CHAPTER 10

Local analytic factorization of polynomials

Recall the idea of normalization for an irreducible algebraic curve \( C \subset \mathbb{P}^2 \): there should exist a Riemann surface \( \tilde{C} \) mapping holomorphically to \( \mathbb{P}^2 \) with \( C \) as its image. In Chapter 7 we did this for non-singular \( C \) by using the holomorphic implicit function theorem to put a complex manifold structure on \( C \) itself. This essentially consisted, for each \( p \in C \), in exhibiting a neighborhood \( \mathcal{N}_p \subset \mathbb{P}^2 \) of \( p \) and a (bi)holomorphic parametrization of \( \mathcal{N}_p \cap C \) by some open set \( U \subset C \). (The holomorphicity of the transition functions was then a consequence.)

Now suppose \( C \) has an ordinary double point (ODP) at \( p \) — recall that this is a singularity with 2 distinct tangent lines. Denoting disjoint union by “\( II \)”, one has

\[
\mathcal{N}_p \cap C \simeq \frac{U_1 II U_2}{0_{U_1} \equiv 0_{U_2}};
\]

that is, \( C \) locally looks like two disks \( U_1, U_2 (\subset C) \) glued together at one point. In order to normalize \( C \), \( U_1 \) and \( U_2 \) must be “detached”:
Our overarching goal is to produce $\tilde{C}$ and $\sigma$ as in this figure. Geometrically it seems clear that the “local analytic curve” $\mathcal{N}_p \cap C$ is reducible, even though the global curve $C$ is not. The first step, then, will be to find an appropriate formalism (in terms of 2-variable power series) for working with $\mathcal{N}_p \cap C$, which one might call “analytic localization.” In this setting, the local equation can be uniquely factored. This will allow us (in the next Chapter) to carry out local normalization — that is, put local coordinates on the irreducible components of $\mathcal{N}_p \cap C$. Finally, we will patch these parametrizations together with those of open subsets of $C \setminus \text{sing}(C)$ to obtain $\tilde{C}$.

There are algebraic approaches to “localization” of $C$ at $p$. For convenience, replace $C$ for the moment by its affinization in $\mathbb{C}^2$. From §9.3, we have the coordinate ring $R = \mathbb{C}[C]$, and to any point $p \in C$ corresponds a maximal ideal in $m \subset R$ (consisting of polynomials vanishing at $p$). Inverting all primes not contained in $m$, or “localizing $R$ at $m$”, replaces polynomial functions by rational functions with poles anywhere but $p$, which roughly corresponds to replacing $C$ by $C$ minus any set of points not including $p$. This is quite different
from intersecting $C$ with an analytic ball at $p$, and will not produce
a local factorization of a globally irreducible $C$. Instead of rational
functions, we need convergent power series. The closest construction
in algebra is something called \textit{completion} (or Henselian localization).
If you are curious (we won’t get into this), a good reference is
the book by D. Eisenbud.

\section{Analytic localization}

It will suffice to think of $C$ as an affine curve \{\(f(x,y) = 0\)\} \subset \mathbb{C}^2
passing through $p = (0,0)$. The defining polynomial \(f \in \mathbb{C}[x,y]\)
is, trivially, a convergent power series; so we may consider how \(f\)
factors in \(O_2 = \mathbb{C}\{x,y\}\) (cf. §7.1). In fact, for purposes of examining
the intersection of $C$ with a small neighborhood of the origin, we will
show that $f$ may be replaced by an element of $\mathbb{C}\{x\}[y] \subset O_2$ in a
particularly nice form:

\subsection{Definition}
The subset\(^1\) $\mathfrak{W} \subset \mathbb{C}\{x\}[y]$ of Weierstrass polynomials comprises elements of the form
\[
y^d + a_1(x)y^{d-1} + \cdots + a_{d-1}(x)y + a_d(x) \quad (d \in \mathbb{Z}_{\geq 0})
\]
where each $a_j(x) \in \mathbb{C}\{x\}$ satisfies
\[
a_j(0) = 0.
\]

\subsection{Lemma}
Let $f \in O_2$ with\(^2\) $f \not\equiv 0$ on the $y$-axis. Then \exists \epsilon, \rho > 0 \text{ such that:}
\begin{enumerate}
\item[(a)] $f \neq 0$ on (i) \{\(|x| < \rho, \ |y| = \epsilon\)\} and (ii) \{\(x = 0, 0 < |y| < \epsilon\)\};
\item[(b)] the number of roots (counted with multiplicity) of $f(x,y)$ in $y$
with $|y| < \epsilon$, is constant in $x$ for $|x| < \rho$.
\end{enumerate}

