Chapter 29: Proof of Normalization Theorem (B)

Several of the main results we have claimed for Riemann surfaces, like \( \dim \mathcal{L}(M) = \text{genus}(M) \) and the Riemann–Roch formula, were actually proved for normalizations of nodal plane curves. We now turn to the demonstration that every (compact) Riemann surface \( M \) arises in this way, assuming a deep analytic result (Lemma 29.1.2). This is used to derive a special case of Riemann–Roch, which then suffices to produce the immersion \( \phi : M \to \mathbb{P}^2 \).

§29.1: An analytic lemma

Let \( M \) be a Riemann surface of genus \( g \). Write \( \omega = \dim \mathcal{L}^1(M) \); note that we may not assume that \( \omega = g \) (or even \( \omega < g \))! However, the proof of Poincaré–Hopf did not use NT(B) and so we know that \( \deg(c_\omega) = 2g - 2 \) for any \( \omega \in \mathcal{K}^1(M)^* \).

Recall (from §23.8, 25.2) that the exterior derivative of given maps between spaces of \( \mathcal{C}^0 \) forms, \( A^0(M) \xrightarrow{\delta} A^1(M) \xrightarrow{\delta} A^2(M) \), with \( \delta \circ \delta = 0 \). We can break \( \delta \) into hole. hole. components, writing (for \( \mathcal{F} \in A^0(M) \))

\[
\delta \mathcal{F} = \frac{\partial \mathcal{F}}{\partial z} dz \wedge \delta + \frac{\partial \mathcal{F}}{\partial \bar{z}} \delta z \wedge = \delta \mathcal{F} + \overline{\delta \mathcal{F}}
\]

in local coordinates, and \( \overline{\delta} (\mathcal{F} \omega) = \overline{\delta \mathcal{F}} \omega + \mathcal{F} \overline{\delta \omega} \) (ditto for \( \delta \)) if \( \omega \in A^1(M) \). Write \( A^0(M) \) resp. \( A^0(M) \subset A^1(M) \) for \( 1 \)-forms with local expressions \( G_{2} \, dz \wedge \delta + G_{x} \delta z \wedge \delta \) (\( G_{x} = \mathcal{C}^0 \text{ functions} \)); clearly
\[ A^1(M) = A^{1,0}(M) \oplus A^{0,1}(M) \text{ and } \partial A^{1,0}(M) = \delta A^{0,1}(M). \] Moreover, \( \overline{\delta} \) such \( A^0(M) \to A^{0,1}(M) \) and \( A^{1,0}(M) \to A^2(M) \), while \( \ker(\overline{\delta}) = A^{1,0}(M) \) is just \( \Omega^1(M) \) (why?). Given \( \eta \in \Omega^1(M) \) and \( F \in A^0(M) \), we have
\[ \delta(F \eta) = \partial F \wedge \eta = \delta F \wedge \eta + \overline{\delta} F \wedge \eta = \overline{\delta} F \wedge \eta \]
since (locally everywhere) \( dt \wedge adx = 0 \).

We will need the following deep analytic result, which requires tools from functional analysis. There is a long proof, but search in [Donaldson, RS].

(29.1.2) Lemma: Given \( A \in A^2(M) \) with \( \int_M A = 0 \), we have \( A \in \overline{\delta} A^0(M) \).

Here it is worth mentioning that \( \overline{\partial} = -\overline{\partial} \) is \( \frac{1}{2i} \Delta \), where
\[ \Delta : A^0(M) \to A^2(M) \]
is the Laplacian given locally by
\[ \Delta g = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) g. \]
The kernel of \( \Delta \) defines harmonic functions on \( M \), which (globally, not locally) are just the constant functions by the maximum principle. Of course, \( 2iA = \Delta g \) is a version of the Poisson equation, which is closely related to electrostatics and heat flow.

The Lemma can be thought of as saying that if the heat sources and sinks (represented by \( A \)) cancel out over \( M \), then there exists a steady-state temperature distribution \( g \).

(29.1.3) Corollary: (i) \( A^{0,1}(M) = \Omega^1(M) \oplus \overline{\delta} A^0(M) \)

(ii) Suppose \( \omega \in A^{0,1}(M) \) satisfying \( \int_M \omega \wedge \omega = 0 \) for every \( \omega \in \Omega^1(M) \). Then \( \omega \in \overline{\delta} A^0(M) \).

