Chapter 33: Abelian varieties

While the 27 lines on a cubic surface are a thing of beauty, they also stand in the way of generalizing the group law on a cubic curve. Given $p, q \in S$ points on a cubic surface, L_{pq} meets S in one other point — unless L_{pq} is one of these lines. In fact, no hypersurface or complete intersection in $\mathbb{P}^N$ of dimension bigger than 1 admits a group law.

So for our second generalization of elliptic curves to higher dimension — smooth projective varieties admitting a group law — it is clear that we will have to take a more "intrinsic" tack, rather than describing them by equations. As we shall soon see, these are all complex tori, but not all complex tori are projective varieties: only those admitting a polarization (a certain kind of Hermitian form on the space of holomorphic 1-forms) may be embedded in projective space. These include all Jacobians of algebraic curves, and — amazingly — one can recover a curve from its (polarized) Jacobian.

All of this turns out to be more natural in arbitrary dimension, though at some points we will focus specifically on the case of surfaces. This includes a beautiful geometric interpretation of the group law on an abelian surface.

§ 33.1 : Abelian varieties are tori

With the understanding that we are speaking only of varieties over $\mathbb{C}$, we begin with the

(33.1.1) <u>Definition</u>: An <u>abelian variety</u> $(A, \cdot)$ is an irreducible smooth projective variety A together with a group structure on its (complex) points, defined by an algebraic morphism "$\cdot$": $A \times A \to A$

Our first result is that A, well, deserves its name:*

(33.1.2) <u>Lemma</u>: The group law on $(A, \cdot)$ is abelian.

<u>Sketch</u>: Define $\xi : A \times A \to A$ by $(a,b) \mapsto [a,b] := aba^{-1}b^{-1}$, and $\pi : A \times A \to A$ by $(a,b) \mapsto b$. Then $\xi(A \times \{1\}) = \{1\} = \xi(\{1\} \times A)$.

Let $U \subset A$ be an affine Zariski-<u>open</u> subset containing 1; then

$$Y := \pi(\xi^{-1}(A \setminus U)) = \{b \in A \mid \xi(A \times \{b\}) \not\subset U\}$$

is <u>closed</u>, and $(1 \in)\ A \setminus Y$ is nonempty open hence dense. For $b \in A \setminus Y$, $A \times \{b\} \xrightarrow{\xi} U$ must be constant as it maps a projective variety into an affine one**; and since $\xi(1,b) = 1$

** The coordinate functions on affine space pull back to "bounded entire functions" on our compact complex manifold.

* We have the compactness to thank here: there are many non-abelian linear-algebraic groups, like $SL_n(\mathbb{C})$, which are nonabelian; but they are also not projective / compact.

we therefore have $\xi(A \times \{b\}) = \{1\}$. As $b \in A \setminus Y$ was arbitrary, this gives $\xi(A \times (A \setminus Y)) = \{1\}$, and $A \times (A \setminus Y) \subset A \times A$ is a dense open subset. So $\xi \equiv 1$ and all commutators vanish. $\square$

(33.1.3) <u>Proposition</u>: $(A, \cdot)$ is a (compact) complex torus.

<u>Sketch</u>: Any topological group admits 1-parameter subgroups, associating to each $v \in T_1 A =: V$ a unique homomorphism $e^{zv} : (\mathbb{C}, +) \to (A, \cdot)$ with differential v at 0. Since $(A, \cdot)$ is abelian, $e^{zv} e^{zw}$ defines a homomorphism, with differential $v+w$ at 0, hence by uniqueness must equal $e^{z(v+w)}$. So $e^{(\cdot)}$ actually gives a homomorphism $\mathbf{e} : (V, +) \to (A, \cdot)$, with $(d\mathbf{e})_0 = \mathrm{id}_V$, onto some (analytic) open subgroup $A_0 \leq A$. The union of the nonidentity cosets of A_0 is thus open, and as its complement, A_0 is closed. But then $A_0 = A$, and so $A \cong V/\ker(\mathbf{e})$. Since $\ker(\mathbf{e}) \leq V$ is a subgroup with compact, Hausdorff quotient, it can only be a full lattice. $\square$

Henceforth we replace $(A, \cdot, 1)$ by the notation $(A, +, 0)$. Note that the group law on A is nonunique, as it depends on a choice of origin; when we speak of an <u>abelian variety</u>, we assume this has been <u>fixed</u>.

