CHAPTER 10

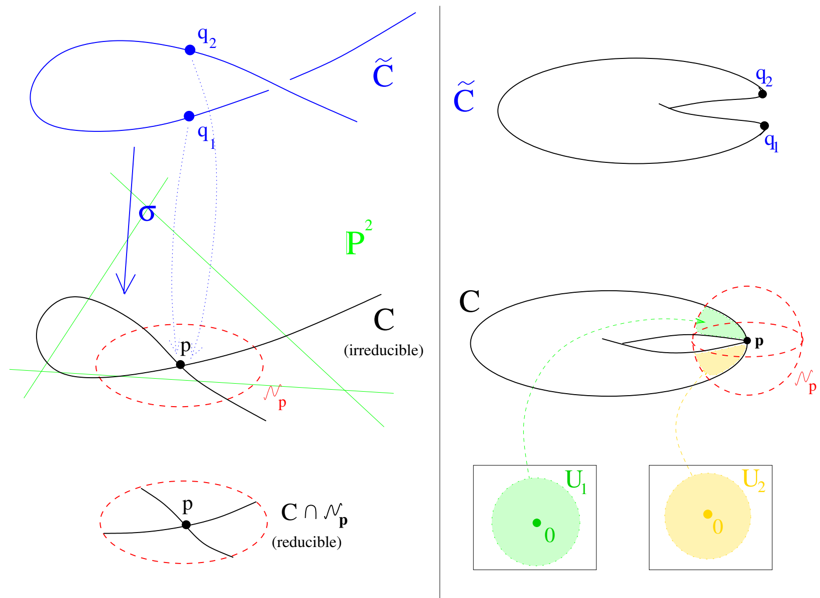
Local analytic factorization of polynomials

Recall the idea of *normalization* for an irreducible algebraic curve $C \subset \mathbb{P}^2$: there should exist a Riemann surface $\tilde{C}$ mapping holomorphically to $\mathbb{P}^2$ with C as its image. In Chapter 7 we *did* this for *nonsingular* C by using the holomorphic implicit function theorem to put a complex manifold structure on C itself. This essentially consisted, for each $p \in C$, in exhibiting a neighborhood $\mathcal{N}_p \subset \mathbb{P}^2$ of p and a (bi)holomorphic parametrization of $\mathcal{N}_p \cap C$ by some open set $U \subset \mathbb{C}$. (The holomorphicity of the transition functions was then a consequence.)

Now suppose C has an *ordinary double point* (ODP) at p — recall that this is a singularity with 2 distinct tangent lines. Denoting disjoint union by "$\amalg$", one has

$$\mathcal{N}_p \cap C \simeq \frac{U_1 \amalg U_2}{0_{U_1} \equiv 0_{U_2}};$$

that is, C locally looks like two disks U_1, U_2 $(\subset \mathbb{C})$ glued together at one point. In order to normalize C, U_1 and U_2 must be "detached":



schematic picture **topological picture**

Our overarching goal is to produce $\tilde{C}$ and σ as in this figure. Geometrically it seems clear that the "local analytic curve" $\mathcal{N}_p \cap C$ is reducible, even though the global curve C is not. The first step, then, will be to find an appropriate formalism (in terms of 2-variable power series) for working with $\mathcal{N}_p \cap C$, which one might call "analytic localization." In this setting, the local equation can be uniquely factored. This will allow us (in the next Chapter) to carry out *local normalization* — that is, put local coordinates on the irreducible components of $\mathcal{N}_p \cap C$. Finally, we will patch these parametrizations together with those of open subsets of $C \backslash \text{sing}(C)$ to obtain $\tilde{C}$.

There are algebraic approaches to "localization" of C at p. For convenience, replace C for the moment by its affinization in $\mathbb{C}^2$. From §9.3, we have the coordinate ring $R = \mathbb{C}[C]$, and to any point $p \in C$ corresponds a maximal ideal in $\mathfrak{m} \subset R$ (consisting of polynomials vanishing at p). Inverting all primes not contained in $\mathfrak{m}$, or "localizing R at $\mathfrak{m}$", replaces polynomial functions by rational functions with poles anywhere but p, which roughly corresponds to replacing C by C minus any set of points not including p. This is quite different from intersecting C with an analytic ball at p, and will not produce a local factorization of a globally irreducible C. Instead of rational functions, we need convergent power series. The closest construction in algebra is something called *completion* (or Henselian localization). If you are curious (we won't get into this), a good reference is the book by D. Eisenbud.