Proof: (i) Given \( \eta \in A^{0,1}(M) \), we have \( \overline{\delta} \eta = \partial \eta \in A^2(M) \), so that
\[ \delta \overline{\delta} \eta = 0. \]
By Lemma, \( \overline{\delta} \eta = \overline{\delta} g \) for some \( g \in A^0(M) \).
whence \( \phi : = \eta - 5g \) has \( \partial \phi = 0 \) \( \Rightarrow \phi \in S^1(M) \). Finally, if \( \delta \not\in \mathcal{D}^0(M) \) belongs also to \( S^1(M) \), then \( 0 = \partial \delta G = 5 \delta G \Rightarrow G \text{ constant} \Rightarrow \delta G = 0 \).

\[ (ii) \text{ Writing } \lambda = \bar{\mu} + \bar{\delta}g, \text{ by } (i), \text{ taking } \omega = \mu \text{ gives} \]
\[ 0 = \int M (\bar{\mu} + \bar{\delta}g) \partial \lambda = \int M \bar{\mu} \partial \lambda + \int M \bar{\delta} g \partial \lambda = \int M \bar{\mu} \partial \lambda, \]

which implies \( \mu = 0 \) as in 29.25.2.

\[ (29.1.4) \text{ Corollary: Given } p_1, \ldots, p_r \in M \text{ and } a_1, \ldots, a_r \in \mathbb{C} \text{ with } \leq a_i = 0, \]
then exists a meromorphic form \( \mu \in K^1_m \) with \( \text{Res}_p(\mu) = \alpha_i \); \( C_v \).

\[ \text{Proof: Consider concentric disks } p \in V_i < U_i < U_i' < M \text{ with all } \]
\( U_i \cap U_j = \emptyset, \) and \( V_i \subset U_i' \), \( U_i' \subset U_i \). Write \( \lambda_i \in A^1_c(U_i) \) for smooth functions which are 1 on \( V_i \) and 0 on \( U_i \setminus V_i \), and set \( \mu_0 := \sum_i \frac{a_i}{z_i} \lambda_i \) \((z_i = \text{local coordinate varying on } p_i)\). Since \( \mu_0 \) is meromorphic on \( V_i \), \( \partial \mu_0 \) is zero there. So clearly \( \bar{\partial} \mu_0 \) is smooth and belongs to \( A^2(M) \), with
\[ \int M \bar{\partial} \mu_0 = \sum_i \int_{U_i \setminus V_i} \bar{\partial} \mu_0 = \sum_i \int_{U_i \setminus V_i} \partial \mu_0 = -\sum_i \int_{U_i} \partial \mu_0 \]
\[ = -\sum_i \text{Res}_p(\frac{a_i}{z_i} \lambda_i) = -2\pi \sum_i a_i \leq 0. \]

By lemma 29.1.1, \( \bar{\partial} \mu_0 \in \mathcal{D}^0(M) \) since \( \bar{\partial} \mu_0 = \bar{\partial} \delta \) for \( \delta \in A^1_0(M) \), so \( \mu = \mu_0 - \epsilon \in \ker(\overline{\partial}) \). Off the \( p_i \), this means \( \mu \) is holomorphic; so it is meromorphic on \( M \).

\[ (29.1.5) \text{ Remark: The same proof shows, more generally, that we can}\]
\( \text{on for arbitrary principal parts of the } \delta_p \) provided the residues cancel.
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An exact sequence of abelian groups $C^i$ is a chain of homomorphisms

$$0 \to C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots \xrightarrow{d_i} C^i \xrightarrow{d_{i+1}} 0$$

in which $\ker(d_{i+1}) = \im(d_i)$ (Hi); in particular, $d_i$ is injective, $d_{i+1}$ is surjective, and $d_{i+2} = 0$. It is a simple exercise to check that when the $C^i$ are vector spaces over a field,

\[ \sum (c_i)^{-1} \partial c_i = 0. \]

Fix a point $p \in M$ and local coordinate $z$ on $U \cap p$. Any $f \in K(M)^{\mathbb{C}}$ with $\nabla f = -k$ can be written locally as $f = \sum c_i z^{-i} + h(z)$, $h$ holomorphic and $\sum c_i z^{-i}$ the principal part of $f$ at $p$; we write $\text{PP}(f) = c$.

Likewise, if $w$ is a form which is holomorphic on $p$, then we shall write $\text{R}(c)(w) = \text{Re}_p(\sum c_i z^{-i} w)$.

**Proposition (29.2.2):** The sequence

$$0 \to \mathcal{L}(k^{(p)}) \xrightarrow{\text{PP}} \mathbb{C}^h \xrightarrow{R} \left( \Omega^1(M)/\mathcal{H}(k^{(p)}) \right) \to 0$$

is exact.