We close this section with a discussion of complex tori more generally. Let $\{\underline{\lambda}^{(1)}, \dots, \underline{\lambda}^{(2g)}\} \subset \mathbb{C}^g$ be an $\mathbb{R}$-linearly

independent set, so that $\Lambda := \mathbb{Z}\langle \underline{\lambda}^{(1)}, \dots, \underline{\lambda}^{(2g)} \rangle \subset \mathbb{C}^g$ is a full lattice ($\cong \mathbb{Z}^{2g}$). Then we have the complex g-torus

$$T := \mathbb{C}^g / \Lambda,$$

with quotient map $\pi : \mathbb{C}^g \twoheadrightarrow T$; and the 1-cycles $\gamma_j := \pi(\overrightarrow{\underline{0}.\underline{\lambda}^{(j)}})$ give a basis of $H_1(T, \mathbb{Z})$ ($\cong \Lambda$). Writing z_i for the complex coordinates on $\mathbb{C}^g$, $\{\omega_i := dz_i\}_{i=1}^g$ provide a basis for $\Omega^1(T)$, and $\lambda_i^{(j)} = \int_{\gamma_j} \omega_i$.* Setting

$$\Pi := (\lambda_i^{(j)}) \in M_{g \times 2g}(\mathbb{C}),$$

the nonsingularity of $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ (cf. Exercise (1)) shows that

$$(33.1.4) \qquad \begin{aligned} \Omega^1(T) \oplus \overline{\Omega^1(T)} &\longrightarrow H_1(T, \mathbb{C})^\vee = H^1(T, \mathbb{C}) \\ (\omega, \bar{\eta}) &\longmapsto \int_{(\cdot)} \omega + \bar{\eta} \end{aligned}$$

is an isomorphism. So we can think of it as a change-of-basis matrix, viz.

$$(33.1.5) \qquad \tilde{\Pi} := \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = {}_{\gamma^\vee}[\mathrm{id}_{H^1}]_{\tilde{\omega}},$$

where $\tilde{\omega} = \{\omega_1, \dots, \omega_g; \bar{\omega}_1, \dots, \omega_g\}$ and $\gamma^\vee$ is the dual basis of $\gamma = \{\gamma_j\}$.

(33.1.6) <u>Remark</u>: Since g columns of the period matrix Π must be

* $\lambda_i^{(j)}$ is simply the ith coordinate of $\underline{\lambda}^{(j)}$, which we think of as a column vector.

independent over $\mathbb{C}$,* we can reindex to make these $\underline{\lambda}^{(1)}, \dots, \underline{\lambda}^{(g)}$; and then a change of basis of $\Omega^1(T)$ (equivalently, of coordinates on $\mathbb{C}^g$) makes $\underline{\lambda}^{(j)} = e^{(j)}$ (standard basis vectors). That is, we have $\Pi = (I \;\; M)$, with $\operatorname{Im} M \in M_g(\mathbb{R})$ nonsingular (why?) and M arbitrary otherwise. At this point, to get a parametrization of complex g-tori up to isomorphism, we should "quotient out" by the action of $GL_{2g}(\mathbb{Z})$ on the $\mathbb{Z}$-basis γ; but this does not act nicely on the above normalization of Π.** In any case, we see that the isomorphism class of T is parametrized by g^2 <u>complex parameters</u> (via the choice of M).

(33.1.7) <u>Remark</u>: If you are acquainted with higher (co)homology groups, they are particularly easy to describe for a complex g-torus. Given any k of the $\{\underline{\lambda}^{(j)}\}$, we can take the parallelepiped they span in $\mathbb{R}^{2g} = \mathbb{C}^g$ and apply π to get a k-cycle. The resulting $\binom{2g}{k}$ homology classes span $H_k(T, \mathbb{Z})$, and dually we have the isomorphisms $H^k(T, \mathbb{C}) \cong \Lambda^k H^1(T, \mathbb{C})$ and $\Omega^k(T) \cong \Lambda^k \Omega^1(T)$. In particular, the 1-dimensionality of $\Omega^g(T) \cong \Lambda^g \Omega^1(T)$ reflects the isomorphism $K_T \; (= \Omega^g_T) \cong \mathcal{O}$ given by the trivializing section $dz_1 \wedge \dots \wedge dz_g$ (cf. Example 32.4.4).

* Again see Exercise (1).

