots, \xi_s + i\eta_s, \xi_s - i\eta_s
\]

do add to \( n \).

Since \( A \) is real, if \( A\vec{v} = \lambda\vec{v} \) for a real \( \vec{v} \neq 0 \) then \( \lambda \) is also real. So if we take now \( \vec{v} \in E_{\xi_k+i\eta_k}(A) \) a nonzero eigenvector, \( \vec{v} \) cannot be real since \( \eta_k \neq 0 \) by hypothesis. Therefore we write \( \vec{v} = \vec{u} + i\vec{w} \), where \( \vec{u} \) and \( \vec{w} \) are real and \( \vec{w} \neq 0 \). We have

\[
\{(\xi_k + i\eta_k)I - A\}(\vec{u} + i\vec{w}) = 0;
\]

taking the complex conjugate \( \Rightarrow \)

\[
\{(\xi_k - i\eta_k)I - A\}(\vec{u} - i\vec{w}) = 0
\]

(where \( A \) is unaffected because it is real). That is,

\[
\vec{v} = \vec{u} - i\vec{w} \in E_{\xi_k-i\eta_k}(A),
\]

and so to each conjugate pair of complex eigenvalues of \( A \) there is at least one conjugate pair of complex eigenvectors.

In fact, if \( A \) is diagonalizable over \( \mathbb{C} \), then there is an eigenbasis

\[
\vec{v}_1, \ldots, \vec{v}_m; \vec{u}_1 + i\vec{w}_1, \vec{u}_1 - i\vec{w}_1, \ldots, \vec{u}_p + i\vec{w}_p, \vec{u}_p - i\vec{w}_p
\]
(where all $\vec{v}_i, \vec{u}_i, \vec{w}_i$ are real) with respective eigenvalues

$$\lambda_1, \ldots, \lambda_m; \zeta_1 + i\eta_1, \zeta_1 - i\eta_1, \ldots, \zeta_p + i\eta_p, \zeta_p - i\eta_p$$

($\lambda_i, \zeta_i, \eta_i \in \mathbb{R}$) where $m + 2p = n$. Of course one can now write a matrix $S$ with the vectors of the (complex) eigenbasis as its columns, and get $A = SDS^{-1}$ where $D$ has the (complex) eigenvalues as its diagonal entries. Such a diagonalization is old news. The new, beautiful fact is that one can decompose $A$ into real $n \times n$ matrices $A = MBM^{-1}$ where

$$M = \begin{pmatrix}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
| & | & | & | & | & | \\
\vec{v}_1 & \ldots & \vec{v}_m & \vec{u}_1 & -\vec{w}_1 & \ldots & \vec{u}_p & -\vec{w}_p \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{pmatrix},$$

$$B = \begin{pmatrix}
\lambda_1 & & & & & & & & \\
& \ddots & & & & & & & \\
& & \zeta_1 - \eta_1 & & & & & & \\
& & \eta_1 & \zeta_1 & & & & & \\
& & & & & \ddots & & & \\
& & & & & & 0 & & \\
& & & & & & & \zeta_p - \eta_p & \\
& & & & & & & \eta_p & \zeta_p
\end{pmatrix}.$$
has eigenvalues $a \pm bi$. To obtain the eigenvectors look at

$$E_{a+bi}(A) = \ker\{(a + bi)I - A\} = \ker\begin{pmatrix} bi & b \\ -b & bi \end{pmatrix}$$

$$= \ker\begin{pmatrix} i & 1 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and

$$E_{a-bi}(A) = \ker\{(a - bi)I - A\} = \ker\begin{pmatrix} -bi & b \\ b & -bi \end{pmatrix}$$

$$= \ker\begin{pmatrix} -i & 1 \end{pmatrix} = \text{span}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$ (Note that you could also just take the conjugate of \(\begin{pmatrix} 1 \\ -i \end{pmatrix}\) to get \(\begin{pmatrix} 1 \\ i \end{pmatrix}\).) Instead of $S$, write

$$N := \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix};$$

$N$ will mean this particular matrix for the remainder of the section. By the usual diagonalization procedure one gets

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = NDN^{-1} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix}. \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$ From this we have immediately the useful relation

$$\begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix} = N^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} N = r \cdot N^{-1}R_\theta N,$$

where $R_\theta$ is the counterclockwise rotation by $\theta = \arctan(b/a)$ and $r = \sqrt{b^2 + a^2}$ the dilation (which commutes with everything). These are just the $\theta$ and $r$ for which $a + bi = re^{i\theta}$.
More generally, let
\[ A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in M_2(\mathbb{R}); \]
then there are three possibilities for the roots of
\[ f_A(\lambda) = \lambda^2 - (\alpha_{11} + \alpha_{22})\lambda + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) = \lambda^2 - (\text{tr}A)\lambda + \det A, \]
based on its discriminant (i.e., the quantity under the radical in the quadratic formula):