**Proof:** Obviously, if $f \in \mathcal{L}(k^{(p)})$ has $\text{PP}(f) = 0$, then $f$ is constant, and vice versa; so the sequence is "exact at $\mathcal{L}(k^{(p)})". Showing that $R$ is surjective and $R \circ \text{PP} = 0$ are straightforward and left as exercises. We need to show that $\ker(R) \subset \im(\text{PP})$. Let $\lambda \in A^0_c(U)$ be as in the proof of Cor. 29.1.4, with $\lambda|_{\partial U} = 0$ and $\lambda|_{U \setminus \partial U} = 0$. 

\[ x \in A^0_c(U) \]
Given $c \in C^k$, define $f_c := \sum j \bar{z}^j z^{-j}$. This is meromorphic on $V$ and smooth on $M \setminus p$. Given $w \in \Omega^1(M)$, we have

$$\text{Res}_p(f_c w) = R(c)(w).$$

Now consider $\delta f_c : \text{on } \overline{V} \cup (M \setminus V)$, this is zero as $f_c$ is meromorphic (or zero) there. Clearly then $\delta f_c \in A^0(M)$; but it may not belong to $\overline{\delta A^0(M)}$ (as $f_c \notin A^0(M)$). In any case, for $w \in \Omega^1(M)$ we have by (29.1.1)

$$\delta f_c \wedge w = \delta f_c \wedge w = -\int_{\partial V} d(f_c \wedge w) = -2i \text{Res}_p(f_c \wedge w).$$

Suppose that $a \leq (\beta_1)$. Then from (29.2.3) and (29.2.4),

$$\int_M \delta f_c \wedge w = 0 \quad \forall w \in \Omega^1(M).$$

By Cor. 29.1.3(ii), $\exists g \in A^0(M)$ with $\delta g = \delta f_c$, and letting $f_c = f_c - g$, we have $\delta f_c = 0 \Rightarrow f_c \in \mathcal{K}(M)$. By construction it is holomorphic off $p$, and so $f_c \in \mathcal{L}(M(p))$; obviously also $\text{PP}(f_c) = c$. 

$$\int \delta f_c \wedge w = 0 \quad \forall w \in \Omega^1(M).$$

**Corollary:** (i) $a_1 < \infty$. (ii) Nonconstant meromorphic functions exist. (iii) For $k > 2g-2$, $\mathcal{L}(M(p)) = k - a_1 - 1$.

**Proof:** Let $k > 2g-2$; then by Poincaré-Hopf, $\mathcal{L}(M(p)) = 0$.

Applying (29.1.1) to the sequence in Prop. 29.2.2 therefore gives $\mathcal{L}(M(p)) + a_1 = k+1$, whence (i) and (iii). For (ii), apply (iii) with $k > \max\{2g-2, a_1\}$. In fact, the proof gives that $a_1 \leq 2g-1$ (by taking $k = 2g-1$).
Let $p_1, \ldots, p_n \in M \setminus \{q\}$ be distinct points, and put $P := \sum_{i=1}^{n} k_i(p_i)$, \( D := k(p) - P \).

\[(29.2.6) \text{ Proposition: } \quad \text{The sequence}
\[
\begin{align*}
D \rightarrow L(D) \overset{\partial P}{\rightarrow} C^1 \overset{R}{\rightarrow} (D(-P)/L(D))^\vee \rightarrow 0
\end{align*}
\]

is exact.

\[\text{Proof: } \quad \text{Again, we claim ker}(R) \subset \text{im}(P), \text{ leaving the rest to you.}
\]

Suppose $x \in \ker(R)$, and define $f_x, g, \text{ and } f_x$ exactly as before.

Since $\bar{\partial} g = \bar{\partial} f_x$ is zero on each $U_i, g|_{U_i}$ is holomorphic; and it suffices to show that $\nu_{pi}(g) \geq k_i$, forcing $f_x \in L(D) \subset L(E)$. To do this, let $W = C^k \otimes \cdots \otimes C^k$ be the vector space consisting of $r$-tuples $(b^{(1)}, \ldots, b^{(r)})$ with $b_1^{(1)} + \cdots + b_r^{(r)} = 0$. By Remark 29.1.5, there exist maps $(f_1, \ldots, f_r) \in W(-P)$ whose principal parts yield a basis of $W$. As before, we have (remembering that $f_x$ resp. $\bar{\partial} f_x$ is only nonzero on $U_i$ resp. $U_i \setminus V_i$)