** No matter how one expresses this quotient, it is nasty: the double-coset space $GL_{2g}(\mathbb{Z}) \backslash GL_{2g}(\mathbb{R}) / GL_g(\mathbb{C})$ is non-Hausdorff. Here the $GL_{2g}(\mathbb{R})$ corresponds to the choice of $2g$ $\mathbb{R}$-linearly-independent vectors in $\mathbb{C}^g \cong \mathbb{R}^{2g}$.

§33.2: But most complex tori are not abelian varieties!

When $g > 1$, a new complication arises: the "general" g-torus doesn't admit nonconstant meromorphic functions, so cannot possibly have a projective embedding. The route we'll take here is to suppose we have a complex torus T which sits inside a projective space $\mathbb{P}^N$, and derive "necessary conditions" on the period matrix Π. It will turn out that, for example,

$$(33.2.1) \qquad \begin{pmatrix} 1 & 0 & \sqrt{-1} & \sqrt{-2} \\ 0 & 1 & \sqrt{-3} & \sqrt{-5} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \sqrt{-1} & 0 \\ 0 & 1 & \sqrt{-2} & \sqrt{-1} \end{pmatrix}$$

<u>cannot</u> be period matrices of an abelian surface, i.e. a 2-torus admitting a projective embedding.

So suppose $T \subset \mathbb{P}^N$, and let $C \overset{?}{\subset} T$ be the curve obtained by intersecting T with $g-1$ general hyperplanes. Then

$$(33.2.2) \qquad B_0(\xi, \xi') := \int_C \iota^*\xi \wedge \iota^*\xi'$$

defines an antisymmetric bilinear form on $H^1(T, \mathbb{C})$, where we think of ξ, ξ' as (classes of) closed 1-forms. On $H^1(T,\mathbb{Z}) \times H^1(T,\mathbb{Z})$, we can interpret B_0 as an intersection number: classes in $H^1(T,\mathbb{Z}) = H_1(T,\mathbb{Z})^\vee$ may be represented by*

* The link between the two interpretations is that, for any closed $(2g-1)$-cycle Γ on T, there is a smooth 1-form η_Γ satisfying $\int_\Gamma \mu = \int_T \eta_\Gamma \wedge \mu$ and $\gamma \cdot \Gamma = \int_\gamma \eta_\Gamma$ for all closed $(2g-1)$-forms μ and closed 1-cycles γ. ... A good place to learn about this is [Bott-Tu].

$(2g-1)$-cycles which give functionals on $H_1(T;\mathbb{Z})$ via the intersection pairing. The "ι^*" intersects them with C to produce 1-cycles, and (33.2.2) takes the intersection number of these 1-cycles on C. Taking $(m) \subset \mathbb{Z}$ the ideal generated by these intersection numbers, we set $\mathcal{B} := \frac{1}{m}\mathcal{B}_0$ and consider the matrix $[\mathcal{B}]_{\gamma^\vee} \in M_{2g}(\mathbb{Z})$. By a result in linear algebra (cf. Exercise (27)), we can choose γ so that

$$(33.2.3) \qquad Q := [\mathcal{B}]_{\gamma^\vee} = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix} \quad \text{with} \quad \begin{cases} \Delta = \operatorname{diag}(\delta_1, \dots, \delta_g) \\ \delta_1 \mid \delta_2 \mid \cdots \mid \delta_g \in \mathbb{Z}_{\geq 0}. \end{cases}$$

Arguing as in §31.1, we now have*

$$(33.2.4) \qquad [\mathcal{B}]_{\tilde{\omega}} = \tilde{\Pi} Q {}^t\tilde{\Pi} = \left(\begin{array}{c|c} \Pi Q {}^t\Pi & \Pi Q {}^t\bar{\Pi} \\ \hline \bar{\Pi} Q {}^t\Pi & \bar{\Pi} Q {}^t\bar{\Pi} \end{array}\right) =: -i \begin{pmatrix} 0 & H \\ -\bar{H} & 0 \end{pmatrix}$$

since $\Pi Q {}^t\Pi$ represents $\frac{1}{m}\int_C \iota^*\omega_i \wedge \iota^*\omega_j = 0$. Moreover, H is Hermitian as

$${}^tH = i\bar{\Pi}\, {}^tQ\, {}^t\Pi = -i\,\bar{\Pi} Q {}^t\Pi = \overline{i\Pi Q {}^t\bar{\Pi}} = \bar{H},$$

and positive-definite** as $\frac{i}{m}\int_C \iota^*\omega \wedge \iota^*\bar{\omega} > 0 \quad \forall \omega \in \Omega^1(T)^{\{0\}}$.

