CHAPTER 11

Proof of the normalization theorem

The purpose of this chapter is twofold: to find a method for explicitly parametrizing neighborhoods of singular points on algebraic curves; and, using this, to completely prove part (A) of Theorem 3.2.1. In fact, we shall prove a stronger result which contains a uniqueness statement:

11.0.1. THEOREM. *Let $C \subset \mathbb{P}^2$ be an irreducible algebraic curve, with $\mathcal{S} = sing(C)$ its set of singular points. Then there exists a Riemann surface $\tilde{C}$ and morphism (of complex manifolds) $\sigma : \tilde{C} \to \mathbb{P}^2$ such that*

(a) $\sigma(\tilde{C}) = C$
(b) $\#\{\sigma^{-1}(\mathcal{S})\} < \infty$
(c) $\sigma : (\tilde{C} \backslash \sigma^{-1}(\mathcal{S})) \to (C \backslash \mathcal{S}) =: C^*$

is injective (hence an isomorphism).

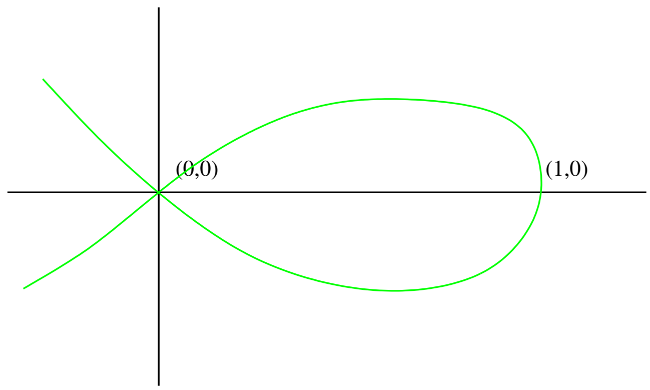
The pair $(\tilde{C}, \sigma)$ is called the normalization *of C, and is unique in the sense that if $(\tilde{C}', \sigma')$ is another, then there exists a morphism $\tau : \tilde{C} \xrightarrow{\cong} \tilde{C}'$ such that $\sigma = \sigma' \circ \tau$.*

We remark that in the correspondence (cf. §9.3) between ideals $I \subset \mathbb{C}[x, y]$, varieties $V = V(I)$, and rings $\mathbb{C}[V] = \frac{\mathbb{C}[x,y]}{I}$, "normalization" means taking the *integral closure of* $\mathbb{C}[V]$ *in* $\mathbb{C}(V)$. Taking "Spec" of the result produces an affine variety $\tilde{V}$ with a morphism to V. This procedure may be carried out for projective varieties by patching affine ones together, and if this is done for curves ($V = C$), then $\tilde{V}$ is really just $\tilde{C}$ constructed algebraically. While this is beyond the scope of our course, it's instructive to look at an example.

11.0.2. EXAMPLE. If we take $V = \{x^3 - x^2 + y^2 = 0\} \subset \mathbb{C}^2$, then the coordinate ring

$$\mathbb{C}[V] = \frac{\mathbb{C}[x,y]}{(x^3 - x^2 + y^2)}$$

is not integrally closed in its fraction field $\mathbb{C}(V)$. That is, the equation $\xi^2 + (x-1) = 0$, while irreducible in $\mathbb{C}[V][\xi]$, is "solved" by $\xi = \frac{y}{x}$, as $\frac{y^2}{x^2} \equiv \frac{x^2 - x^3}{x^2} = 1 - x$ in $\mathbb{C}(V)$. A schematic picture of the irreducible cubic curve V is



and $\frac{y}{x}$ can be viewed as "separating the branches" of V at the singular point $(0,0)$. The sense in which adjoining $\frac{y}{x}$ to $\mathbb{C}[V]$ produces its integral closure and normalizes V is considered in Exercise (3).

