Chapter 32: Cubic surfaces

In the next three chapters we explore three different classes of algebraic surfaces which "generalize" elliptic curves in different ways:

- smooth cubic hypersurfaces in $\mathbb{P}^3$;
- abelian surfaces (i.e. algebraic compact complex 2-tori); and
- K3 surfaces (and Calabi-Yau varieties more generally).

Of course, cubic surfaces "generalize" cubic curves by virtue of their degree (which makes their smooth hyperplane sections elliptic). But they also possess beautiful "enumerative" properties analogous to the 9 flexes on a cubic curve, the most famous of which is the discovery by Cayley and Salmon in 1849 that every smooth cubic surface has exactly 27 lines.*

Their discovery began with the simple observation that lines in $\mathbb{P}^3$ have 4 degrees of freedom, and for a line L to lie on a given cubic places 4 conditions on L. (Namely, the 4 coefficients of the pullback of the cubic polynomial to L must vanish.) This led them to expect that there would be only finitely many lines. There are many ways to count them, the most attractive of which involves describing the cubic as a "blow-up" of $\mathbb{P}^2$ at 6 points, hence as a "rational surface". So we shall begin with some preliminaries about blow-ups and some simpler examples before passing to cubics.

* However, these lines are not where the cubic surface meets its Hessian quartic

§32.1: Some birational geometry

A projective surface is any projective (algebraic) variety of dimension 2. Obviously $\mathbb{P}^2$ is one, as are hypersurfaces in $\mathbb{P}^3$. Here are two more ways of getting more examples.

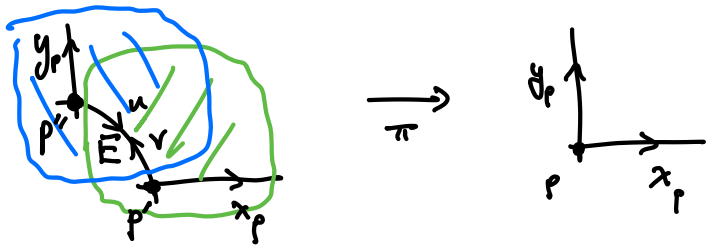
(32.1.1) Ex/ Products of projective curves: given $C \subset \mathbb{P}^m$, $D \subset \mathbb{P}^n$, we take the image of $C \times D$ under the Segre embedding $\theta_{mn} : \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{(m+1)(n+1)-1}$ defined by sending $([\{X_i\}_{i=0}^m], [\{Y_j\}_{j=0}^n])$ to $[\{Z_{ij}(\underline{X}, \underline{Y}) := X_i Y_j\}]$. One can either argue by GAGA that, as a compact analytic variety, it is projective algebraic, or explicitly produce the homogeneous equations cutting out $\theta_{mn}(C \times D)$. For instance, $\mathrm{im}(\theta_{mn})$ itself is the zero locus of the quadric polynomials $Z_{ij} Z_{kl} - Z_{ik} Z_{jl}$. //

(32.1.2) Ex/ Blowing up a projective surface: given $S \subset \mathbb{P}^n$ (smooth and irreducible) and $p \in S$, the blowup $\mathrm{Bl}_p S$ is a surface which "replaces p by $\mathbb{P}(T_p S)$", the $\mathbb{P}^1$ of tangent lines through p. This comes with a natural blowdown morphism* $\pi : \mathrm{Bl}_p S \twoheadrightarrow S$, which collapses this $\mathbb{P}^1$ to p and is an isomorphism outside this. The construction can be iterated (independently of order) to produce $\pi : \mathrm{Bl}_{\{p_1, \dots, p_k\}} S \twoheadrightarrow S$, and $E = E_1 + \dots + E_k := \pi^{-1}(\{p_1, \dots, p_k\})$ is called the exceptional divisor of the blowup. (On a surface, divisors are formal sums of curves.)

* We often refer to π also as a blowup. Notice that it immediately shows that the principle we often invoked for curves (dimension 1), that morphisms of degree 1 are isomorphisms, is not valid in dimension 2.

If $S = \mathbb{P}^2$ and $p = [1:0:0]$, $Bl_p S$ is the subvariety of* $\mathbb{P}^2_{[Z:X:Y]} \times \mathbb{P}^1_{[U:V]}$ defined by $XV - YU = 0$. Notice that (only) at $X = Y = 0$, $[U:V]$ is not determined. More generally, if $R = \frac{S_n}{J}$ is the affine coordinate ring of $S = V(J) \subset \mathbb{C}^n$, and $m_p = (x_p, y_p) \subset R_p$ the maximal ideal in its localization at p, then $(J, x_p V - y_p U)$ defines the blowup Zariski-locally in $S \times \mathbb{P}^1$ (whose closure yields $Bl_p S \subset \mathbb{P}^n \times \mathbb{P}^1$).

Practically speaking, the main point is that the local coordinate chart (x_p, y_p) at p is replaced by two local charts:

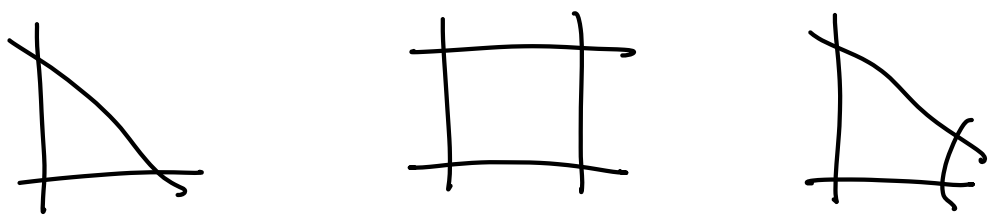


in which π sends

(32.1.3) $$(x_p, v) \mapsto (x_p, v x_p) \quad \text{and} \quad (u, y_p) \mapsto (u y_p, y_p),$$

and $u = \frac{U}{V}$, $v = \frac{V}{U}$ in the previous notation. //

Now we would like to compare these surfaces. For instance, you might expect that $\mathbb{P}^2$ is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ but wonder about $Bl_{[0:1:0]} \mathbb{P}^2$, since all 3 have an embedded $\mathbb{C}^* \times \mathbb{C}^*$ but the last two have 4 lines in the complement:



$\mathbb{P}^2$ $\qquad$ $\mathbb{P}^1 \times \mathbb{P}^1$ $\qquad$ $Bl_{[0:1:0]} \mathbb{P}^2$.

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* hence of $\mathbb{P}^5$, via Segre

One way to do this is by comparing "self-intersection numbers" of curves. To make sense of this requires the broader context of intersection #'s of divisors. We write $\mathrm{Div}(S)$ for the free abelian group of curves on a smooth S, and again $D \underset{rat}{\equiv} D' \overset{def.}{\iff} D'-D = (f)$ for some $f \in k(S)^*$ (= rational functions $\mathbb{C}(S)$, cf. Ex. 7.3.5). If D and D' are rationally equivalent divisors intersecting a curve C properly — i.e., in points — then $D\cdot C,\ D'\cdot C \in \mathrm{Div}(C)$ are defined, and satisfy $D'\cdot C - D\cdot C = (f|_C)$. So we have $D\cdot C \underset{rat}{\equiv} D'\cdot C$ on C, and the intersection numbers $(D\cdot C)$ and $(D'\cdot C)$ are equal. If (say) D' does <u>not</u> meet C properly, then we <u>define</u>* $(D'\cdot C)$ to be $(D\cdot C)$. Extending the definition linearly lets us replace C by a divisor, and writing $\mathrm{Pic}(S) = \mathrm{Div}(S)/(k(S)^*)$, this intersection product descends to a pairing

[32.1.4) $$(\cdot)_S : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}.$$

(32.1.5) Ex/($S = \mathbb{P}^2$) For any curve C of degree $d(\geq 1)$, $(C\cdot C)_{\mathbb{P}^2} = d^2$.

(32.1.6) Ex/($S = \mathbb{P}^1\times\mathbb{P}^1$) A curve $C \subset S$ is cut out by a polynomial which is <u>bihomogeneous</u> of some <u>bidegree</u> (d_1, d_2): $f(\lambda_1[\underline{X}], \lambda_2[\underline{Y}]) = \lambda_1^{d_1}\lambda_2^{d_2} f([\underline{X}],[\underline{Y}])$. (Here d_1 and d_2 are nonnegative, and not both zero.) So C is rationally equivalent to the divisors $d_1[\{X_0=0\}] + d_2[\{Y_0=0\}]$ and $d_1[\{X_1=0\}] + d_2[\{Y_1=0\}]$. Clearly these have intersection number $2d_1d_2$, and so $(C\cdot C)_S = 2d_1d_2$.