(33.2.5) <u>Remark</u>: The last statement is, of course, equivalent to injectivity of $\iota^*: \Omega^1(T) \to \Omega^1(C)$. This follows from a general theorem of Lefschetz on hyperplane sections, but can also be seen directly here. For instance, we can choose C to pass through $0 \in T$ and to have any tangent vector v there, by varying the hyperplanes. By Exercise (3),

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* $\tilde{\Pi}$ as in (33.1.5), but <u>not</u> normalized as in Remark 33.1.6.

** It follows immediately that $\det[\mathcal{B}]_\omega \neq 0 \Rightarrow \det Q \neq 0 \Rightarrow$ all $\delta_i > 0$ (and $\delta_1 = 1$).

a curve through 0 is contained in a (compact) proper complex subtorus $W/W\cap\Lambda \subsetneq T$ if $\ker(\iota^*) \neq \{0\}$. But these all arise from subgroups $\Lambda_0 \subsetneq \Lambda$ whose real span $W = \mathbb{R}\langle\Lambda_0\rangle$ is a complex subspace of $\mathbb{C}^g$, and there are at most countably many of these. Taking $v \in \mathbb{C}^g$ outside of these subspaces, we have $\ker(\iota^*) = \{0\}$. $\square$

Writing $\mathbb{H}$ for our positive-definite Hermitian form on $\Omega^1(T)$ (with matrix H), consider the (injective) composition $j: H^1(T,\mathbb{Z}) \hookrightarrow H^1(T,\mathbb{C}) \twoheadrightarrow \Omega^1(T)$, so that for $\xi \in H^1(T,\mathbb{Z})$, we have $\xi = j(\xi) + \overline{j(\xi)}$. Then

$$\begin{aligned} 2i\,\mathrm{Im}\,\mathbb{H}(j(\xi), j(\xi')) &= \mathbb{H}(j(\xi), j(\xi')) - \overline{\mathbb{H}(j(\xi), j(\xi'))} \\ &= \frac{i}{m}\int_C \left(\iota^* j(\xi) \wedge \iota^* \overline{j(\xi')} + \iota^* \overline{j(\xi)} \wedge \iota^* j(\xi')\right) \\ &= \frac{i}{m}\int \iota^*\xi \wedge \iota^*\overline{\xi'} = i\,\mathcal{B}(\xi,\xi') \in i\mathbb{Z}, \end{aligned}$$

so that $2\,\mathrm{Im}\,\mathbb{H}$ on $j(H^1(T,\mathbb{Z}))$ recovers $\mathcal{B}$. We also have a dual Hermitian form $\mathbb{H}^\vee$ on $\Omega^1(T)^\vee$ (represented by ${}^tH^{-1}$ in the dual basis); and by "the same" computation $2\,\mathrm{Im}\,\mathbb{H}^\vee$ recovers the dual bilinear form (represented by ${}^tQ^{-1}$) on the image of $H_1(T,\mathbb{Z}) \xrightarrow{\tilde{j}} \Omega^1(T)^\vee \cong \mathbb{C}^g$,* which is nothing but the period lattice Λ. Rescaling if necessary by an integer (to make Δ^{-1} integral), we have an $\mathcal{H}$ as in the following

(33.2.6) <u>Definition</u>: A <u>polarization</u> of T is a positive-definite Hermitian form $\mathcal{H}$ on $\Omega^1(T)^\vee$, with $2\,\mathrm{Im}\,\mathcal{H}$ taking integer values on $H_1(T,\mathbb{Z})$. Equivalently, it is a matrix $\mathcal{H} \in M_g(\mathbb{C})$ with ${}^t\mathcal{H} = \overline{\mathcal{H}}$, $\mathcal{H} > 0$, and

* The isomorphism is given by evaluation of functionals on $\{\omega_1, \dots, \omega_g\}$, and the functionals in the image are the integrals over 1-cycles.

(33.2.7) $$2\,\mathrm{Im}({}^t\Pi\,\mathcal{H}\,\overline{\Pi}) \in M_{2g}(\mathbb{Z})$$

(since Π is the matrix of $\tilde{j}$). The torus T is said to be <u>polarized</u> by $\mathcal{H}$, and the pair $(T, \mathcal{H})$ is called a <u>polarized torus</u>.