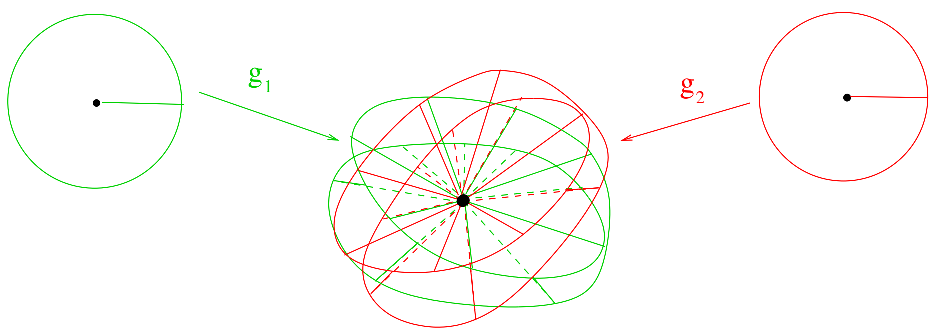
There is a closely related concept known as *blowing up* (already briefly mentioned in Example 7.3.4), which (algebro-)geometrizes the adjunction of elements of the fraction field. In this case, we need to blow up $\mathbb{C}^2$ at the origin, which replaces $(0,0)$ by a $\mathbb{P}^1$ with coordinate "$\frac{y}{x}$" (parametrizing slopes of lines through the origin). More precisely, we take two copies of $\mathbb{C}^2$ — namely, U_0 with coordinates (x,u), and U_1 with coordinates (v,y) — and glue them by the map sending $\{u \neq 0\} \subset U_0$ to $\{v \neq 0\} \subset U_1$ via $(x,u) \overset{\phi}{\mapsto} (u^{-1}, xu)$. The resulting complex manifold $\widetilde{\mathbb{C}^2} = U_0 \underset{\phi}{\cup} U_1$ maps onto $\mathbb{C}^2$ by sending $(x,u) \mapsto (x, xu)$ and $(v,y) \mapsto (vy, y)$. Exercises (4)-(6) explore the use of blowups to normalize singular curves, starting with the above curve V.

Returning to our main subject, in Exercise (1) of Chapter 10 you were asked to carry out the (local) analytic approach for the last example, proving that $x^3 - x^2 + y^2$ is irreducible in $\mathbb{C}[x,y]$ but *reducible* in $\mathbb{C}\{x\}[y]$. Here is another such example.

11.0.3. EXAMPLE. Consider the equation $y^4 + x^3 - x^2 (= 0)$, which is irreducible in $\mathbb{C}[x,y]$ but reducible in $\mathbb{C}\{x\}[y]$, into the product of Weierstrass polynomials

$$(y^2 - x\sqrt{1-x})(y^2 + x\sqrt{1-x}).$$

Here $x\sqrt{1-x}$ is regarded as a convergent power series (in $\mathbb{C}\{x\}$) vanishing at $x = 0$. The local picture (near $(0,0)$) described by this factorization is of two "parking lots" (topologically, these are just disks) attached at their centers:



We need a general procedure that produces the indicated holomorphic parametrizations of these two branches.

11.1. Overview

Informally, here is the main idea of the proof of Theorem 11.0.1. Given an irreducible algebraic curve C with singular point p, we may use a $\mathrm{PGL}(3,\mathbb{C})$-transformation (i.e. a projectivity) of $\mathbb{P}^2$ to move $p \mapsto [1:0:0]$. By another linear transformation of coordinates (cf. §8.2), we can put the affine equation in the form

$$f = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)\,(=0)\,, \quad a_j(x) \in \mathbb{C}[x]. \tag{11.1.1}$$

Now we want to normalize a neighborhood of the singularity $(0,0)$. Since f is irreducible in $\mathbb{C}[x][y]$, its discriminant $\mathcal{D}(f)(x)$ is not identically zero in $\mathbb{C}[x]$. Hence the "local factorization" $f = f_1^{m_1} \cdots f_\ell^{m_\ell}$ into irreducibles in $\mathbb{C}\{x\}[y]$ will have no repeated factors (all $m_i = 1$). Writing $\Delta = \{|x| < \rho\}$ and $W_\Delta = \Delta \times \{|y| < \epsilon\}$ for sufficiently small $\rho, \epsilon > 0$, this corresponds to the decomposition of $C \cap W_\Delta$ into $C_1^\Delta \cup \cdots \cup C_\ell^\Delta$, where each C_i^Δ is homeomorphic to a disk and the union attaches them only at their centers.)