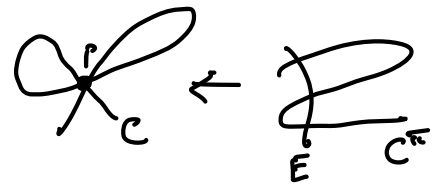
Since $\mathbb{P}^1\times\mathbb{P}^1$ has a curve with self-intersection 0, it is not isomorphic to $\mathbb{P}^2$ — even though they have the same rational function field $\mathbb{C}(x,y)$.

* <u>Remark</u>: How do we know that we can always move C "off itself" (hence any D' off C) by subtracting some (f)? Since S is smooth (at any $p \in C$ in particular), the localization $\mathcal{O}_p$ of its affine coordinate ring is a regular local ring, hence UFD. The curve C defines a prime ideal of height 1 in the coordinate ring and thus in $\mathcal{O}_p$. Since ideals of height 1 in a UFD are principal, there exists $f \in \mathcal{O}_p \subset \mathbb{C}(S)$ which is a local equation for C at p, hence on a Zariski open containing p. From this we see that $C-(f) \in \mathrm{Div}(S)$ meets C properly.

How do intersection numbers change under blowup? The key is to use divisors of pullbacks of functions under π to replace an improper intersection by a proper one. We will need the following bit of terminology:

(32.1.7) <u>Definition</u>: Given $C \subset S$ a curve on a smooth surface, the <u>total transform</u> $\pi^*(C) \in \mathrm{Div}(Bl_p(S))$ of C under $\pi : Bl_p(S) \twoheadrightarrow S$ is obtained <u>locally</u> everywhere by writing C as $f=0$ and taking the divisor of $f \circ \pi$. (We can linearly extend this definition to divisors.)

The <u>strict transform</u> C^π is given by taking the closure of $\pi^{-1}(C \setminus \{p\})$ if $p \in C$ and by $\pi^{-1}(C)$ if $p \notin C$.



(32.1.8) <u>Proposition</u>: Let S be a smooth projective surface, $p \in S$ a point, and $C \subset S$ a curve with $\mathrm{ord}_p C = k$. (If $p \notin C$, $k=0$; if $p \in C$ and C is smooth, $k=1$.) Then on $Bl_p S$ we have

(a) $\pi^* C = C^\pi + kE$, and $(C^\pi \cdot E) = k$.

(b) $(C^\pi \cdot C^\pi) = (C \cdot C)_S - k^2$.

(c) $(E \cdot E) = -1$.

<u>Proof</u>: (a) Take $f = \sum c_{\underline{m}} x_p^{m_1} y_p^{m_2}$ a local defining equation for C and (in view of (32.1.3)) replace y_p by ux_p to get $f \circ \pi$ in the local chart at p'. Clearly $x_p = 0$ defines E in this chart and $\mathrm{ord}_E(f \circ \pi)$ is the smallest $|\underline{m}|$ (namely, k) for which some $c_{\underline{m}} \neq 0$. Moreover, $C^\pi \cdot E$ is given by $\sum_{|m|=k} c_{\underline{m}} x_p^{m_1} y_p^{k-m_1} = 0$ in $\mathbb{P}^1 (= E)$.

(b) Let $\tilde{f}$ be the extension of f to a rational/meromorphic function on S. Write $(\tilde{f}) = C + D$ (where $p \notin |D|$) and $(\tilde{f} \circ \pi) = C^\pi + kE + \pi^* D$ (by (a)). Since the divisor of a function is 0 in Pic, its intersection number with anything is 0. So $(C \cdot C)_S = -(C \cdot D)_S = -(C^\pi \cdot \pi^* D)$ (why?), and

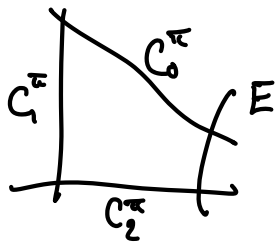
$$0 = ((\tilde{f}\circ\pi)\cdot C^\pi) = (C^\pi\cdot C^\pi) + k(E\cdot C^\pi) + (C^\pi\cdot \pi^*D)$$

gives the result

(c) If C is a smooth (red.) curve passing through p, then $k=1$ and $0 = ((\tilde{f}\circ\pi), E) = (C^\pi\cdot E) + (E\cdot E) + 0 = 1 + (E\cdot E)$. $\square$

(32.1.9) Ex/[$S = Bl_p \mathbb{P}^2$, also known as the <u>first Hirzebruch surface</u> $\mathbb{F}_1$]

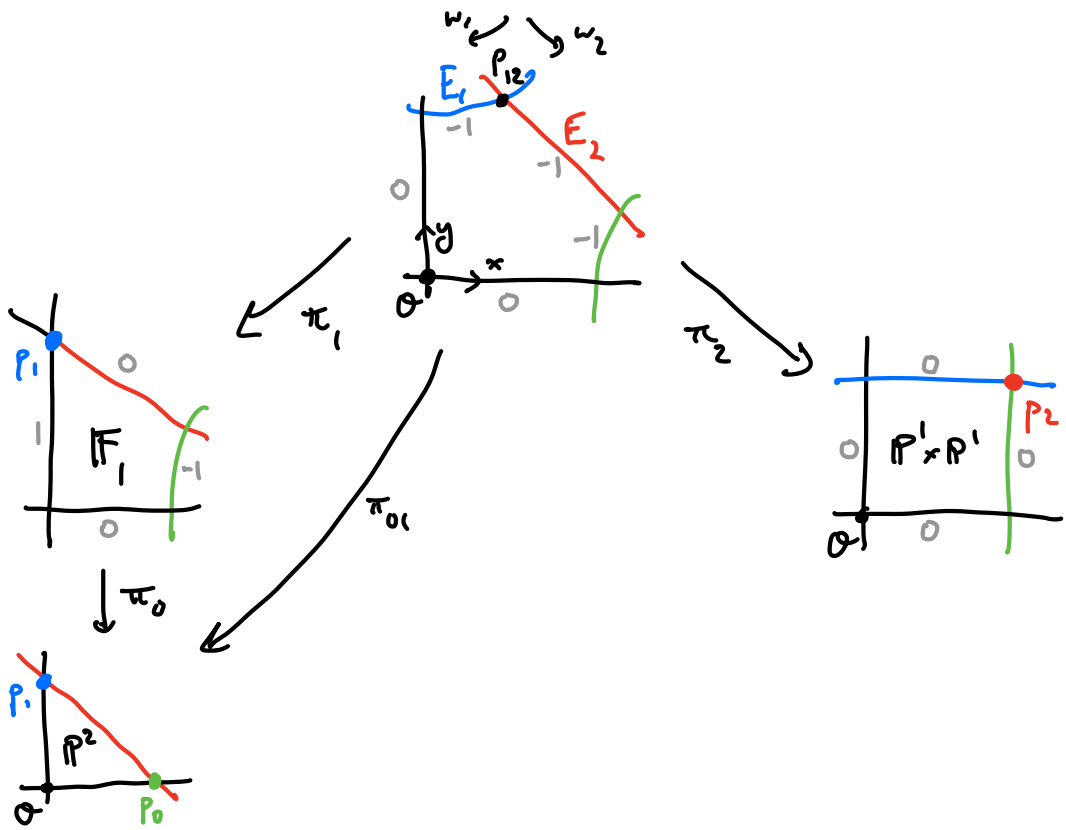
Though it doesn't matter, take $p = [0:1:0]$ for convenience. Writing $C_0 = \{z=0\}$, $C_1 = \{x=0\}$, $C_2 = \{y=0\}$, we have $(C_i\cdot C_j)_{\mathbb{P}^2} = 1$ $(\forall i,j)$. On S,



- C_0^π and C_2^π have intersection number 1 with E and C_1^π
- $(E\cdot E) = -1$, $(C_1^\pi, C_1^\pi) = 1$, and $(C_0^\pi, C_0^\pi) = 0 = (C_2^\pi, C_2^\pi)$. //