(i) \((\text{tr}A)^2 > 4 \det A\): distinct real roots \( \implies A \) is diagonalizable over \( \mathbb{R} \);
(ii) \((\text{tr}A)^2 = 4 \det A\): repeated real root \( \implies \) ??? \((A \) may not be diagonalizable even over \( \mathbb{C} \));
(iii) \((\text{tr}A)^2 < 4 \det A\): (distinct) conjugate complex roots \( \implies A \) diagonalizable over \( \mathbb{C} \) but not over \( \mathbb{R} \) \((f_A(\lambda) \) is an “irreducible” quadratic).

We are interested in case (iii). Call the roots \( a + bi, a - bi \) (noting that \( b \neq 0 \)), and let
\[ \vec{v} = \vec{u} + i\vec{w} \in E_{a+bi}(A) \]
be a nonzero eigenvector (in particular, \( \vec{w} \neq 0 \) as before). Then
\[ \vec{u} - i\vec{w} \in E_{a-ib}(A) \]

since (noting that \( A \) is real, so \( A = \bar{A} \))
\[
A(\vec{u} + i\vec{w}) = (a + bi)(\vec{u} + i\vec{w}) \quad \implies \quad A(\vec{u} - i\vec{w}) = (a - bi)(\vec{u} - i\vec{w}),
\]
and \( \{\vec{u} + i\vec{w}, \vec{u} - i\vec{w}\} \) give an \( A \)-eigenbasis\(^1\) for \( \mathbb{C}^2 \). Taking
\[ S = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} + i\vec{w} & \vec{u} - i\vec{w} \\ \downarrow & \downarrow \end{pmatrix} \]

\(^1\)They are independent because they have distinct eigenvalues; note also that their independence implies independence of \( \vec{u} \) and \( \vec{w} \) (why?).
we have (using the boxed relation above)

\[ A = S \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix} S^{-1} = SN^{-1} \begin{pmatrix} a - b \\ b & a \end{pmatrix} NS^{-1}; \]

if we set

\[ M = SN^{-1} = \frac{1}{2} \begin{pmatrix} \uparrow \uparrow \\ \uparrow \downarrow \end{pmatrix} \begin{pmatrix} \bar{u} + i\bar{w} & \bar{u} - i\bar{w} \\ \downarrow \downarrow \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \]

\[ = \frac{1}{2} \begin{pmatrix} \uparrow \uparrow \\ \downarrow \downarrow \end{pmatrix} \begin{pmatrix} (\bar{u} + i\bar{w}) & i(\bar{u} + i\bar{w}) \\ +(\bar{u} - i\bar{w}) & -i(\bar{u} - i\bar{w}) \end{pmatrix} = \begin{pmatrix} \uparrow \uparrow \\ \downarrow \downarrow \end{pmatrix}, \]

then we have the decomposition

\[ A = M \begin{pmatrix} a & -b \\ b & a \end{pmatrix} M^{-1} = r \cdot MR_{\theta}M^{-1}, \quad M = \begin{pmatrix} \uparrow \uparrow \\ \downarrow \downarrow \end{pmatrix} \]

completely over \( \mathbb{R} \)!! This plays exactly the same role in solving linear dynamical systems /\( \mathbb{R} \) when the eigenvalues are conjugate complex, as the decomposition \( A = SDS^{-1} \) (diagonalization) did in the case of distinct real eigenvalues.

**Example 2.** Take

\[ A = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}, \]

so that

\[ f_A(\lambda) = \det \begin{pmatrix} \lambda - 3 & 5 \\ -1 & \lambda + 1 \end{pmatrix} = \lambda^2 - 2\lambda + 2 \]

has conjugate complex roots \( \lambda = 1 \pm i =: a \pm ib \), which is to say \( a = b = 1 \). Moreover

\[ E_{1+i}(A) = \ker \begin{pmatrix} -2 + i & 5 \\ -1 & 2 + i \end{pmatrix} \]
\[= \ker \begin{pmatrix} -1 & 2 + i \\ \end{pmatrix} = \text{span} \begin{pmatrix} 2 + i \\ 1 \end{pmatrix},\]

and so we have the eigenvector
\[\vec{v} = \begin{pmatrix} 2 + i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: \vec{u} + i\vec{w};\]
its conjugate \(\vec{u} - i\vec{w}\) is the other one. Set
\[M = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & -\vec{w} \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\]
and note that
\[M^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix};\]
then according to our result above, we have the real decomposition
\[A = M \begin{pmatrix} a & -b \\ b & a \end{pmatrix} M^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.\]