More precisely, writing

$$f = uw_1 \cdots w_\ell$$

as in §, the C_i^Δ are the zero-loci $\{w_i = 0\}$ of irreducible Weierstrass polynomials. If we can write down 1-to-1 holomorphic maps $\tilde{\varphi}_i : \tilde{\Delta} \to \mathbb{C}^2$ ($\tilde{\Delta}$ is some other disk related to Δ) with image $\tilde{\varphi}_i(\tilde{\Delta}) = C_i^\Delta$, and repeat this procedure over all singular points, then the normalization $\tilde{C}$ can be constructed as follows. On $C^* = C \backslash \text{sing}(C)$, we have a covering by holomorphic parametrizations $\varphi_\alpha = z_\alpha^{-1}$ (from §7.2). Composing $\tilde{\varphi}_i$ with z_α whenever $C_i^\Delta \cap U_\alpha$ is nonempty yields holomorphic transition functions. Thinking of C^* as an abstract complex 1-manifold, these transition functions indicate how to attach each C_i^Δ to C^* to yield a new complex 1-manifold $\tilde{C}$. To obtain C (topologically) from this, we just reattach the centers of the C_i^Δ.

The first step indicated in this outline, which we do not yet know how to do, was the construction of the $\{\tilde{\varphi}_i\}$. We shall now do this.

11.2. Irreducible local normalization

Let $w = y^k + b_1(x)y^{k-1} + \cdots + b_k(x)$ be a Weierstrass polynomial, irreducible in $\mathbb{C}\{x\}[y]$. Unless $k = 1$, the discriminant $(\mathcal{D}(w))(x)$ has a zero at $x = 0$. Since $\mathcal{D}(w)$ is not identically zero, this zero is isolated, and we can take ρ small enough that $x = 0$ is its only zero on $\Delta = \{|x| < \rho\}$.

Now, there is a factorization $w = \prod_{\nu=1}^k (y - y_\nu(x))$ which is valid in the sense of §8.2, but *not* in $\mathbb{C}\{x\}[y]$. Namely, the $\{y_\nu(x)\}$ are "multivalued" on Δ,[1] but become well-defined on Δ minus a slit.

[1] except at 0, since all $y_\nu(0) = 0$

(Another, more algebraic, way to think of this factorization, if $0 < |x_0| < \rho$, is as taking place in $\mathbb{C}\{x - x_0\}[y]$.) The multivaluedness is manifested as follows: by the heredity principle, going once counterclockwise around the origin in Δ^*, permutes the roots of w by $y_\nu(x) \mapsto y_{\tau(\nu)}(x)$ where $\tau \in \mathfrak{S}_k$(=the symmetric group on k elements). This permutation must be transitive, i.e. a k-cycle: otherwise, it splits into a product of (smaller) cycles, each of which gives rise to an irreducible proper factor of w in $\mathbb{C}\{x\}[y]$, in contradiction to its irreducibility.

Here, then, is how to parametrize the set $\{w = 0\} \subset W_\Delta$:

11.2.1. PROPOSITION. *Let $w \in \mathfrak{W}$ be irreducible of degree k, and pick any $\nu \in \{1, \ldots, k\}$. Then writing $\tilde{\Delta} := \{t \in \mathbb{C} \mid |t| < \rho^{\frac{1}{k}}\}$,*

$$g : \tilde{\Delta} \to \mathbb{C}^2$$

$$t \mapsto (t^k, \tilde{y}_\nu(t^k))$$

is well-defined and injective,[2] *with image the local analytic curve*

$$C^\Delta := \{(x, y) \mid w(x, y) = 0,\ |x| < \rho,\ |y| < \epsilon\},$$

and gives a biholomorphism (of complex 1-manifolds)

$$\tilde{\Delta} \backslash \{0\} \overset{\cong}{\to} C^\Delta \backslash \{(0, 0)\}.$$

11.2.2. REMARK. Here $C^\Delta \backslash \{(0, 0)\}$ is a complex 1-manifold by the holomorphic implicit function theorem as in §7.2, and is covered by neighborhoods with local holomorphic coordinate x. One can regard the last biholomorphism as giving the transition function between $(\tilde{\Delta}, t)$ and (more generally) *any* open set in C^* with holomorphic coordinate x.