The presence of a (-1)-curve (i.e. a genus-0 curve with self-intersection -1) on $\mathbb{F}_1$ means it cannot be isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. We have to blow them up once more: relabeling $\pi, p \mapsto \pi_0, p_0$, I claim that

(32.1.10) $$(Bl_{\{p_0,p_1\}}(\mathbb{P}^2) \cong)\ Bl_{p_1}(\mathbb{F}_1) \cong Bl_{p_2}(\mathbb{P}^1\times\mathbb{P}^1):$$



The self-intersection numbers on curves "in the complement of $\mathbb{C}^* \times \mathbb{C}^*$" certainly make it plausible. To prove it, one writes out the transition functions on coordinate charts on $Bl_{\{p_0,p_1\}}(\mathbb{P}^2)$ implied by (32.1.3) and constructs a compatible blow-down map to $\mathbb{P}^1 \times \mathbb{P}^1$. For instance, coordinates (w_1, w_2) in a neighborhood of p_{12} map to $(w_1 w_2, w_2)$ $\left(= \left(\frac{Z}{Y}, \frac{X}{Y}\right)\right)$ via π_{01} and to $(w_1, w_1 w_2)$ $\left(= \left(\frac{X_0}{X_1}, \frac{Y_0}{Y_1}\right)\right)$ via π_2. This is compatible with the matching of rational functions $\left\{\frac{X}{Z}, \frac{Y}{Z}\right\} \in K(\mathbb{P}^2)^*$ and $\left\{\frac{X_1}{X_0}, \frac{Y_1}{Y_0}\right\} \in K(\mathbb{P}^1 \times \mathbb{P}^1)^*$. (Details are left as an exercise.) Since all choices of $p_2 \in \mathbb{P}^1 \times \mathbb{P}^1$ are equivalent, (32.1.10) leads at once to the

(32.1.11) <u>Proposition</u>: The blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at $k \geq 1$ points is isomorphic to a blowup of $\mathbb{P}^2$ at $k+1$ points.

Notice that $\pi_2 \circ \pi_{01}^{-1}$ is well-defined as a morphism from $\mathbb{P}^2 \setminus \{p_0, p_1\}$ to $\mathbb{P}^1 \times \mathbb{P}^1$. On the level of homogeneous coordinates, it sends $[Z:X:Y] \mapsto ([Z:X], [Z:Y])$, which is <u>not</u> well-defined at $p_0 = [0:1:0]$ and $p_1 = [0:0:1]$.

(32.1.12) <u>Definition</u>: (a) A <u>rational map</u>* $\mu: X \dashrightarrow Y$ between varieties is a morphism from a Zariski open subset of X to Y.

(b) A <u>birational</u> $\left\{\begin{matrix}\text{isomorphism}\\ \text{map}\end{matrix}\right.$ is a $\left\{\begin{matrix}\text{morphism}\\ \text{rational map}\end{matrix}\right.$ μ with a rational inverse $\mu^{-1}: Y \dashrightarrow X$. In this case, X and Y are said to be <u>birational</u>. Blowups are birational morphisms. Birational varieties have isomorphic function fields (and the converse also holds).

(c) A variety is <u>rational</u> if it is birational to a projective space.

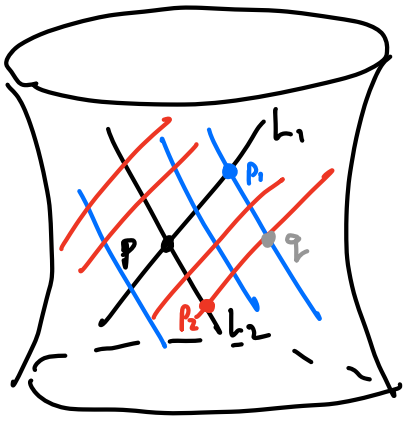
* The point is, if you take $n+1$ <u>rational</u> functions $f_i \in K(X)^*$, then $(f_0 : f_1 : \cdots : f_n)$ defines a rational map $X \dashrightarrow \mathbb{P}^n$.

So any surface obtained from $\mathbb{P}^2$ by a sequence of blowups and blowdowns is rational, and has function field $\cong \mathbb{C}(x,y)$.** It can be shown that the 3 properties are equivalent.

Before turning to cubics, here is one more easy

(32.1.13) Ex / Let $S \subset \mathbb{P}^3$ be a smooth quadric hypersurface. In Corollary 6.2.2, we saw that a projective transformation brings its equation into the form $X^2+Y^2+Z^2+W^2=0$, which it is easy to further transform into $XY - ZW = 0$. This is the image of the Segre map θ_{11} hence isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (cf. Exercise (8) of Chapter 6).

Here is a more geometric approach which is related to our approach to cubics later: given $p \in S$, $T_pS \cap S \subset T_pS \cong \mathbb{P}^2$ is a quadric cone singular at p. Since it can't be a double line (S would be singular), it is a pair of lines $L_1 \cup L_2$ meeting at p, and these are all the lines on S through p (cf. Exercise (3)).



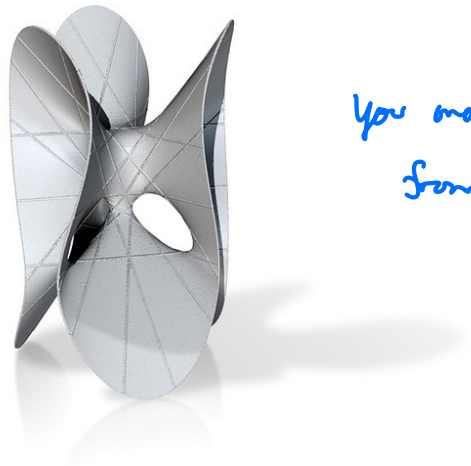
Now given $(p_1, p_2) \in L_1 \times L_2$, we have $L_i \subset T_{p_i}S$. So the line $T_{p_1}S \cap T_{p_2}S$ intersects S in p and <u>one other</u> point $\theta(p_1,p_2)$ (Bézout). This defines a map $\theta : L_1 \times L_2 \to S$. Conversely, given $q \in S$, the intersection of the plane P_{qL_i} with S has degree 2 and contains L_i. Hence it contains one other line $l_i(q)$ (through q), which must meet L_i in some point; call this $p_i(q)$. Evidently $q \mapsto (p_1(q), p_2(q))$ <u>furnishes an inverse to θ</u> (for instance, $P_{qL_i} = T_{p_i(q)}S$), which is thus an isomorphism.

We conclude that S contains two distinct infinite families of lines, called <u>rulings</u> of S; each is parametrized by a $\mathbb{P}^1$. //

** This is the motivation for the name.

§ 32.2. The Clebsch cubic

We begin with a very special smooth cubic whose 27 lines can be written down — and even visualized, since they are all defined over $\mathbb{R}$.



You may recognize this from our course webpage...

It is usually given in "symmetric form" as the vanishing locus of $\sum_{i=0}^{4} z_i$ and $\sum_{i=0}^{4} z_i^3$ in $\mathbb{P}^4$, but writing $z_4 = -\sum_{i=0}^{3} z_i$ eliminates the first equation, leaving us with

(32.2.1) $$S := \{ \underbrace{\textstyle\sum_{i=0}^{3} z_i^3 - \left(\sum_{i=0}^{3} z_i\right)^3}_{F} = 0 \} \subset \mathbb{P}^3.$$

The lines are easier to describe in the first form, where we have an action of $\mathfrak{S}_5$ on the coordinates: they are

(32.2.2) the $\mathfrak{S}_5$-orbit of the line through $[1:-1:0:0:0]$ and $[0:0:1:-1:0]$ (15 lines), and

(32.2.3) the $\mathfrak{S}_5$-orbit of the line through $[1:\zeta_5:\zeta_5^2:\zeta_5^3:\zeta_5^4]$ and $[1:\bar{\zeta}_5:\bar{\zeta}_5^2:\bar{\zeta}_5^3:\bar{\zeta}_5^4]$ (12 lines).

But how do we know (32.2.2)–(32.2.3) are <u>all</u> the lines?

We first settle an easier question:

(32.2.4) <u>Proposition</u>: S has finitely many lines.