\footnote{technically, a submonoid – you can multiply (but not add) elements, and it has
the identity element 1; the notions of “irreducible element” and “uniqueness of
factorization” still have meaning. Since \(\mathfrak{W}\) is inside a UFD (see proof of Thm.
10.2.2) and has 1 as its sole unit, it does indeed have unique factorization in a very
strong sense. (See the discussion after the proof of Thm. 10.2.2.)}

\footnote{In general, if \(S\) is some subset of the domain of definition of a function \(f\),
one should read “\(f \not\equiv 0\) on \(S\)” as “\(f\) is not identically zero on \(S\),” and “\(f \not\equiv 0\) on \(S\)”
as “\(f\) does not vanish on \(S\)” (i.e. \(f\) is zero at no point of \(S\)) — two very different
meanings. Henceforth the symbols will be used with no further explanation.}
PROOF. The zeroes of \( f(0, y) \) are isolated: otherwise they would have a limit point, forcing \( f \) to be identically zero. We may therefore choose \( \epsilon \) so that \( f'(0, y) \neq 0 \) for \( 0 < |y| \leq \epsilon \). To get (a)(i) from this, just use continuity and choose \( \rho \) sufficiently small. The number of roots in (b) is computed by

\[
\frac{1}{2\pi i} \oint_{|y|=\epsilon} \frac{f_y(x, y)}{f(x, y)} \, dy \in \mathbb{Z},
\]

which is continuous in \( x \) and therefore constant. \( \square \)

10.13. LEMMA. For \( f \) as in Lemma 10.1.2, let \( \{y_v(x)\}_{v=1,\ldots,d} \) be the roots described in (b).\(^3\) Denote the elementary symmetric polynomials in them by \( e_j(x) \left( = \sum_{v_1<\ldots<v_j} y_{v_1}(x) \cdots y_{v_j}(x) \right) \). Then

\[
w := y^d - e_1(x)y^{d-1} + \cdots + (-1)^de_d(x)
\]

is a Weierstrass polynomial.

PROOF. Note that for each \( v \), \( y_v(0) = 0 \) from Lemma 10.1.2(a)(i). Clearly then the \( e_j(x) \) are well-defined and satisfy \( e_j(0) = 0 \); we must show that they are holomorphic on \( \{|x| < \rho\} \). First we have

\[
\frac{1}{2\pi i} \oint_{|y|=\epsilon} y^k \frac{f_y(x, y)}{f(x, y)} \, dy = \sum_v (y_v(x))^k =: \sigma_k(x),
\]

since the residue at each \( y_v(x) \) of the argument is

\[
(y_v(x))^k \cdot \text{Res}_{y_v(x)} \left( \frac{f_y}{f} \right) = (y_v(x))^k \cdot \text{ord}_{y_v(x)} (f(x, \cdot)).
\]

Here the Newton symmetric polynomials \( \sigma_k(x) \) generate the same algebra over \( \mathbb{C} \) as the \( e_j(x) \); that is, they can be expressed as polynomials in each other.\(^4\) From the integral expression, the \( \sigma_k \) are evidently holomorphic, and therefore so are the \( e_j \). \( \square \)

\(^3\)These may well be multivalued on \( \{|x| < \rho\} \) — in particular, one should expect them to be permuted as \( x \) goes about 0. So the \( y_v(x) \) are really only well-defined on some simply-connected subset of the disk \( \{|x| < \rho\} \) (e.g., deleting the positive real numbers gives a slit disk).

\(^4\)See the exercises; in abstract algebra one shows that they both generate the ring of symmetric polynomials in the \( \{y_v\} \).
Let $\Omega := \mathcal{O}_2^+ \subset \mathcal{O}_2$ denote the units, which are just the invertible convergent power series, or equivalently the convergent power series with nonzero constant term. (That is, given $g \in \mathcal{O}_2$, $g \in \Omega \iff \frac{1}{g} \in \mathcal{O}_2$.)

10.1.4. LEMMA. For $f$ and $w$ as above, there exists a unique $u \in \Omega$ such that $uw = f$, and this holds on all of $V := \{|x| < \rho \text{ and } |y| \leq e\}$.

PROOF. Write $\tilde{u} := \frac{f}{w} \in \mathcal{O}(V \setminus \{w = 0\})$. For fixed $x$, $w(x, y) = \prod_{v=1}^{d}(y - y_v(x))$, as multiplying this out gives the $e_j(x)$ as coefficients. Consequently, for each fixed $x$ (with $|x| < \rho$), $w(x, y)$ and $f(x, y)$ have the same roots (in $y$). Therefore $\tilde{u} \neq 0$ on $V$, and $\tilde{u}(x, y)$ is (for each $x$) holomorphic in $y$. Now, for any given $y_0$ with $|y_0| < e$,

$$\tilde{u}(x, y_0) = \frac{1}{2\pi i} \int_{|y|=e} \frac{\tilde{u}(x, y)}{y-y_0} dy.$$ 