PROOF OF PROP. 11.2.1. Recall that $y_\nu(x)$ is well-defined on the slit disk $\Delta^- := \Delta \backslash \{x \in \mathbb{R}_{\geq 0}\}$. Analytic continuation of $y_\nu(x)$ once counterclockwise around $x = 0$ yields $y_{\tau(\nu)}(x)$; going around once more gives $y_{\tau^2(\nu)}(x)$, and so on. Since τ is a k-cycle, $\tau^k(\nu) = \nu$ and *going around zero k times* returns us to $y_\nu(x)$. But t^k does *precisely this*

[2]The meaning of "$\tilde{y}_\nu(t^k)$" will be defined in proof.

when t goes around 0 once, and so $y_\nu(t^k)$ extends to a well-defined analytic function on $\tilde{\Delta}$.

A bit more carefully, we subdivide $\tilde{\Delta}^* = \cup_{j=0}^{k-1}\tilde{\Delta}^{(j)}$ into pie-slices $\tilde{\Delta}^{(j)} := \{0 < |t| < \rho^{\frac{1}{k}}$ and $\frac{j}{2\pi} \leq \arg(t) \leq \frac{j+1}{2\pi}\}$. On the interior of each slice (that is, where $\frac{j}{2\pi} < \arg(t) < \frac{j+1}{2\pi}$), we can define a holomorphic function by $y_{\tau^j(\nu)}(t^k)$, since $t \mapsto t^k$ maps this interior (isomorphically) to Δ^- where $y_{\tau^j(\nu)}(x)$ is defined. Extending these functions continuously to $\tilde{\Delta}^{(j)}$, they patch together (in fact, analytically continue into one another) to yield a single holomorphic function $\tilde{y}_\nu(t^k)$ on $\tilde{\Delta}^*$. This is bounded exactly as in §8.2, and so extends to $\mathcal{O}(\tilde{\Delta})$ by the removable singularity theorem.

Let $\zeta_k := e^{\frac{2\pi\sqrt{-1}}{k}}$. If $(t_1^k, \tilde{y}_\nu(t^k)) = (t_2^k, \tilde{y}_\nu(t_2^k))$ then

$$t_2 = (\zeta_k)^\ell t_1$$

for some $\ell \in \mathbb{Z}$, and

$$y_{\tau^\ell(\nu)}(t_1^k) = y_\nu(t_1^k).$$

Since the $\{y_\nu\}$ are all distinct away from 0, the last equation is impossible unless $k|\ell$, which implies $(\zeta_k)^\ell = 1$ so that $t_1 = t_2$. This proves that g is injective.

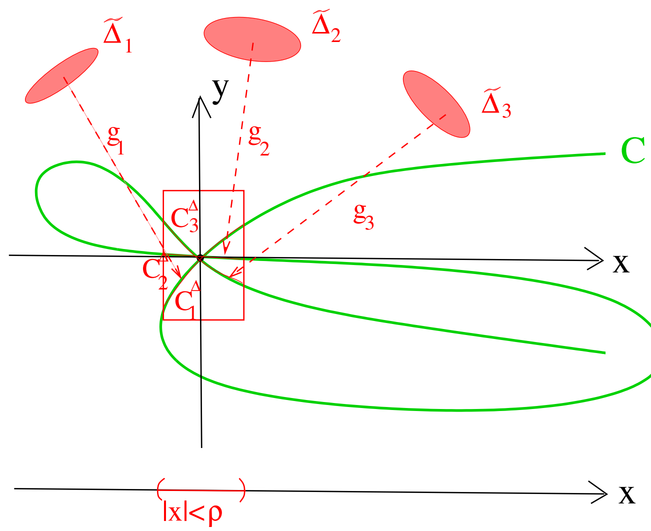
Since τ is transitive, g maps surjectively onto C^Δ. It gives, finally, a holomorphic map of Riemann surfaces on the complement of 0 since in local coordinates t (on $\tilde{\Delta}^*$) and x (for open subsets covering $C^\Delta \backslash \{(0,0)\}$) we have $x =$"$g(t)$'s x-coordinate"$= t^k$. □

11.3. Finishing local normalization

Referring back to §11.1, for each of the irreducible factors w_j of f we now apply Proposition 11.2.1. This yields normalizations

$$g_j : \tilde{\Delta}_j \to C_j^{\Delta}$$

of the irreducible components of the local analytic curve $C \cap W_{\Delta}$:



Each restriction

$$g_j^{\circ} : \tilde{\Delta}_j^* \xrightarrow{\cong} (C_j^{\Delta} \backslash \{(0,0)\}) \hookrightarrow C^*$$

is biholomorphic with respect to local coordinates on $\tilde{\Delta}_j$ and an open covering of $C_j^{\Delta} \backslash \{(0,0)\}$. In fact, it takes the form $t \mapsto t^k (= x)$ as indicated at the end of the last proof. These may be regarded as the "gluing" maps that will attach each $\tilde{\Delta}_j$ to C^* thereby plugging the holes in C^*, which is what we do next.