Proof: Let $\mathbb{G} = G(2,4)$ denote the Grassmannian variety of 2-planes through the origin in $\mathbb{C}^4$, or equivalently lines in $\mathbb{P}^3$. As a complex manifold, $\mathbb{G}$ is compact. Given a line $L \subset S$, we can apply a projectivity to move both L and S so that $L = \{z_2 = z_3 = 0\}$. *

The local coordinates about $L \in \mathbb{G}$ are then given by $\underline{a} = (a_0, a_1, a_2, a_3)$ $\mapsto L_{\underline{a}} :=$ the projective span of the rows of $\begin{pmatrix} 1 & 0 & a_0 & a_2 \\ 0 & 1 & a_1 & a_3 \end{pmatrix}$, with $L = L_{\underline{0}}$.

We have $L_{\underline{a}} \subset S \iff$

(32.2.5) $\quad 0 = F(s, t, a_0 s + a_1 t, a_2 s + a_3 t) =: \sum_{i=0}^{3} s^{3-i} t^i G_i(\underline{a}) \quad (\forall [s:t] \in \mathbb{P}^1)$

$\iff G_i(\underline{a}) = 0 \ (\forall i)$. Here we can think of $\underline{G} = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \end{pmatrix} : \mathbb{G}^0 \to \mathbb{C}^4$ as a map on the Zariski-open subset $\mathbb{G}^0 \subset \mathbb{G}$ parametrized by $\underline{a}$. Its zero-locus is the set of lines (in $\mathbb{G}^0$) contained in S. I claim that its Jacobian matrix $\left(\frac{\partial G_i}{\partial a_j}\right)$ is nonsingular at $\underline{a} = \underline{0}$.

Suppose otherwise. Writing ${}^t\partial_{\underline{a}} = \left(\frac{\partial}{\partial a_0}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}\right)$, we compute

${}^t\partial_{\underline{a}} (32.2.5) \Big|_{\underline{a} = \underline{0}} =$

(32.2.6) $$\left(s \frac{\partial F}{\partial z_2}(s,t,0,0),\ t \frac{\partial F}{\partial t_2}(s,t,0,0),\ s \frac{\partial F}{\partial z_3}(s,t,0,0),\ t \frac{\partial F}{\partial z_3}(s,t,0,0) \right).$$

Now for a homogeneous polynomial P of degree 3 in s, t, write $[P]$ for the 4-vector of coefficients of s^3, s^2t, st^2, t^3. Clearly $P = 0$ as a polynomial $\iff [P] = [0]$. The columns of $\left(\frac{\partial G_i}{\partial a_j}\right)$ are the $[\cdot]$'s of the entries of (32.2.6). Their assumed dependence

* This will of course change F; as we'll see, this won't matter.

is equivalent to the triviality of $(\lambda_0 s + \lambda_1 t)\frac{\partial F}{\partial z_2}(s,t,0,0) + (\lambda_2 s + \lambda_3 t)\frac{\partial F}{\partial z_3}(s,t,0,0)$ <u>as a polynomial</u> for some $\{\lambda_i\} \subset \mathbb{C}$, not all 0. By the Fundamental Theorem of Algebra, we can write

$$(\lambda_0 s + \lambda_1 t)\frac{\partial F}{\partial z_2}(s,t,0,0) = \prod_{k=1}^{3}(\alpha_k s + \beta_k t) = -(\lambda_2 s + \lambda_3 t)\frac{\partial F}{\partial z_3}(s,t,0,0)$$

whence (by unique factorization) $\frac{\partial F}{\partial z_2}(s,t,0,0)$ and $\frac{\partial F}{\partial z_3}(s,t,0,0)$ have a common factor (or one of them is zero). So there exists a $p_0 = (s_0, t_0, 0, 0) \in L$ at which $\frac{\partial F}{\partial z_2}$ and $\frac{\partial F}{\partial z_3}$ both vanish. Since $L \subset S$, $\frac{\partial F}{\partial z_0}$ and $\frac{\partial F}{\partial z_1}$ vanish at p_0 as well; this makes p_0 a singular point of S, in contradiction to its smoothness.

The inverse function theorem now tells us that $\underline{G}$ is invertible on some analytic open $U \subset \mathbb{G}^\circ$ about L (i.e. $\underline{a} = \underline{0}$). Hence the zero-locus of $\underline{G}|_U$ is just the single "point" L, and no other lines in U belong to S.

Finally, the set $\mathcal{L}_S \subset \mathbb{G}$ of lines on S is defined by polynomial equations, so is a (Zariski and analytically) closed subvariety. Thus, given $L \notin \mathcal{L}_S$, there is an analytic open $U_L \subset \mathbb{G}$ about it not meeting $\mathcal{L}_S$. On the other hand, we have shown that for $L \in \mathcal{L}_S$ there is a U_L with $U_L \cap \mathcal{L}_S = \{L\}$. The family $\{U_L\}_{L \in \mathbb{G}}$ has a finite subcover since $\mathbb{G}$ is compact, and so $\mathcal{L}_S$ is finite. $\square$

§32.3 The general cubic

Cubic surfaces in $\mathbb{P}^3$ are parametrized by their defining polynomials up to multiplication by a constant — that is, by $\mathbb{P}S^3_4 \cong \mathbb{P}^{19}$. Let $\mathcal{V} \subset \mathbb{P}S^3_4$ be the (Zariski) open set parametrizing smooth cubics.*

As you may have noticed, there was nothing specific to the Clebsch cubic about the last proof. A slight modification actually shows the

(32.3.1) <u>Lemma</u>: Every smooth cubic surface $S \in \mathcal{V}$ contains the same number N of lines, with $1 \leq N < \infty$.**

<u>Proof</u>: Consider the closed subvariety $\mathcal{I} = \{(S,L) \mid L \subset S\} \subset \mathcal{V} \times \mathbb{G}$. Fix $(S_0, L_0) \in \mathcal{I}$, and let $V_{L_0} \subset \mathcal{V}$ and $U_{L_0} \subset \mathbb{G}$ be suff. small analytic opens about S_0 and L_0. Repeating the proof of Prop. 32.2.4, but allowing the cubic (and its equation F) to vary about S_0, produces $\underline{G}: V_{L_0} \times U_{L_0} \to \mathbb{C}^4$ with zero locus $\mathcal{I}_{L_0} := \mathcal{I} \cap (V_{L_0} \times U_{L_0})$ and $\left(\frac{\partial G_i}{\partial a_j}\right)$ again nonsingular. The implicit function theorem then says that $\mathcal{I}_{L_0}$ is the graph of a smooth function from V_{L_0} to U_{L_0}.

Now given <u>any</u> $L \in \mathbb{G}$, there are two possibilities: either (i) $(S_0, L) \in \mathcal{I}$ as above, and $\mathcal{I}$ is locally a graph on some $V_L \times U_L$; or (ii) $(S_0, L) \notin \mathcal{I}$, and $\exists\, V_L \times U_L$ not meeting $\mathcal{I}$. The resulting $\{U_L\}_{L \in \mathbb{G}}$ cover $\mathbb{G}$; write $\{U_{L_j}\}_{j=1}^M$ for a finite subcover, and $N\ (\leq M)$ for the number of $\{L_j\}$ of type (i). Clearly, every cubic $S \in \bigcap_{j=1}^M V_{L_j}$ contains exactly N lines.

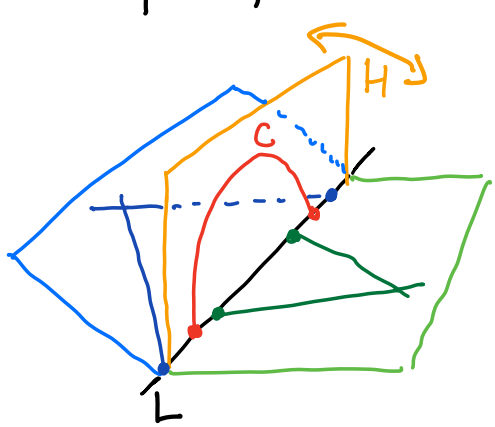
* $\mathcal{V}$ is the complement of the zero locus of a polynomial generalizing the discriminant. If we go modulo projectivities, we get the <u>moduli space</u> $\mathcal{M} := \mathcal{V}/SL_4(\mathbb{C})$ (of dimension $19 - 15 = 4$), which parametrizes smooth cubics up to isomorphism.