When (33.2.7) is unimodular, $\mathcal{H}$ is called a <u>principal polarization</u>, and T is <u>principally polarized</u>; this corresponds to $\Delta = I$ in the above scenario. $\square$

We can put polarized tori in an especially nice form by reasoning exactly as in (31.1.4)–(31.1.7): writing $\Pi = (A\ B)$, we deduce that A is invertible, and change ω-basis by $\Delta^{-1}A^{-1}$ to get $\Pi = (\Delta^{-1}\ Z)$. From $\Pi Q^t\Pi = 0$ and $i\Pi Q^t\overline{\Pi} > 0$ we immediately have $Z = {}^tZ$ and $\mathrm{Im}\,Z > 0$, so that Z belongs to the Siegel upper half-space $\mathfrak{h}_g$. (Conversely, you can use the criterion of Definition 33.2.6 to check that every complex torus with Π of this form is polarizable, see Exercise (4).) This also makes clear that every polarized torus is a finite unramified quotient of (or cover of) a principally polarized one.

The upshot of our discussion is the following

(33.2.8) <u>Proposition</u>: An abelian variety A is a polarized complex torus, with polarization arising from the projective embedding. It can thus be expressed as $\mathbb{C}^g/\Lambda$, with Λ the full lattice generated by columns of a matrix of the form $(\Delta^{-1}\ Z)$, with Δ as above and $Z \in \mathfrak{h}_g$.

Going back to the beginning of the section, how might we show that a given torus T is <u>not polarizable</u>, hence cannot be an abelian variety? This is a little tricky: if $\Pi = (I \ Z)$ and $Z \notin h_g$, that does <u>not</u> imply T isn't polarizable; e.g., Z could arise from permuting the columns of a matrix in h_g. For 2-tori with $\Pi = (I \ iM)$, $M \in M_2(\mathbb{R})$, a criterion for checking non-polarizability is developed and applied to (33.2.1) in Ex. (5). On the other hand, it is evident by dimension counting that <u>most g-tori aren't polarizable</u>, since $g^2 > \binom{g+1}{2} = \dim h_g$ once $g > 1$.

In the next two sections, we turn to the converse of Proposition 33.2.8, and show that every polarized torus is an abelian variety, i.e. embeds in some $\mathbb{P}^N$. For this it suffices to consider principally polarized tori, since the quotient of a projective variety by a finite group is projective. (I won't prove this, but it arises from taking G-invariants in the coordinate ring.) But before proceeding, we should consider what sort of embedding this can be.

For instance, how large is N? Can abelian g-folds be embedded as hypersurfaces in $\mathbb{P}^{g+1}$, and can we use intersections with lines to define the group law geometrically? This, unfortunately, cannot be. The theorem of Lefschetz mentioned in Remark 33.2.5 also implies that for any complete intersection $X \subset \mathbb{P}^N$ of dimension > 1, $H_1(X, \mathbb{Z})$ and $\pi_1(X)$ are trivial. (Here X need not even be smooth!) So with the exception of elliptic curves, abelian varieties cannot be projective complete intersections, let alone hypersurfaces.

(33.2.9) <u>Remark</u>: If you are comfortable with a bit of algebraic topology, here is a thought experiment that shows we can't have an abelian surface embedded in $\mathbb{P}^3$. Such a hypersurface $A \overset{\iota}{\subset} \mathbb{P}^3$ would be cut out by a single homogeneous polynomial of degree d. If we let $\mathcal{H} \subset \mathbb{P}^3$ be a general hypersurface of degree d, so that $C := A \cap \mathcal{H} \overset{j}{\subset} A$ is smooth, then one obtains a commutative diagram

$$\begin{tikzcd}
H^1(A) \arrow[r, "j^*"] \arrow[d, "\iota_*"'] \arrow[dr, "\cup[C]", "\cup[A\cdot A]"'] & H^1(C) \arrow[d, "j_*"] \\
H^3(\mathbb{P}^3) \arrow[r, "\iota^*"'] & H^3(A) .
\end{tikzcd}$$

One then argues that $j_* \circ j^*$ is injective since $i \int_A j_* j^* \omega \cup \bar{\omega} = i \int_C j^* \omega \wedge j^* \bar{\omega} > 0$ for every nonzero $\omega \in \Omega^1(A)$. On the other hand, since $H^3(\mathbb{P}^3) = \{0\}$, $j_* \circ j^* = \iota^* \circ \iota_* = 0$; together with the injectivity, this forces $H^1(A) = \{0\}$, which is not true for an abelian surface.