10.1. Analytic localization

It will suffice to think of C as an affine curve $\{f(x,y)=0\} \subset \mathbb{C}^2$ passing through $p = (0,0)$. The defining polynomial $f \in \mathbb{C}[x,y]$ is, trivially, a convergent power series; so we may consider how f factors in $\mathcal{O}_2 = \mathbb{C}\{x,y\}$ (cf. §7.1). In fact, for purposes of examining the intersection of C with a small neighborhood of the origin, we will show that f may be replaced by an element of $\mathbb{C}\{x\}[y]$ ($\subset \mathcal{O}_2$) in a particularly nice form:

10.1.1. DEFINITION. The subset[1] $\mathfrak{W} \subset \mathbb{C}\{x\}[y]$ of *Weierstrass polynomials* comprises elements of the form

$$y^d + a_1(x)y^{d-1} + \cdots + a_{d-1}(x)y + a_d(x) \qquad (d \in \mathbb{Z}_{\geq 0})$$

where each $a_j(x) \in \mathbb{C}\{x\}$ satisfies

$$a_j(0) = 0.$$

10.1.2. LEMMA. *Let $f \in \mathcal{O}_2$ with[2] $f \not\equiv 0$ on the y-axis. Then $\exists$ $\epsilon, \rho > 0$ such that:*

(a) $f \neq 0$ on (i) $\{|x| < \rho,\ |y| = \epsilon\}$ and (ii) $\{x = 0,\ 0 < |y| < \epsilon\}$;

(b) the number of roots (counted with multiplicity) of $f(x,y)$ in y with $|y| < \epsilon$, is constant in x for $|x| < \rho$.

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[1]technically, a submonoid – you can multiply (but not add) elements, and it has the identity element 1; the notions of "irreducible element" and "uniqueness of factorization" still have meaning. Since $\mathfrak{W}$ is inside a UFD (see proof of Thm. 10.2.2) and has 1 as its sole unit, it does indeed have unique factorization in a very strong sense. (See the discussion after the proof of Thm. 10.2.2.)

[2]In general, if $\mathcal{S}$ is some subset of the domain of definition of a function f, one should read "$f \not\equiv 0$ on $\mathcal{S}$" as "f is not identically zero on $\mathcal{S}$", and "$f \neq 0$ on $\mathcal{S}$" as "f does not vanish on $\mathcal{S}$" (i.e. f is zero at no point of $\mathcal{S}$) — two very different meanings. Henceforth the symbols will be used with no further explanation.

PROOF. The zeroes of $f(0,y)$ are isolated: otherwise they would have a limit point, forcing f to be identically zero. We may therefore choose ϵ so that $f(0,y) \neq 0$ for $0 < |y| \leq \epsilon$. To get (a)(i) from this, just use continuity and choose ρ sufficiently small. The number of roots in (b) is computed by

$$\frac{1}{2\pi\sqrt{-1}} \oint_{|y|=\epsilon} \frac{f_y(x,y)}{f(x,y)} dy \ \in\ \mathbb{Z},$$

which is continuous in x and therefore constant. □

10.1.3. LEMMA. *For f as in Lemma 10.1.2, let $\{y_\nu(x)\}_{\nu=1,\ldots,d}$ be the roots described in (b).*[3] *Denote the elementary symmetric polynomials in them by $e_j(x)$ ($= \sum_{\nu_1<\cdots<\nu_j} y_{\nu_1}(x) \cdot \cdots \cdot y_{\nu_j}(x)$). Then*

$$w := y^d - e_1(x)y^{d-1} + \cdots + (-1)^d e_d(x)$$

is a Weierstrass polynomial.