** Of course §32.2 shows $N \geq 27$, but we won't proceed in this way.

Finally, letting S_0 vary, we see that the number of lines is locally constant — hence globally constant, as $\mathcal{V}$ is connected (why?). Since the Clebsch cubic has at least one line, so does any smooth cubic. $\square$

(32.3.2) <u>Remark</u>: (off-ramp #1) At this point, we could conclude that <u>every</u> smooth cubic has 27 lines, if we can establish it for just <u>one</u>. See Exercise (5) for how to do this. But this misses out on the beautiful geometry that comes next!

We now know that a smooth cubic surface S contains a line L. Here is how to find more: the idea is to take a (hyper)plane $H \subset \mathbb{P}^3$ through L, and "rotate it around L" while intersecting with S. Since S is cubic, $H \cap S$ is a cubic curve containing L — generically, the union of L and a <u>smooth</u> conic C. But for 5 choices of H, C degenerates to a pair of distinct lines $L' \cup L''$ (both different from L). Such an H is called a <u>tritangent plane</u> since it is tangent to S at the three* intersection points of L, L' and L''.



* If the 3 lines are concurrent, there is just one intersection point; H is still called a tritangent plane.

(32.3.3) <u>Lemma</u>: S contains exactly 10 other lines meeting L.

<u>Proof</u>: We may assume (applying a projectivity) that $L = \{z_2 = z_3 = 0\}$. Smoothness of S dictates that $H \cap S$ cannot contain a double line;* so the options are as described above. Writing $H = \{t z_2 = s z_3\}$, we ask: for which values of $[s:t] \in \mathbb{P}^1$ is C singular?

Since $S \supset L$, its equation cannot have any $z_0^3, z_0^2 z_1, z_0 z_1^2$, or z_1^3 terms; so it takes the form

(32.3.4)
$$0 = (1 \;\; z_0 \;\; z_1) \underbrace{\begin{pmatrix} A_3 & B_2 & C_2 \\ B_2 & D_1 & E_1 \\ C_2 & E_1 & F_1 \end{pmatrix}}_{M(z_2, z_3)} \begin{pmatrix} 1 \\ z_0 \\ z_1 \end{pmatrix}$$

with the entries of M homogeneous in z_2, z_3 of the indicated degrees. Taking $[z_2 : z_3] = [sz : tz]$ in (32.3.4) gives the equation of $C \cup L$ (in $[z : z_0 : z_1]$), and dividing by z yields that of C. In fact, we can just set $[z : z_0 : z_1] = [1 : x : y]$ (as $C \not\supset L$) to get the affine equation of C:

$$0 = (1 \;\; x \;\; y)\, M(s,t) \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}.$$

So C is singular iff the homogeneous <u>quintic</u> $\det M(z_2, z_3)$ vanishes at $[s:t]$.

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* Up to a projectivity, we would have $H = \{z_0 = 0\}$ and $S = \{F := z_3 Q + z_0^2 z_1 = 0\}$ with Q quadratic. The $\frac{\partial F}{\partial z_i}$ all vanish on $z_0 = z_3 = Q = 0$, making S singular.

It remains to show that $\det M(z_2, z_3)$ has no <u>repeated</u> roots. Say $[1:0]$ is a root (i.e. $z_3 \mid \det M$), so that $\{z_3 = 0\} \cap S = L \cup L' \cup L''$, with equation $R := z_0 z_1 z_2 = 0$ or $R := z_2(z_2^2 - z_1^2) = 0$ (up to projectivity). Then that of S is $R + z_3 Q = 0$, and z_3 divides all entries of M except E_1 resp. $\{A_3, D_1\}$. As S is smooth (in particular, at $[0:0:1:0]$ resp. $[1:0:0:0]$), one calculates that z_3^2 doesn't divide A_3 resp. B_2, and concludes that $z_3^2 \nmid \det M$. $\square$

(32.3.5) <u>Remark</u>: (off-ramp #2) We can now count to 27 as Salmon did in his 1849 letter to Cayley. Let H be a tritangent plane, with $H \cap S = L^{(1)} \cup L^{(2)} \cup L^{(3)}$. There are 5 tritangent planes $\{H_j^{(k)}\}_{j=1}^{5}$ through $L^{(k)}$ for each k, with $H_1^{(k)} = H$ $(\forall k)$. The intersections $H_j^{(k)} \cap S =: L^{(k)} \cup (L_j^{(k)})' \cup (L_j^{(k)})''$ yield 24 <u>new</u> lines $\{(L_j^{(k)})', (L_j^{(k)})''\}_{\substack{2 \le j \le 5 \\ 1 \le k \le 3}}$, which are all distinct (see Exercise (3)). Now let $\tilde{L} \subset S$ be any line. It meets H, hence one of the $L^{(k)}$; and then the plane spanned by $\tilde{L}$ and $L^{(k)}$ must be an $H_j^{(k)}$ (why?) and $\tilde{L}$ an $(L_j^{(k)})'$ or $(L_j^{(k)})''$. So there are 27 lines. But there is a still more beautiful way to account for them, as we'll see.

(32.3.6) <u>Corollary</u>: Any smooth cubic S contains a skew (disjoint) pair of lines.

<u>Proof</u>: Let $L \subset S$ be a line, and H_1 and H_2 distinct tritangent planes through L, with $H_j \cap S = L \cup L_j' \cup L_j''$. I claim that $L_1' \cap L_2' = \emptyset$. Otherwise, they meet at a point $p \in L$. But then (by Exercise (3)) L, L_1' and L_2' are coplanar. This is a contradiction, since the projective spans of $\{L, L_1'\}$ and $\{L, L_2'\}$ are H_1 and H_2, which are distinct. $\square$

(32.3.7) <u>Corollary</u>: Given skew lines L_1, L_2 on a smooth cubic S, there are exactly 5 lines on S meeting both L_1 and L_2.

<u>Proof</u>: Let $\{H_j\}_{j=1}^{5}$ be the tritangent planes containing L_1, with $H_j \cap S = L_1 \cup L_j' \cup L_j''$. Since a plane and a line in $\mathbb{P}^3$ meet in one point, L_2 meets $H_j \cap S$. It doesn't meet L_1 (by assumption), and it can't hit $L_j' \cap L_j''$ (Exercise (3) again), so it meets either L_j' or L_j''. $\square$

§ 32.4. Invertible sheaves and adjunction

Let M be a complex manifold, $\{U_\alpha\}$ an (analytic) open cover. Assign to each U_α a free, rank-one $\mathcal{O}(U_\alpha)$-module $\mathcal{S}(U_\alpha)$ with generator σ_α,* and to each $U_{\alpha\beta}$ $(= U_\alpha \cap U_\beta)$ a nonvanishing holomorphic "transition function" $\sigma_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$, such that $\sigma_{\alpha\beta}\sigma_{\beta\gamma}\sigma_{\gamma\alpha} = 1$ on $U_{\alpha\beta\gamma}$. For each open $U \subset M$ we define an abelian group

(32.4.1) $$\mathcal{S}(U) := \left\{ \text{collections } \{g_\alpha \sigma_\alpha|_{U_\alpha \cap U} \in \mathcal{S}(U_\alpha \cap U)\} \;\middle|\; g_\alpha \sigma_{\alpha\beta} = g_\beta \text{ on } U_{\alpha\beta} \cap U \right\}$$

with obvious restriction maps $\mathcal{S}(U) \xrightarrow{|_V} \mathcal{S}(V)$ for $V \subset U$.

(32.4.2) <u>Definition</u>: The assignment $U \mapsto \mathcal{S}(U)$ (or simply "$\mathcal{S}$") is called an <u>invertible sheaf</u>,** and $\mathcal{S}(U)$ its (abelian) group of <u>sections over U</u>. We write $\Gamma(\mathcal{S}) := \mathcal{S}(M)$ for (global holomorphic) <u>sections</u>, and $\mathcal{M}(\mathcal{S})$ for the (global) <u>meromorphic sections</u> obtained by allowing $g_\alpha \in \mathcal{K}(U_\alpha)$ instead of just $\mathcal{O}(U_\alpha)$. The <u>inverse</u> $\mathcal{S}^{-1}$ is given by replacing each $\sigma_{\alpha\beta}$ by $\sigma_{\alpha\beta}^{-1}$.