\[
D = \sum_{i=1}^{n} (\bar{\partial} f_x | (U \setminus V_i)) = \text{Res}_{p_i} (f_x | (U \setminus V_i)) = \frac{1}{2\pi i} \int_{U \setminus V_i} \bar{\partial} f_x \wedge \omega_i,
\]

which now becomes (writing $g = \sum_{i=1}^{n} \sum_{j=1}^{a_i^{(i)}} g_j^i \omega_i^j$ on $U_i$)

\[
D = \frac{1}{2\pi i} \int_{U \setminus V_i} \bar{\partial} g \wedge \omega_i = \frac{1}{2\pi i} \int_{U \setminus V_i} \partial (g \omega_i^j) = \frac{1}{2\pi i} \int_{U \setminus V_i} g \bar{\partial} \omega_i^j
\]

for all $j$. Writing $g|_{U_i} = \sum_{i=1}^{a_i^{(i)}} g_j^i \omega_i^j$, this forces $a_j^{(i)} = 0$ for all $i$ and $1 \leq j \leq k_i$, and $a_i^{(i)}$ all equal to some common $a$.

We get $a = 0$ by adding a constant to $g$. \[\square\]

\[\text{Assume the } U_i \text{ (from proof of Cor. 29.1.4) and } U \text{ do not meet.}\]
29.2.7 Corollary: Let $p_1, \ldots, p_r \in M \setminus \Sigma^0$, not necessarily distinct, be given, and write $D = \mathcal{L}(p_i) - \sum_{i=1}^r \mathcal{L}(p_i).$ Then for $k - r > 2g - 2,$

$$\lambda(D) = k - r - g + 1.$$  

Proof: Since deg $(D) > 2g - 2,$ $\lambda(D) = 2g$ by Poincaré-Hopf. By Rem. 29.1.5, the sequence

$$0 \to \mathcal{L}(M) \to \lambda(-p) \xrightarrow{\delta} W \to 0$$

is exact, where $\delta$ records the principal parts at the $\{p_i\}$. Applying (29.2.1) gives $\lambda(-p) = g + r - 1$, and applying it again to the sequence in Prop. 29.2.6 gives the result. \hfill \Box

\underline{29.3: Projective embeddings and immersions.}

Taking $k$ at least $2g + 1$, we note that for any $q \in M$ Cor. 29.2.7 gives $\lambda(k(p)) = \lambda(k(p) - p_j) > \lambda(k(p) - p_j - p_i) (\text{and } \lambda(k(p) - 2q_i))$

These are exactly the inequalities required for the argument in \S\ 28.2 to produce an embedding $\varphi : M \hookrightarrow \mathbb{P}^{k-g}$. To prove part (A) of the Normalization Theorem, we proceed by way of the equally striking

(29.3.1) Theorem: There exists a holomorphic embedding $\varphi : M \hookrightarrow \mathbb{P}^d$ with smooth image.

Proof: Of course, we are done if $g \leq 1$. Now let $L = \mathcal{L}(2g+3)(p)$; this is a vector space of dimension $2g - a_g + 1 \quad (25)$. We proceed by choosing 4 (independent) elements which provide the embedding, starting with $f_0 = 1$ (constant function).

* By cancelling $(p_i)(p)$ to zero, the result also applies when some $p_i = p$.  

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\* By cancelling $(p_i)(p)$ to zero, the result also applies when some $p_i = p$.  

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Next, take any $f_i \in \mathcal{L}(2g+2)[p]$ (which is nonempty by 32.9.2).

Then $f = [f_0 : f_i] : M \to \mathbb{P}^1$ is well-defined and nonconstant, sending $p \mapsto (0 : 1)$
since $f_i$ has a pole there (of order $2g+3$, which is also the mapping degree of $f$).

The next choice is... complicated. Take $f_2$ to be any element of
that avoids a (finite) bunch of codimension-1 subsets:

(I) $\bigcup \{ \mathcal{L}(2g+3)[p] - 2\{v_i\} \otimes \mathbb{C} \}$, where $\{v_i\}$ are the (finitely many)
ramification points of $f$.

(II) $\bigcup \{ \mathcal{L}(2g+3)[p] - [p_i] - [p_j] \otimes \mathbb{C} \}$ for $\{p_i\}$ the preimage points
of some fixed noncritical value of $f$.

(III) $\bigcup_{q_i \in M} \{ \mathcal{L}(2g+3)[p] - [q] - [p_i] - [q_j] \otimes \mathbb{C} \}$ ; each subspace in
$f(q) = f(q_i) = f(q_j)$ this union has dimension $2g - g + 2$, but the
union is permuted by a finite cover of $M$ (which
adds 1).