PROOF. Note that for each ν, $y_\nu(0) = 0$ from Lemma 10.1.2(a)(i). Clearly then the $e_j(x)$ are well-defined and satisfy $e_j(0) = 0$; we must show that they are holomorphic on $\{|x| < \rho\}$. First we have

$$\frac{1}{2\pi\sqrt{-1}} \oint_{|y|=\epsilon} y^k \frac{f_y(x,y)}{f(x,y)} dy \ =\ \sum_\nu (y_\nu(x))^k \ =:\ \sigma_k(x),$$

since the residue at each $y_\nu(x)$ of the argument is

$$(y_\nu(x))^k \cdot Res_{y_\nu(x)}(\frac{f_y}{f}) = (y_\nu(x))^k \cdot \mathrm{ord}_{y_\nu(x)}(f(x,\cdot)).$$

Here the *Newton symmetric polynomials* $\sigma_k(x)$ generate the same algebra over $\mathbb{C}$ as the $e_j(x)$; that is, they can be expressed as polynomials in each other.[4] From the integral expression, the σ_k are evidently holomorphic, and therefore so are the e_j. □

[3]These may well be multivalued on $\{|x| < \rho\}$ — in particular, one should expect them to be permuted as x goes about 0. So the $y_\nu(x)$ are really only well-defined on some simply-connected subset of the disk $\{|x| < \rho\}$ (e.g., deleting the positive real numbers gives a slit disk).

[4]See the exercises; in abstract algebra one shows that they both generate the ring of symmetric polynomials in the $\{y_\nu\}$.

Let $\mathfrak{U} := \mathcal{O}_2^* \subset \mathcal{O}_2$ denote the units, which are just the invertible convergent power series, or equivalently the convergent power series with nonzero constant term. (That is, given $g \in \mathcal{O}_2$, $g \in \mathfrak{U} \iff \frac{1}{g} \in \mathcal{O}_2$.)

10.1.4. LEMMA. *For f and w as above, there exists a unique $u \in \mathfrak{U}$ such that $uw = f$, and this holds on all of $V := \{|x| < \rho$ and $|y| \leq \epsilon\}$.*

PROOF. Write $\tilde{u} := \frac{f}{w} \in \mathcal{O}(V \backslash \{w = 0\})$. For fixed x, $w(x,y) = \prod_{\nu=1}^{d}(y - y_\nu(x))$, as mutliplying this out gives the $e_j(x)$ as coefficients. Consequently, for each fixed x (with $|x| < \rho$), $w(x,y)$ and $f(x,y)$ have the same roots (in y). Therefore $\tilde{u} \neq 0$ on V, and $\tilde{u}(x,y)$ is (for each x) holomorphic in y. Now, for any given y_0 with $|y_0| < \epsilon$,

$$\tilde{u}(x,y_0) = \frac{1}{2\pi\sqrt{-1}} \oint_{|y|=\epsilon} \frac{\tilde{u}(x,y)}{y - y_0} dy\,.$$

Since $\tilde{u}(x,y)$ is holomorphic on a neighborhood of $|y| = \epsilon$, this formula shows $\tilde{u}(x,y_0)$ is holomorphic in x. By Osgood's lemma, we have $\tilde{u} \in \mathcal{O}(V)$. Since $\tilde{u} \neq 0$, it has nonzero constant term $\tilde{u}(0,0)$, and is thus a unit. Uniqueness is clear since $\tilde{u}w = f$ and $uw = f \implies (\tilde{u} - u)w = 0 \implies u - \tilde{u} = 0$. □

10.2. Uniqueness of local factorization

The uniqueness of u in the last Lemma was trivial. A slightly less trivial uniqueness question would be: can we write f as a product of a unit and a Weierstrass polynomial in two different ways – i.e., with a different w *and* u? We cannot:

10.2.1. LEMMA. *Given $f \in \mathcal{O}_2$ (with $f \not\equiv 0$ on the y-axis), the decomposition $f = wu$ in Lemma 10.1.4 (i.e., into $w \in \mathfrak{W}$ and $u \in \mathfrak{U}$) is unique.*

PROOF. Since any unit u has $u(0,0) \neq 0$, shrinking ϵ, ρ (hence V) if necessary, we have $u \neq 0$ on V. Thus if $f = wu$, the zeroes of f and w are the same. This forces $w = \prod(y - y_\nu(x)) = y^d - e_1(x)y^{d-1} + \cdots + (-1)^d e_d(x)$, which makes w (hence u) unique. □