Outline of remaining material covered (but not written up):

- Theta sheaf, Riemann theta function, & theta divisor
- Lefschetz embedding theorem (polarized tori embed in $\mathbb{P}^N$ hence are abelian varieties)
- period map for curves, Torelli's theorem, proof for $g=2$
- K3 surfaces (Kummer K3s, quartic surfaces, toric hypersurfaces)

Exercises

(1) If the columns of $\Pi \in M_{g\times 2g}(\mathbb{C})$ are $\mathbb{R}$-linearly independent,

(a) show that $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ is nonsingular, and that this makes (33.1.4) an isomorphism. (b) Check that there exist g columns of Π which are $\mathbb{C}$-linearly independent.

(2) Prove that, given $M \in M_g(\mathbb{Z})$ with ${}^tM = -M$, there exists $S \in GL_g(\mathbb{Z})$ with ${}^tSMS = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_g)$, and $\delta_1 | \delta_2 | \cdots | \delta_g$.

[Hint: think of M as the matrix of a bilinear form, and consider the (necessarily principal) ideal comprising all of its values on pairs of vectors in $\mathbb{Z}^g$. Choose a pair with minimal nonzero value, and consider its "orthogonal complement". Restrict the form to this and reason inductively.]

(3) Let $C \overset{2}{\subset} A$ be an irreducible (smooth, projective) curve in an abelian variety, with $0 \in C$, and $\pi : \mathbb{C}^g \twoheadrightarrow \mathbb{C}^g/\Lambda = A$ the projection. Write $\tilde{C} \subset \mathbb{C}^g$ for the connected component of $\pi^{-1}(C)$ through $\underline{0}$, and $W \subset \mathbb{C}^g$ for the smallest $\mathbb{C}$-linear subspace containing $\tilde{C}$.

(a) Show that W is actually the smallest $\mathbb{R}$-linear subspace of $\mathbb{C}^g = \mathbb{R}^{2g}$ containing $\tilde{C}$. [Hint: any real coordinate is the real part of a complex one.]

(b) Prove that $W \cap \Lambda$ is a full lattice in W, so that $\pi(W) = W/W\cap\Lambda$ is a (compact) complex k-torus for some $k \leq g$. [Hint: let $W_0 = \mathbb{R}\langle W \cap \Lambda \rangle \subset W$. Show that if $W_0 \neq W$, there is a nonconstant harmonic function on C.]

(c) Conclude that $\ker(\iota^* : \Omega^1(A) \to \Omega^1(C))$ is nonzero iff C is contained in a proper sub-abelian variety. [Hint: identify W with $(\Omega^1(A)/\ker(\iota^*))^\vee$.]

④ Let $T = \mathbb{C}^g/\Lambda$, Λ = span of columns of $\Pi = (I \; Z)$, with $Z \in h_g$. Show there is an $\mathcal{H} \in M_g(\mathbb{C})$ satisfying the criteria of Definition 33.2.6, hence polarizing T.

⑤ Let T be a 2-torus with period matrix $\Pi = (I \; \vdots \; M)$, $M \in M_2(\mathbb{R})$. (a) Show that if T is polarizable, there exists $\mathcal{H} = R + i\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}$ with $v \in \mathbb{Z}$, $R = {}^tR \in M_2(\mathbb{R})$, $RM \in M_2(\mathbb{Z})$, $v \det M \in \mathbb{Z}$, $\det R > 0$, and $\operatorname{tr} R > 0$.

(b) Writing $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, deduce that if $\dim_{\mathbb{Q}} \langle a, b, c, d \rangle = 4$ and $ad - bc \notin \mathbb{Q}$, then T cannot be algebraic. Apply to the first matrix of (33.2.1)

(c) The second matrix of (33.2.1) represents a 2-torus T that is a nontrivial "extension" of two complex 1-tori, viz. $0 \to E_1 \to T \to E_2 \to 0$ as abelian groups. While the E_i are algebraic, use (a) to show that T is not.*

* Unlike the first example in (33.2.1), this torus does have some nonconstant meromorphic functions — precisely the ones pulled back from E_1. Of course, this isn't enough to separate points of T.