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* That is, $\mathcal{S}(U_\alpha) = \{g_\alpha \sigma_\alpha \mid g_\alpha \in \mathcal{O}(U_\alpha)\}$ as a set; for each open $V_\alpha \subset U_\alpha$ we also write $\mathcal{S}(V_\alpha) = \{h_\alpha \sigma_\alpha|_{V_\alpha} \mid h_\alpha \in \mathcal{O}(V_\alpha)\}$, and define restriction maps $\mathcal{S}(U_\alpha) \to \mathcal{S}(V_\alpha)$ by sending $g_\alpha \sigma_\alpha \mapsto g_\alpha|_{V_\alpha} \sigma_\alpha|_{V_\alpha}$.

** If you are familiar with line bundles, there is a bijection between the two notions, but the language of invertible sheaves is more convenient for our purposes.

More informally, (32.4.1) is telling us to think of $\sigma_{\alpha\beta}$ as $\dfrac{\sigma_\alpha|_{U_{\alpha\beta}}}{\sigma_\beta|_{U_{\alpha\beta}}}$ and to require the $\{g_\alpha\sigma_\alpha\}$ to agree on overlaps.

(32.4.3) <u>Definition</u>: A <u>morphism</u> $\varphi : \mathcal{S}' \to \mathcal{S}$ of invertible sheaves is given by $\mathcal{O}(U)$-module homomorphisms $\varphi_U : \mathcal{S}'(U) \to \mathcal{S}(U)$ $(\forall\, U \subset M)$ with $\varphi_V \circ |_V = |_V \circ \varphi_U$ $(\forall\, V \subset U)$. In terms of the above data,* it suffices to take $\varphi_\alpha(\sigma'_\alpha) := h_\alpha \sigma_\alpha$ with $h_\alpha \in \mathcal{O}(U_\alpha)$ satisfying $h_\alpha \dfrac{\sigma_{\alpha\beta}}{\sigma'_{\alpha\beta}} = h_\beta$ $\forall \alpha\beta$; and φ is an isomorphism (i.e. invertible) iff all $h_\alpha \in \mathcal{O}^*(U_\alpha)$.

When M is a (smooth) projective variety Y, it is a manifestation of Serre's GAGA principle (cf. Chapter 25) that every invertible sheaf is algebraic — that is, isomorphic to one produced using Zariski open $\{U_\alpha\}$ and rational $\{\sigma_{\alpha\beta}\}$. Given a meromorphic section $\sigma = \{g_\alpha\sigma_\alpha\} \in \mathcal{M}(\mathcal{S})$, the <u>divisor</u> $(\sigma) \in \mathrm{Div}(Y)$ is obtained by patching together the (g_α), which agree on overlaps since $(\sigma_{\alpha\beta}) = 0$. Divisors of (holomorphic) sections $\sigma \in \Gamma(\mathcal{S})$ are effective.

* Here one should pass to a common refinement of the open covers used to define $\mathcal{S}'$ and $\mathcal{S}$.

(32.4.4) <u>Example</u>: The most obvious example is $\mathcal{O}$ itself — the sheaf of holomorphic functions. Any $\mathcal{L}$ admitting a nowhere-vanishing section is isomorphic to $\mathcal{O}$ (and said to be "trivial"), see Exercise (6). (We sometimes write $\mathcal{O}_M$ for clarity.)

In the projective case, we can generalize $\mathcal{O}$ by taking $D \in \mathrm{Div}(Y)$ and choosing a Zariski open cover $\{U_\alpha\}$ on which $\sigma_\alpha \in K(U_\alpha)$ with $(\sigma_\alpha) = -D|_{U_\alpha}$ exist. The resulting sheaf (with $\sigma_{\alpha\beta} := \frac{\sigma_\alpha}{\sigma_\beta} \in \mathcal{O}^*(U_{\alpha\beta})$) is denoted $\mathcal{O}(D)$ (or $\mathcal{O}_Y(D)$). One easily checks (Exercise (7)) that sending $g \mapsto \left\{ \frac{g|_{U_\alpha}}{\sigma_\alpha} \sigma_\alpha \right\}$ induces an isomorphism

(32.4.5) $$\Gamma(\mathcal{O}(D)) \xleftarrow{\cong} \mathcal{I}(D) := \{ g \in K(Y)^* \mid (g) + D \geq 0 \} ;$$

and given $D' \underset{rat}{\equiv} D$ (with $(F) = D - D'$), that multiplication by F yields

(32.4.6) $$\mathcal{O}(D) \xrightarrow{\cong} \mathcal{O}(D').$$

Moreover, the divisor of any $\sigma \in \mathcal{M}(\mathcal{O}(D))$ is rationally equivalent to D: σ takes the form $\sigma_f = \left\{ \frac{f|_{U_\alpha}}{\sigma_\alpha} \sigma_\alpha \right\}$ for some $f \in K(Y)^*$, whence $(\sigma)|_{U_\alpha} = \left(\frac{f|_{U_\alpha}}{\sigma_\alpha} \right) = (f)|_{U_\alpha} + D|_{U_\alpha} \Rightarrow (\sigma) = (f) + D$.

(32.4.7) <u>Example</u>: Another natural example is given by holomorphic forms of "top degree" $n = \dim_{\mathbb{C}} M$. Writing $\{U_\alpha, (z_{\alpha 1}, \dots, z_{\alpha n})\}$ for a covering by holomorphic coordinate charts, we take $\sigma_\alpha = dz_{\alpha 1} \wedge \dots \wedge dz_{\alpha n}$ and $\sigma_{\alpha\beta} = \det\left(\frac{\partial z_{\alpha i}}{\partial z_{\beta j}} \right)$; the resulting <u>canonical sheaf</u> is denoted Ω^n_M. If $M = Y$

is projective, then the divisor of any meromorphic section is called a <u>canonical divisor</u>, written K_Y (and well-defined in $\mathrm{Pic}(Y)$).

(32.4.8) <u>Example</u>: We can take <u>tensor products</u> of invertible sheaves by multiplying transition functions, e.g. $\mathcal{O}(D)\otimes\mathcal{O}(D') = \mathcal{O}(D+D')$ and $\Omega^n_Y \otimes \mathcal{O}(D) =: \Omega^n_Y(D)$. If $Y = C$ is a smooth curve, $\Gamma(\Omega^1_C(-D))$ is exactly what we called $\mathcal{K}(D)$ before.

Given any $\mathcal{L}$ on Y and $\sigma \in \mathcal{M}(\mathcal{L})$, with $(\sigma) \underset{rat}{\equiv} D$, take $\psi \in \mathcal{M}(\mathcal{O}(D))$ and consider $\frac{\sigma}{\psi} \in \mathcal{M}(\mathcal{L}(-D))$. Then we have $(\psi) \underset{rat}{\equiv} D \Rightarrow \left(\frac{\sigma}{\psi}\right) \underset{rat}{\equiv} 0$, and multiplying ψ by $f \in \mathcal{K}(Y)^*$ we can make $\left(\frac{\sigma}{\psi}\right) = 0$. But then by Exercise (6) $\mathcal{L}(-D) \cong \mathcal{O}$, i.e. $\mathcal{L} \cong \mathcal{O}(D)$. So we see that invertible sheaves are parametrized, up to isomorphism, by their <u>divisor classes</u> in $\mathrm{Pic}(Y)$.

Now let $Y \subset X$, with X smooth projective of dimension $n+1$; and let $\{\tilde{U}_\alpha, (z_{\alpha 0}, z_{\alpha 1}, \dots, z_{\alpha n})\}$ be holomorphic coordinate charts on X covering Y, with $U_\alpha := Y \cap \tilde{U}_\alpha = \{z_{\alpha 0} = 0\}$. The "normal" direction in this situation is given by $\frac{\partial}{\partial z_{\alpha 0}}$, with dual ("conormal") $dz_{\alpha 0} = \dfrac{dz_{\alpha 0} \wedge dz_{\alpha 1} \wedge \dots \wedge dz_{\alpha n}}{dz_{\alpha 1} \wedge \dots \wedge dz_{\alpha n}}$. With this

in mind we define* the <u>conormal sheaf</u> by $\mathcal{N}^*_{Y/X} := \Omega^{n+1}_X|_Y \otimes (\Omega^n_Y)^{-1}$; then the <u>normal sheaf</u> is $\mathcal{N}_{Y/X} := (\mathcal{N}^*_{Y/X})^{-1}$, and

(32.4.9) $$\Omega^{n+1}_X|_Y \otimes \mathcal{N}_{Y/X} \cong \Omega^n_Y .$$

Denoting by $Y\cdot Y$ the class in $\mathrm{Pic}(Y)$ obtained by writing $Y \underset{\text{rat}}{\equiv} D$ on X (with $|D|$ not containing Y) and taking $Y\cdot D$, we have the

(32.4.10) <u>Proposition</u>: $\mathcal{N}_{Y/X} \cong \mathcal{O}_Y(Y\cdot Y)$.