One may conceptualize this in terms of transformations as follows: if \(T: \mathbb{R}^2 \to \mathbb{R}^2\) is the transformation with matrix \([T]_\hat{e} = A\) relative to the standard basis \(\{\hat{e}_1, \hat{e}_2\}\) of \(\mathbb{R}^2\), then in terms of \(B = \{\vec{u}, -\vec{w}\}\)
\[[T]_B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2} \cdot R_{\pi/4} =: r \cdot R_\theta.\]
So \(T\) is a sort of “skew” rotation-dilation.

**Real dynamical systems with complex eigenvalues.**

1. A discrete dynamical system. Let \(A\) remain the matrix of the above example. We will work out the solution to
\[\vec{x}(\tau + 1) = A\vec{x}(\tau)\]
abstractly and then substitute in the numerical information. let
\[\vec{\gamma}(\tau) = M^{-1}\vec{x}(\tau), \quad \vec{\gamma}_0 = M^{-1}\vec{x}_0\]
be the present analogue of the eigen-coordinates \((\vec{c} = S^{-1} \vec{x})\). These are the coordinates with respect to \(\vec{u}\) and \(-\vec{w}\): that is, \(\vec{x} = \gamma_1 \vec{u} - \gamma_2 \vec{w}\).

Using our real decomposition of \(A\) (because we want a real solution),

\[
\vec{x}(\tau) = A^T \vec{x}_0 = (r MR_\theta M^{-1})^T \vec{x}_0 = r^T M(R_\theta)^T M^{-1} \vec{x}_0 = r^T M R_\theta M^{-1} \vec{x}_0.
\]

Multiplying both sides by \(M^{-1}\), this becomes

\[
M^{-1} \vec{x}(t) = r^T R_\theta M^{-1} \vec{x}_0
\]

or

\[
\vec{\gamma}(t) = r^T R_\theta \vec{\gamma}_0.
\]

Suppose we start at \(\vec{u}\), i.e take \(\vec{x}_0 = \vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\). Then \(\vec{\gamma}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) (why?) and

\[
\vec{\gamma}(t) = (\sqrt{2})^T R_{\frac{\pi}{4} \tau}(1) = \sqrt{2}^T \begin{pmatrix} \cos \frac{\pi}{4} \tau \\ \sin \frac{\pi}{4} \tau \end{pmatrix},
\]

or in \(\hat{c}\)-coordinates

\[
\vec{x}(\tau) = M \vec{\gamma}(t) = \sqrt{2}^T \left\{ \cos \left( \frac{\pi}{4} \tau \right) \vec{u} - \sin \left( \frac{\pi}{4} \tau \right) \vec{w} \right\}.
\]

II. A continuous dynamical system. Again assuming \(A = \) the matrix from the example, let’s investigate the continuous dynamical system

\[
\frac{d \vec{x}}{d \tau} = A \vec{x}(\tau).
\]

This turns out to be unexpectedly complicated; we will employ both sets of coordinates. Put

\[
\vec{\gamma}(\tau) = M^{-1} \vec{x}(\tau), \quad \vec{c}(\tau) = S^{-1} \vec{x}(\tau)
\]