Making use of the last two lemmas, we now show that $f \in \mathcal{O}_2$ factors uniquely (up to units) into irreducibles $f_i \in \mathcal{O}_2$. If f began its life as a polynomial defining an irreducible algebraic curve $C = \{f = 0\} \subset \mathbb{C}^2$, then the local piece $C \cap V$ breaks (uniquely) into irreducible components $\{f_i = 0\}$. Provided there is more than one of them, the f_i are no longer polynomials, for that would contradict (global) irreducibility of C.

10.2.2. THEOREM. *$\mathcal{O}_2$ is a UFD.*

PROOF. We need to demonstrate that $f \in \mathcal{O}_2$ factors into irreducibles $f_1 \cdots f_\ell$ uniquely up to order and units.

First, note that $\mathcal{O}_1 = \mathbb{C}\{x\}$ is a UFD: given $g \in \mathcal{O}_1$, we have a unique decomposition $g(x) = x^{\nu_0(f)}h(x)$, where h is a unit (convergent power series with $h(0) \neq 0$) and $\nu_0(f) \in \mathbb{Z}$. The irreducibles in this case are just the factors of x.

By the Gauss lemma, it follows that $\mathbb{C}\{x\}[y]$ is a UFD.

Next, suppose that $f(x,y) = \sum_{a,b} \alpha_{ab} x^a y^b \in \mathcal{O}_2$ vanishes identically on the y-axis; that is, $0 \equiv f(0,y) = \sum_b \alpha_{0b} y^b$. It follows that all $\alpha_{0b} = 0$ for all b, so that $f = x^\nu f_0$ where $\nu > 0$ and $f_0(0,y) \not\equiv 0$. We must prove unique factorization for f_0.

Let $f \in \mathcal{O}_2$ with $f(0,y) \not\equiv 0$. Lemmas 10.1.4 and 10.2.1 give $f = uw$ uniquely. Since w belongs to the UFD $\mathbb{C}\{x\}[y]$, we have a unique decomposition $w = h_1 \cdot \cdots \cdot h_\ell$ into irreducibles $h_j \in \mathbb{C}\{x\}[y]$. Clearly also $h_j(0,y) \not\equiv 0$, and so Lemma 10.1.4 applied to each h_j gives uniquely $h_j = u_j w_j$, with each w_j a Weierstrass polynomial irreducible in $\mathbb{C}\{x\}[y]$ (since h_j is). This yields $w = (u_1 w_1) \cdot \cdots \cdot (u_\ell w_\ell) = (u_1 \cdots u_\ell) w_1 \cdots w_\ell =: \tilde{u}\tilde{w}$, and by Lemma 10.2.1 $\tilde{u}$ must be 1. So far we have $f = u w_1 \cdots w_\ell$.

We do not know yet whether w_j is irreducible in $\mathcal{O}_2$. If $w_j = v'v''$ ($v', v'' \in \mathcal{O}_2$), then $w_j(0,y) \not\equiv 0 \implies$ the same thing for v', v''. Lemma 10.1.4 applies to yield $v' = u'w'$ and $v'' = u''w''$, so that $w_j = (u'u'')(w'w'')$; applying Lemma 10.2.1 yet again gives $u'u'' = 1 \implies w_j = w'w''$. But $w', w'' \in \mathfrak{W} \subset \mathbb{C}\{x\}[y]$, contradicting irreducibility of w_j in $\mathbb{C}\{x\}[y]$.