<u>Proof</u>: We do this for the case of a curve C on a surface S (both smooth projective, of course). Choose $f \in \mathcal{K}(S)^*$ so that $(f) = C - D$, with $|D| \not\supset C$; and pick $p \in C$. There exists a Zariski open $U \subset Y$ about p and $z, w \in \mathcal{O}(U)$, with $(w) = C \cap U$ and $z|_C$ a local coordinate at p; we may assume also that $|D| \cap C \cap U$ is either empty or $\{p\}$. Setting $g := \frac{f}{w}$, we have $g|_C \underset{\text{loc}}{=} z^k$ + higher-order terms, where $k = -(D\cdot C)_p$.

Consider "$\wedge df|_C$" as a meromorphic section of

$$\mathrm{Hom}(\Omega^1_C, \Omega^2_S|_C) \subseteq (\Omega^1_C)^{-1} \otimes \Omega^2_S|_C \cong \mathcal{N}^*_{C/S} \ (= \mathcal{N}^{-1}_{C/S}).$$

Wedging df with dz kills its dz term, and so locally $(\wedge df|_C)$ is the divisor of (the restriction to C of) the dw term. Since $f|_U = wg$, $df|_U = (wg_w + g)dw + wg_z dz$ and

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* A more standard approach is to define $\mathcal{N}_{Y/X}$ as the sheaf of sections of the quotient of the holomorphic tangent bundle $T^{1,0}X|_Y$ by $T^{1,0}Y$; then (32.4.9) is an easy <u>result</u> rather than by definition.

$\wedge df|_{C\cap U} = g|_C\, dw \sim z^k dw$, with divisor $k[p] = -D\cdot C|_U$.
Conclude that $(\wedge df|_C) = -D\cdot C \equiv -C\cdot C$, so that $\mathcal{N}^*_{C/S} \cong \mathcal{O}(-C\cdot C)$ and $\mathcal{N}_{C/S} \cong \mathcal{O}(C\cdot C)$. $\square$

In view of the Proposition, taking divisor classes on both sides of (32.4.9) yields

(32.4.11) $$(K_X + Y)\cdot Y = K_Y$$

in $\mathrm{Pic}(Y)$. Both (32.4.9) and (32.4.11) are called the <u>adjunction formula</u>. We can use them to calculate the canonical sheaves of hypersurfaces in projective space, in view of the easy

(32.4.12) <u>Lemma</u>: $\Omega^n_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-(n+1)H)$, where $H \subset \mathbb{P}^n$ is a hyperplane; that is, $K_{\mathbb{P}^n} \equiv -(n+1)H$.

<u>Proof</u>: Write $D_i = \{z_i = 0\}$, $D = \sum_{i=0}^n D_i$, and $z_i = \frac{Z_i}{Z_0}$. Then $\frac{dz_1}{z_1}\wedge\cdots\wedge\frac{dz_n}{z_n}$ is a meromorphic section of $\Omega^n_{\mathbb{P}^n}$, with divisor $-D \underset{rat}{\equiv} -(n+1)H$. $\square$

(32.4.13) <u>Remark</u>: Part of the point here is that $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$: not only are any two hyperplanes rationally equivalent; a degree-d hypersurface is $\underset{rat}{\equiv} dH$, and so any divisor is $\underset{rat}{\equiv} kH$. So $\mathcal{O}_{\mathbb{P}^n}(kH)$ is typically written as $\mathcal{O}_{\mathbb{P}^n}(k)$, and all invertible sheaves on $\mathbb{P}^n$ take this form.

One outcome of this discussion is that for a surface $S \subset \mathbb{P}^3$ of degree d, if $H_S := H \cap S$ is a hyperplane section, we have $S \underset{\mathbb{P}^3}{\equiv} dH \Rightarrow K_{\mathbb{P}^3} + S \equiv (d-4)H \Rightarrow$ $K_S = (d-4)H_S \Rightarrow \Omega^2_S \cong \mathcal{O}((d-4)H_S)$. Thus $\Omega^2(S) := \Gamma(\Omega^2_S) \cong \mathcal{L}((d-4)H_S)$ is zero for $d \leq 3$. On the other hand, in this range the <u>anticanonical sheaf</u> $(\Omega^2_S)^{-1} \cong \mathcal{O}_{\mathbb{P}^3}((4-d)H)|_S$ has plenty of sections; e.g. for $d=3$, taking $H = \{z_0 = 0\}$, the embedding of S in $\mathbb{P}^3$ is "tautologically" given by the sections $1, \frac{z_1}{z_0}\Big|_S, \frac{z_2}{z_0}\Big|_S, \frac{z_3}{z_0}\Big|_S$.

(32.4.14) <u>Definition</u>: An invertible sheaf $\mathcal{L}$ (or its divisor) on a smooth projective variety Y is said to be <u>ample</u> if $\Gamma(\mathcal{L}^{\otimes k})$ produces an embedding of Y in a projective space for some $k \in \mathbb{N}$. If the anticanonical divisor $-K_Y$ is ample, Y is said to be <u>Fano</u>.

In dimension 1, the only Fano curve is $\mathbb{P}^1$.
In dimension 2, Fano surfaces are also known as <u>del Pezzo surfaces</u>; as we have seen, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and cubic surfaces are del Pezzo. It turns out that all del Pezzo surfaces are rational, but not conversely.

§ 32.5. The 27 lines.

We are now ready to put everything together for a smooth cubic surface $S \subset \mathbb{P}^3$. First we show it is rational:

(32.5.1) <u>Theorem</u>: We have $S \cong Bl_{\{p_1,\dots,p_6\}}\mathbb{P}^2$, where $p_1,\dots,p_6$ are not concentric and no three of them are collinear.

<u>Proof</u>: Let $L_1, L_2 \subset S$ be the disjoint pair of lines guaranteed by Corollary 32.3.6. Define a morphism $\mu : S \to L_1 \times L_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ by sending $p \in S \setminus (L_1 \cup L_2)$ to $(P_{pL_2} \cap L_1, P_{pL_1} \cap L_2)$, $p \in L_1$ to $(p, T_pS \cap L_2)$, and $p \in L_2$ to $(T_pS \cap L_1, p)$. This has a rational inverse $\mu^{-1} : L_1 \times L_2 \dashrightarrow S$ given by sending (q_1, q_2) to the third intersection point of $L_{q_1q_2}$ with S. Indeed, the plane through $\mu^{-1}(q_1,q_2)$ and (say) L_2 meets L_1 in p_1; while if $(q_1,q_2) = \mu(p)$, then $L_{q_1q_2} = P_{pL_2} \cap P_{pL_1}$ certainly contains p. So μ is a birational morphism.

The crucial observation is now that μ fails to be well-defined precisely when $L_{p_1p_2} \subset S$. By Corollary 32.3.7, we know that this happens for exactly 5 pairs. So μ expresses S as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at these 5 points, hence by Theorem 32.1.11 as the blowup of $\mathbb{P}^2$ at 6 points $p_1,\dots,p_6$. Write $\pi : S \to \mathbb{P}^2$ for this map, and $E_i := \pi^{-1}(p_i)$, $E := \sum_{i=1}^{6} E_i$.

Choose a line (i.e. hyperplane) $H \subset \mathbb{P}^2$ avoiding the p_i,

so that $\pi^* H = H^\pi$. By Lemma 32.4.12, $K_{\mathbb{P}^2} = -3H$; and by Exercise (10), $K_S = -3H^\pi + E$. From the end of §32.4, we know that sections of $(\Omega^2_S)^{-1} \cong \mathcal{O}(-K_S)$ embed $S \hookrightarrow \mathbb{P}^3$. So their restriction to any curve on S embeds <u>it</u> in $\mathbb{P}^3$.