Since $\tilde{u}(x, y)$ is holomorphic on a neighborhood of $|y| = e$, this formula shows $\tilde{u}(x, y_0)$ is holomorphic in $x$. By Osgood’s lemma, we have $\tilde{u} \in \mathcal{O}(V)$. Since $\tilde{u} \neq 0$, it has nonzero constant term $\tilde{u}(0, 0)$, and is thus a unit. Uniqueness is clear since $\tilde{u}w = f$ and $uw = f \implies (\tilde{u} - u)w = 0 \implies u - \tilde{u} = 0$. □

10.2. Uniqueness of local factorization

The uniqueness of $u$ in the last Lemma was trivial. A slightly less trivial uniqueness question would be: can we write $f$ as a product of a unit and a Weierstrass polynomial in two different ways – i.e., with a different $w$ and $u$? We cannot:

10.2.1. LEMMA. Given $f \in \mathcal{O}_2$ (with $f \neq 0$ on the $y$-axis), the decomposition $f = wu$ in Lemma 10.1.4 (i.e., into $w \in \mathcal{W}$ and $u \in \Omega$) is unique.

PROOF. Since any unit $u$ has $u(0, 0) \neq 0$, shrinking $\epsilon, \rho$ (hence $V$) if necessary, we have $u \neq 0$ on $V$. Thus if $f = wu$, the zeroes of $f$ and $w$ are the same. This forces $w = \prod(y - y_v(x)) = y^d - e_1(x)y^{d-1} + \cdots + (-1)^de_d(x)$, which makes $w$ (hence $u$) unique. □
Making use of the last two lemmas, we now show that \( f \in \mathcal{O}_2 \) factors uniquely (up to units) into irreducibles \( f_i \in \mathcal{O}_2 \). If \( f \) began its life as a polynomial defining an irreducible algebraic curve \( C = \{ f = 0 \} \subset \mathbb{C}^2 \), then the local piece \( C \cap V \) breaks (uniquely) into irreducible components \( \{ f_i = 0 \} \). Provided there is more than one of them, the \( f_i \) are no longer polynomials, for that would contradict (global) irreducibility of \( C \).

10.2.2. Theorem. \( \mathcal{O}_2 \) is a UFD.

Proof. We need to demonstrate that \( f \in \mathcal{O}_2 \) factors into irreducibles \( f_1 \cdots f_\ell \) uniquely up to order and units.

First, note that \( \mathcal{O}_1 = \mathbb{C}\{x\} \) is a UFD: given \( g \in \mathcal{O}_1 \), we have a unique decomposition \( g(x) = x^{v_0(f)}h(x) \), where \( h \) is a unit (convergent power series with \( h(0) \neq 0 \)) and \( v_0(f) \in \mathbb{Z} \). The irreducibles in this case are just the factors of \( x \).

By the Gauss lemma, it follows that \( \mathbb{C}\{x\}[y] \) is a UFD.

Next, suppose that \( f(x,y) = \sum_{a,b} a_{ab} x^a y^b \in \mathcal{O}_2 \) vanishes identically on the \( y \)-axis; that is, \( 0 \equiv f(0,y) = \sum_b a_{0b} y^b \). It follows that all \( a_{0b} = 0 \) for all \( b \), so that \( f = x^v f_0 \) where \( v > 0 \) and \( f_0(0,y) \neq 0 \). We must prove unique factorization for \( f_0 \).

Let \( f \in \mathcal{O}_2 \) with \( f(0,y) \neq 0 \). Lemmas 10.1.4 and 10.2.1 give \( f = u w \) uniquely. Since \( w \) belongs to the UFD \( \mathbb{C}\{x\}[y] \), we have a unique decomposition \( w = h_1 \cdots h_\ell \) into irreducibles \( h_j \in \mathbb{C}\{x\}[y] \). Clearly also \( h_j(0,y) \neq 0 \), and so Lemma 10.1.4 applied to each \( h_j \) gives uniquely \( h_j = u_j w_j \), with each \( w_j \) a Weierstrass polynomial irreducible in \( \mathbb{C}\{x\}[y] \) (since \( h_j \) is). This yields \( w = (u_1 w_1) \cdots (u_\ell w_\ell) = u_1 \cdots u_\ell w_1 \cdots w_\ell =: \tilde{u} \tilde{w} \), and by Lemma 10.2.1 \( \tilde{u} \) must be 1. So far we have \( f = u w_1 \cdots w_\ell \).