where

\[
M = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} & -\vec{w} \end{pmatrix}, \quad S = \begin{pmatrix} \uparrow & \uparrow \\ \vec{u} + i\vec{w} & \vec{u} - i\vec{w} \end{pmatrix}.
\]
First use the eigencoordinates $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ to transform
\[
\frac{d\vec{x}}{d\tau} = S \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix} S^{-1} \vec{x}(\tau)
\]
to
\[
\frac{d\vec{c}}{d\tau} = \begin{pmatrix} a + bi & 0 \\ 0 & a - bi \end{pmatrix} \vec{c}(t).
\]
The solution to this is just
\[
\vec{c}(\tau) = \begin{pmatrix} e^{(a+bi)\tau} & 0 \\ 0 & e^{(a-bi)\tau} \end{pmatrix} \vec{c}_0 = e^{a\tau} \begin{pmatrix} e^{ib\tau} & 0 \\ 0 & e^{-ib\tau} \end{pmatrix} \vec{c}_0,
\]
i.e.
\[
S^{-1} \vec{x}(\tau) = e^{a\tau} \begin{pmatrix} e^{ib\tau} & 0 \\ 0 & e^{-ib\tau} \end{pmatrix} S^{-1} \vec{x}_0
\]
or
\[
\vec{x}(\tau) = e^{a\tau} S \begin{pmatrix} e^{ib\tau} & 0 \\ 0 & e^{-ib\tau} \end{pmatrix} S^{-1} \vec{x}_0
\]
\[
= e^{a\tau} S \begin{pmatrix} \cos b\tau + i \sin b\tau & 0 \\ 0 & \cos b\tau - i \sin b\tau \end{pmatrix} S^{-1} \vec{x}_0
\]
\[
= e^{a\tau} S N^{-1} \begin{pmatrix} \cos b\tau & - \sin b\tau \\ \sin b\tau & \cos b\tau \end{pmatrix} NS^{-1} \vec{x}_0 = e^{a\tau} M R_{b\tau} M^{-1} \vec{x}_0.
\]
In order to decouple (and solve) the differential equations we had to diagonalize, but that introduced complex numbers. To then get rid of them, we had to be more clever than in the last three types of dynamical systems (discrete with real or complex $\lambda$, continuous with real $\lambda$).

Now all we have to do is transform to $\{\vec{u}, -\vec{\omega}\}$ coordinates:
\[
M^{-1} \vec{x}(\tau) = e^{a\tau} R_{b\tau} M^{-1} \vec{x}_0 \quad \rightarrow \quad \vec{\gamma}(\tau) = e^{a\tau} R_{b\tau} \vec{\gamma}_0.
Again let \( \vec{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \leftrightarrow \vec{\gamma}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) be the initial state, so that the solution looks like

\[
\vec{\gamma}(\tau) = e^{a\tau} \begin{pmatrix} \cos b\tau & -\sin b\tau \\ \sin b\tau & \cos b\tau \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{a\tau} \begin{pmatrix} \cos b\tau \\ \sin b\tau \end{pmatrix}
\]
or

\[
\vec{x}(\tau) = M\vec{\gamma}(\tau) = e^{a\tau} \begin{pmatrix} \uparrow & \uparrow \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} \cos b\tau \\ \sin b\tau \end{pmatrix} = e^{a\tau} \{\cos(b\tau)\vec{u} - \sin(b\tau)\vec{w}\}
\]

\[
= e^\tau \left\{ \cos(\tau) \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \sin(\tau) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.
\]

This corresponds\(^2\) to the picture

\[\text{Although for this particular matrix there are not dramatic differences between the discrete and continuous systems as far as the qualitative phase diagram goes, note the distinctions in the dilations}
\]

\[r^\tau = \sqrt{a^2 + b^2\tau} \quad \text{vs.} \quad e^{a\tau}\]

\(2\)I make no claims to quantitative correctness here. The spiral probably unfolds at a different rate than what is drawn.
and the rotations

\[ R_{\arctan(b/a)} \quad \text{vs.} \quad R_t b. \]

This leads to qualitative differences for certain values of \( a \) and \( b \).

**Exercises**

(1) (a) Show

\[
A = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

cannot be diagonalized over \( \mathbb{R} \). (A 90° rotation in 3 space has only one real eigenvector – the one it rotates around.)

(b) Now diagonalize it over \( \mathbb{C} \).

(2) Decouple the linear system

\[
\begin{align*}
\frac{\partial u}{\partial t} + 8 \frac{\partial v}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + 2 \frac{\partial u}{\partial x} &= 0
\end{align*}
\]

of PDEs and find the solution given initial conditions \( u(x,0) = \sin x \), \( v(x,0) = x^2 \).

(3) Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the transformation given in the standard basis by

\[
A = \begin{pmatrix}
-4 & 10 \\
-4 & 8
\end{pmatrix}
\]

(a) Diagonalize \( A \) over \( \mathbb{C} \).

(b) Now show that \( A \) is similar (\( A = R^{-1}BR \)) to a rotation-dilation matrix (both \( R \) and \( B \) are supposed to be real).

(c) Give a closed-form solution to the discrete dynamical system

\( \vec{x}(t + 1) = A \vec{x}(t) \) if \( \vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \) and draw a phase portrait for the system.
(4) “Partially diagonalize” the matrix

\[ A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 3 \end{pmatrix} \]

over \( \mathbb{R} \): that is, write it as \( MBM^{-1} \) where

\[ B = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix} \]

with \( a, b, \lambda \in \mathbb{R} \).