Now suppose $C \subset \mathbb{P}^2$ is a conic passing through m of the $\{p_i\}$. Then $(H^\pi \cdot C^\pi) = (H \cdot C) = 2$ and $(C^\pi \cdot E) = m \Rightarrow$ $K_S \cdot C = m - 6 \Rightarrow (\Omega^2_S)^{-1}|_C \cong \mathcal{O}_C(-K_S \cdot C) \cong \mathcal{O}_{\mathbb{P}^1}(6-m)$. This cannot embed $C\ (\cong \mathbb{P}^1)$ in $\mathbb{P}^3$ if $m=6$ because there would only be the constant section. If we replace C by a line L, then $(\Omega^2_S)^{-1}|_L \cong \mathcal{O}_{\mathbb{P}^1}(3-m)$ and we must have $m \leq 2$. $\square$

We can use this to reduce the problem of finding lines on S to something more manageable. By a (-1)-curve on a surface, we mean a smooth rational curve $(\cong \mathbb{P}^1)$ with self-intersection -1.

(32.5.2) <u>Lemma</u>: The lines on $S \subset \mathbb{P}^3$ are exactly the (-1)-curves on S (or equivalently, on $Bl_{\{p_1,\dots,p_6\}}\mathbb{P}^2$).

<u>Proof</u>: Let C be a nonsingular rational curve on S. By adjunction, $K_C = K_S \cdot C + C \cdot C = -H_S \cdot C + C \cdot C$ (H_S a hyperplane section). Writing d_C for the degree of C under $C \hookrightarrow S \hookrightarrow \mathbb{P}^3$, we have $(H_S \cdot C) = d_C$; C is a line iff $d_C = 1$. But since $C \cong \mathbb{P}^1$, we have $-2 = \deg K_C = -d_C + (C \cdot C)$, and so $d_C = 1 \Leftrightarrow (C \cdot C) = -1$. $\square$

(32.5.3) <u>Theorem</u>: $\mathbb{B} := Bl_{\{p_1,\dots,p_6\}}\mathbb{P}^2$ has exactly 27 (-1)-curves.

<u>Proof</u>: It's easy to find 27, using Proposition 32.1.8. Certainly the 6 $\{E_i\}$ are. Next, there are 15 lines $L_{ij} := L_{p_ip_j} \subset \mathbb{P}^2$, and $(L_{ij}^\pi \cdot L_{ij}^\pi)_{\mathbb{B}} = (L_{ij}\cdot L_{ij})_{\mathbb{P}^2} - 2 = -1$. Finally, there are the 6 conics $C_i \subset \mathbb{P}^2$ through the $\{p_j\}_{j\neq i}$; and $(C_i^\pi \cdot C_i^\pi)_{\mathbb{B}} = (C_i\cdot C_i)_{\mathbb{P}^2} - 5 = 4-5 = -1$. So how do we know we have found them all?

Any divisor on $\mathbb{B}$ is rationally equivalent to one of the form $D = aH^\pi - \sum_{i=1}^{6} b_iE_i$, by using pullbacks of meromorphic functions from $\mathbb{P}^2$. If C_1 and C_2 are distinct (-1)-curves and both $\underset{rat}{\equiv}$ to the same D, we have $-1 = (C_1\cdot C_1) = (C_1\cdot D) = (C_1\cdot C_2) \geq 0$ (since C_1 and C_2 intersect properly), a contradiction. So it is enough to show that there are only 27 solutions to*

(32.5.4) $$-1 = (D\cdot D)_{\mathbb{B}} = a^2(H^\pi\cdot H^\pi) + \sum b_i^2(E_i\cdot E_i) = a^2 - \sum b_i^2$$

that can possibly represent irreducible rational (-1)-curves. So assume $D \underset{rat}{\equiv} C$ for such a curve.

Now $(D\cdot E_j) = b_j$, so if some $b_i < 0$ then $D\cdot E_i < 0$. So C cannot meet E_i properly, and hence must <u>be</u> E_i. In this case we must have $a=0$, $b_i = -1$, and $b_j = 0$ $(\forall j\neq i)$; we get 6 solutions in this way.

Next let all $b_j \geq 0$. Since C is a (-1)-curve,

* Here I used the disjointness of $\{H^\pi, E_1,\dots,E_6\}$.

by the proof of lemma 32.5.2 we have

(32.5.6) $$1 = d_C = (-K_S \cdot C) = ((3H^\pi - \sum E_i) \cdot D) = 3a - \sum_{i=1}^{6} b_i .$$

Applying Cauchy-Schwarz to the vectors $\underline{b}$ and $\underline{1} = (1, \dots, 1)$ gives

$$(\underline{b} \cdot \underline{1})^2 \leq \|\underline{b}\|^2 \|\underline{1}\|^2 = 6\, \underline{b} \cdot \underline{b} \Rightarrow \sum b_i^2 \geq \tfrac{1}{6} (\sum b_i)^2 \Rightarrow$$

$$a^2 = \sum b_i^2 - 1 \geq \tfrac{1}{6} (\sum b_i)^2 - 1 = \tfrac{1}{6}(3a-1)^2 - 1 = \tfrac{3}{2}a^2 - a - \tfrac{5}{6}$$

$\Rightarrow a = 0, 1,$ or 2. Since all $b_j \geq 0$, we can rule out 0 by (32.5.6). If $a = 1$, then $\sum b_i = 2 = \sum b_i^2 \Rightarrow$ two $b_i = 1$ and the rest are 0 ($\binom{6}{2} = 15$ solutions). If $a = 2$, then $\sum b_i = 5 = \sum b_i^2 \Rightarrow$ one $b_i = 0$ and the other $b_j = 1$ (6 solutions). So we have $6 + 15 + 6 = 27$ solutions! $\square$

(32.5.4) <u>Corollary</u>: S has exactly 27 lines.

Exercises

① Let $\pi : \tilde{S} \twoheadrightarrow S$ be a birational morphism of smooth projective surfaces (composition of blowups), and $\mathcal{D} \in \mathrm{Pic}(S)$, $D \in \mathrm{Pic}(\tilde{S})$. Show $(D \cdot \pi^* \mathcal{D}) = (\mathcal{D} \cdot \pi_* D)$, where π_* takes the image* of a divisor under π.
[Hint: see the proof of Prop. 32.1.8.]

② Complete the suggested proof of (32.1.10).

③ Let S be a smooth surface, and $p \in S$. Show that any line on S through p lies in $T_p S$, thus that all such lines are coplanar.
(Also, check how this is used in Remark 32.3.5 & Cors. 32.3.6–7.)

④ Check that the lines (32.2.2–3) lie on the Clebsch cubic and are defined over $\mathbb{R}$.

⑤ Consider the Fermat cubic surface $S_F := \{ z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0 \}$.
Up to permutation of coordinates, any line can be written in the form $L_{\underline{a}}$ from the proof of (32.2.4). Use this to show that S_F has <u>exactly</u> 27 lines.

⑥ Prove that an invertible sheaf $\mathcal{L}$ (on a complex manifold M) admitting a nowhere-vanishing section is $\cong \mathcal{O}$. [Hint: use Definition 32.4.3.]

⑦ Check (32.4.5)–(32.4.6).

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* If π maps a curve E to a point, we define $\pi_* E = 0$.

(8) Use adjunction (together with Poincaré-Hopf) to give a <u>quick</u> proof of the degree-genus formula for a smooth curve in $\mathbb{P}^2$.

(9) Determine the number of tritangent planes of a smooth cubic surface.

(10) Let S be a smooth projective surface, and $\tilde{S} := Bl_{\{p_1,\dots,p_k\}} S \xrightarrow{\pi} S$ the blow-up, with exceptional divisor E. Show that $K_{\tilde{S}} = \pi^* K_S + E$.

[Hint: write K_S for the divisor of a meromorphic section $\omega \in \mathcal{M}(\Omega^2_S)$, with the $p_i \notin |K_S|$. Locally compute $\pi^*\omega \in \mathcal{M}(\Omega^2_{\tilde{S}})$ and its divisor in a neighborhood of an E_i.]

(11) A <u>double-six</u> on a cubic threefold is a configuration of 12 lines $A_1, \dots, A_6, B_1, \dots, B_6$ where each A_i meets the 5 $\{B_j\}_{j \neq i}$, each B_i meets the 5 $\{A_j\}_{j \neq i}$, and these are the only intersections.

Using the explicit description of (-1)-curves on $Bl_{\{p_1,\dots,p_6\}} \mathbb{P}^2$, construct a double-six.

[(12) Check that the Clebsch cubic surface is smooth.]