For the map $\sigma_0 : M \to \mathbb{P}^2$ defined by $[f_0 : f_1 : f_2]$, we find that

- $f_1$ avoids (I) $\implies \sigma_0$ is an immersion : the spaces in (I)
  comprise functions which locally take the form $t_i^{m^2} + \text{const.} \ (\text{i.e.}
  \text{pow})$ at some $t_i$. So $f_1$ is locally nonramification points,
  hence $\sigma_0$ has nonvanishing derivative everywhere.

- $f_2$ avoids (II) $\implies \sigma_0$ is generically \* injective : because $f_2$
  then takes distinct values at the $\{p_i\}$, thereby separating those points
  not separated by $f$.

\* i.e. has always constant functions, which adds 1 to the
dimension computed by Con 29.2.7.

\** i.e. off a finite point set; the usual "discriminant" construction
$T_{ij} = (f_2(p_i) - f_2(p_j))$ is not identically zero, yield a meromorphic function
domains on $\sigma^1$ with finitely many zeroes.
\* f_2 avoids (III) \implies 6_0 \text{ does not send 3 (or more) points of } M \text{ to the same point of } P^2: \text{ because } f_2 \text{ doesn't take the same value on any triple of points in a "fiber" } f^{-1}(q) \text{ of } f.

With \{q_i, q'_i\} for the (finitely many) pairs of points identified by 6_0, and choose \( f_3 \in L \) in the complement of the codimension - 1 subset

\[
\bigcup_i \{ \mathcal{L}(2g+3)(P) - (q_i) - (E_i) \} \oplus C_q.
\]

Then \( f_3(q_i) \neq f_3(q'_i) \), and \( \{6_0: f_i: f_2: f_3\} = 6 \) separates these pairs.

\[\square\]

Proof of Thm. 3.2.1 (B): In the last proof, we may also choose \( f_2 \) to write

(IV) \[ \bigcup_{j \neq i} \{ \mathcal{L}(2g+3)(P) - (q_j) - (E_i) \} \oplus C_q \] and

(V) \[ \bigcup_{i, j \neq k} \left\{ g \in \left[ \mathcal{L}(2g+3)(P) - (q_j) - (E_i) \right] \oplus C \left| g'(e) = g'(s) \frac{f_2'(e)}{f_2'(s)} \right. \right\} \text{ if } f_2'(e) \neq 0.
\]

In (V), for each pair of non-ramification points of \( f \), the nonemptiness of \( \mathcal{L}(2g+3)(P) - (E_i) - (E_j) \) shows that the condition on diverging cuts out a proper subspace of dimension \( 2g-2+2 \), so that the union is codim. 1 in \( L \). Now

- \( f_2 \) avoids (IV) \implies 6_0 \text{ separates ramification points of } f
- \( f_2 \) avoids (V) \implies at non-ramification points of \( f \) identified by \( f \), the derivatives \( 6_0'(s) \neq 6_0'(t) \text{ in } P^1 \) (tangent lines have distinct slopes).

We already know that \( 6_0 \) identifies at most finitely many pairs of points; since tangent lines are distinct when this happens, they are nodes in \( 6_0(M) \).
Finally, \( \mathcal{O}_\mathbb{P}(1) \subset \mathbb{P}^2 \) is an algebraic curve by Exercise (2) of Chapter 16.

At this point we really get \( g=0 \) and all the other results that make use of this embedding.

**Exercise**

1. Check that \( \overline{\Omega} = -\Delta = \frac{i}{2\pi} \Delta \).

2. Prove Lemma 29.1.2 on \( M = \mathbb{H} \times \mathbb{H} \) by writing
   \( \mathcal{A} = \mathcal{F}(x,y) \) and expressing \( \mathcal{F} \) as a Fourier series.
   [You may assume that for a smooth function, the Fourier series converges uniformly, and that smoothness is equivalent to the Fourier coefficients \( c_m \) satisfying \( \sum_{n=-\infty}^{\infty} (1 + m^2 + n^2)^s |c_m| \leq \infty \) for all \( s > 0 \).]

3. Check (29.2.1) for exact sequences of vector spaces.

4. Complete the proof of Prop. 29.2.2 by showing that \( R \) is surjective and \( R \circ PP = 0 \).

5. For an arbitrary Riemann surface of arbitrary genus \( g \), write the degree \( d \) and number of nodes \( \delta \) for some curve \( C \subset \mathbb{P}^2 \) that it normalizes.