We do not know yet whether \( w_j \) is irreducible in \( \mathcal{O}_2 \). If \( w_j = v' v'' \) (\( v', v'' \in \mathcal{O}_2 \)), then \( w_j(0,y) \neq 0 \implies \) the same thing for \( v', v'' \). Lemma 10.1.4 applies to yield \( v' = u' w' \) and \( v'' = u'' w'' \), so that \( w_j = (u' u'')(w' w'') \); applying Lemma 10.2.1 yet again gives \( u' u'' = 1 \implies w_j = w' w'' \). But \( w', w'' \in \mathcal{M} \subset \mathbb{C}\{x\}[y] \), contradicting irreducibility of \( w_j \) in \( \mathbb{C}\{x\}[y] \).
To see uniqueness, write factorizations $f = f_1 \cdots f_\ell = g_1 \cdots g_k$ into irreducibles in $O_2$; we may assume $f(0, y) \neq 0$. Then Lemma 10.1.4 gives $f_j = u_j w_j$ and $g_i = \tilde{u}_i \tilde{w}_i$ with $w_j, \tilde{w}_i$ irreducible Weierstrass polynomials. We then have

$$(u_1 \cdots u_\ell)(w_1 \cdots w_\ell) = (\tilde{u}_1 \cdots \tilde{u}_k)(\tilde{w}_1 \cdots \tilde{w}_k),$$

so that by Lemma 10.2.1 $u_1 \cdots u_\ell = \tilde{u}_1 \cdots \tilde{u}_k$ and $w_1 \cdots w_\ell = \tilde{w}_1 \cdots \tilde{w}_k$. By uniqueness of factorization in $C\{x\}[y]$ (and Lemma 10.2.1), the \{w_j\} and \{\tilde{w}_i\} are the same (up to reordering), and $\ell = k$. \hfill $\square$

Note the key statement that comes out of this proof: given $f \in O_2$ with $f(0, y) \neq 0$, we have

$$f = uw_1 \cdots w_\ell,$$

where $u \in \mathfrak{M}$ and $w_i$ are Weierstrass polynomials which are irreducible (as Weierstrass polynomials, as elements of $C\{x\}[y]$, and as elements of $O_2$). Moreover, this decomposition is completely unique, up to reordering of the $w_i$. Finally – this also comes out of the proof – if $f$ was a Weierstrass polynomial, then $u = 1$ in (10.2.3), and $\deg_y(f) = \sum_{i=1}^\ell \deg_y(w_i)$. This will be useful in working the following problems.

**Exercises**

1. Show $f(x, y) = x^3 - x^2 + y^2$ is (a) irreducible in $C[x, y]$ and (b) reducible in $C\{x\}[y]$.
2. Show $g(x, y) = x^3 - y^2$ is irreducible in $C\{x\}[y]$.
3. What form does the factorization in $C\{x\}[y]$ of the Weierstrass polynomial $f = y^4 + xy + x^4$ take? (You don’t have to compute the power series!) How does this correspond to the Newton polygon connecting the $(a, b)$ which appear as exponents (viz., $x^a y^b$) in monomial terms of $f$? 5

5 More precisely, the Newton polygon is the boundary of the convex hull of the sets $(a, b) + \mathbb{R}_{\geq 0}$.
(4) By considering a few more examples, e.g. $y^4 + xy^2 + x^5$, $x^5 + y^5$, and $y^8 + xy^4 + x^2y^2 + x^4y + x^8$, can you formulate a conjecture connecting the Newton polygon to the factorization into irreducibles in $\mathbb{C}\{x\}[y]$?

(5) The proof of Lemma 10.1.3 used the fact that the elementary symmetric polynomials $e_k$ in $\{y_i\}_{i=1}^m$ can be expressed as polynomials in the Newton symmetric polynomials $\sigma_k = \sum_{i=1}^m y_i^k$. Prove this by establishing that $-ke_k = \sum_{i=1}^k (-1)^i e_{k-i}p_i$. [Hint: substitute $x = y_j$ in $\prod_{i=1}^m (x - y_i) = \sum_{i=0}^m (-1)^{k-i} e_{k-i}x^i$, then sum over $j$.]

(6) Adapt the proof of Proposition 8.2.7 to show that any (closed) complex analytic curve $C \subset \mathbb{P}^2$ (i.e., a subset which in a neighborhood of any point is cut out by the vanishing of a nonconstant holomorphic function) is in fact algebraic (cut out by a homogeneous polynomial). [Suggestion: by applying a projectivity, you may assume that $[0:1:0]$ is not in $C$. Note that the intersection of $C$ with any vertical line $x = x_0$ is finite; in a neighborhood of each intersection point $(x_0, y_0)$, $C$ can be described by a Weierstrass polynomial in $y - y_0$. Multiply these together to get an element of $\mathbb{C}\{x - x_0\}[y]$, monic in $y$, cutting out $C$ for $|x - x_0| < \rho$ (and all $y$). Argue that these local elements patch together to give an element $"\prod_{\lambda=1}^m (y - y_{\lambda}(x))"$ in $\mathcal{O}(C)[y]$, and then show that $\mathcal{O}(C)$ can be replaced by $\mathbb{C}[x]$.]