To see uniqueness, write factorizations $f = f_1 \cdots f_\ell = g_1 \cdots g_k$ into irreducibles in $\mathcal{O}_2$; we may assume $f(0,y) \not\equiv 0$. Then Lemma 10.1.4 gives $f_j = u_j w_j$ and $g_i = \tilde{u}_i \tilde{w}_i$ with $w_j, \tilde{w}_i$ irreducible Weierstrass polynomials. We then have

$$(u_1 \cdots u_\ell)(w_1 \cdots w_\ell) = (\tilde{u}_1 \cdots \tilde{u}_k)(\tilde{w}_1 \cdots \tilde{w}_k),$$

so that by Lemma 10.2.1 $u_1 \cdots u_\ell = \tilde{u}_1 \cdots \tilde{u}_k$ and $w_1 \cdots w_\ell = \tilde{w}_1 \cdots \tilde{w}_k$. By uniqueness of factorization in $\mathbb{C}\{x\}[y]$ (and Lemma 10.2.1), the $\{w_j\}$ and $\{\tilde{w}_i\}$ are the same (up to reordering), and $\ell = k$. □

Note the key statement that comes out of this proof: given $f \in \mathcal{O}_2$ with $f(0,y) \not\equiv 0$, we have

$$f = u w_1 \cdots w_\ell, \tag{10.2.3}$$

where $u \in \mathfrak{U}$ and w_i are Weierstrass polynomials which are irreducible (as Weierstrass polynomials, as elements of $\mathbb{C}\{x\}[y]$, and as elements of $\mathcal{O}_2$). Moreover, this decomposition is completely unique, up to reordering of the w_i. Finally – this also comes out of the proof – if f was a Weierstrass polynomial, then $u = 1$ in (10.2.3), and $\deg_y(f) = \sum_{i=1}^{\ell} \deg_y(w_i)$. This will be useful in working the following problems.

Exercises

(1) Show $f(x,y) = x^3 - x^2 + y^2$ is (a) irreducible in $\mathbb{C}[x,y]$ and (b) reducible in $\mathbb{C}\{x\}[y]$.
(2) Show $g(x,y) = x^3 - y^2$ is irreducible in $\mathbb{C}\{x\}[y]$.
(3) What form does the factorization in $\mathbb{C}\{x\}[y]$ of the Weierstrass polynomial $f = y^4 + xy + x^4$ take? (You don't have to compute the power series!) How does this correspond to the *Newton polygon* connecting the (a,b) which appear as exponents (viz., $x^a y^b$) in monomial terms of f?[5]

[5]More precisely, the Newton polygon is the boundary of the convex hull of the sets $(a,b) + \mathbb{R}^2_{\geq 0}$.

(4) By considering a few more examples, e.g. $y^4 + xy^2 + x^5$, $x^5 + y^5$, and $y^8 + xy^4 + x^2y^2 + x^4y + x^8$, can you formulate a conjecture connecting the Newton polygon to the factorization into irreducibles in $\mathbb{C}\{x\}[y]$?

(5) The proof of Lemma 10.1.3 used the fact that the elementary symmetric polynomials e_k in $\{y_i\}_{i=1}^m$ can be expressed as polynomials in the Newton symmetric polynomials $\sigma_k = \sum_{i=1}^m y_i^k$. Prove this by establishing that $-ke_k = \sum_{i=1}^k (-1)^i e_{k-i} p_i$. [Hint: substitute $x = y_j$ in $\prod_{\ell=1}^m (x - y_\ell) = \sum_{i=0}^m (-1)^{k-i} e_{k-i} x^i$, then sum over j.]

(6) Adapt the proof of Proposition 8.2.7 to show that *any* (closed) *complex analytic curve* $\mathcal{C} \subset \mathbb{P}^2$ (i.e., a subset which in a neighborhood of any point is cut out by the vanishing of a nonconstant holomorphic function) *is in fact algebraic* (cut out by a homogeneous polynomial). [Suggestion: by applying a projectivity, you may assume that [0:1:0] is not in $\mathcal{C}$. Note that the intersection of $\mathcal{C}$ with any vertical line $x = x_0$ is finite; in a neighborhood of each intersection point (x_0, y_0), $\mathcal{C}$ can be described by a Weierstrass polynomial in $y - y_0$. Multiply these together to get an element of $\mathbb{C}\{x - x_0\}[y]$, monic in y, cutting out $\mathcal{C}$ for $|x - x_0| < \rho$ (and all y). Argue that these local elements patch together to give an element "$\prod_{\lambda=1}^m (y - y_\lambda(x))$" in $\mathcal{O}(\mathbb{C})[y]$, and then show that $\mathcal{O}(\mathbb{C})$ can be replaced by $\mathbb{C}[x]$.]