Before that, we just note that one should carry out the construction of g_j's as we have done near $p = (0,0)$, at all the other singular points of C.

11.4. Global normalization (patching)

Suppose for the moment $(0,0)$ is the *only* singular point of C, so that $C^* = C \backslash \{(0,0)\}$. Then we put

$$\tilde{C} := C^* \underset{g_1^\circ}{\cup} \tilde{\Delta}_1 \underset{g_2^\circ}{\cup} \tilde{\Delta}_2 \cup \cdots \underset{g_\ell^\circ}{\cup} \Delta_\ell \ ,$$

where $C^* \underset{g_1^\circ}{\cup} \tilde{\Delta}_1$ means

$$\frac{C^* \amalg \tilde{\Delta}_1}{g_1^\circ(p) \sim p \ \ (\forall p \in \tilde{\Delta}_1^*)},$$

$C^* \underset{g_1^\circ}{\cup} \tilde{\Delta}_1 \underset{g_2^\circ}{\cup} \tilde{\Delta}_2$ means

$$\frac{C^* \amalg \tilde{\Delta}_1 \amalg \tilde{\Delta}_2}{g_1^\circ(p) \sim p\,,\ g_2^\circ(q) \sim q \ \ (\forall p \in \tilde{\Delta}_1^*, q \in \tilde{\Delta}_2^*)},$$

and so forth.

If there are more singularities, then repeat this patching at each point in $\mathcal{S} = \mathrm{sing}(C)$.

To get a map $\sigma : \tilde{C} \to \mathbb{P}^2$ with image C, set

$$\sigma(c) := \begin{cases} c \ \ , \text{ for } c \in C^* \\ g_j(c), \text{ for } c \in \tilde{\Delta}_j \end{cases} .$$

These two prescriptions are compatible with the patching.

To see that $\tilde{C}$ is compact: given an open cover $\{U_\alpha\}$ of $\tilde{C}$, pick one $U_{\alpha(q)}$ containing each $q \in \sigma^{-1}(\mathcal{S})$. The complement $\tilde{C}'$ of these in $\tilde{C}$ is isomorphic to a closed subset of C, since σ is bijective away from $\sigma^{-1}(\mathcal{S})$. Now a closed subset of C is a closed subset in $\mathbb{P}^2$, $\mathbb{P}^2$ is compact, and a closed subset of a compact set is compact. So $\tilde{C}'$ is compact and $\{U_\alpha \cap \tilde{C}'\}$ has a finite subcover $\{U_i \cap \tilde{C}'\}$. The $\{U_i\}$ together with the $\{U_{\alpha(q)}\}$ then furnish a finite subcover of $\tilde{C}$.

We have now proved all but the uniqueness part of Theorem 11.0.1 and it is time to backtrack and get explicit.

11.5. Examples of local normalization

11.5.1. EXAMPLE. Assuming $\gcd(k, a) = 1$,

$$y^k - x^a$$

is irreducible in $\mathbb{C}\{x\}[y]$, and we shall apply the procedure of Proposition 11.2.1. The k (multivalued) roots of $y^k - x^a = 0$ in y are

$$y_1(x) = \sqrt[k]{x^a}\,,\ y_2(x) = \zeta_k \sqrt[k]{x^a}\,,\ \ldots\,,\ y_k(x) = (\zeta_k)^{k-1}\sqrt[k]{x^a};$$

they are well defined on the slit disk $\{0 < |x| < \rho,\ \arg(x) \in (0, 2\pi)\}$. If we plug t^k into $y_1(x)$ and analytically continue, we get

$$\tilde{y}_1(t^k) = t^a.$$

Hence by definition

$$g(t) = (t^k, t^a).$$

We should check that the image of g lies in $y^k - x^a = 0$: this is just the statement that $(t^k)^a = (t^a)^k$.

11.5.2. EXAMPLE. Here is a more complicated example where there is more than one g_j (as in §11.3):

$$f = y^8 + y^4 - x^6 + x^3 - x^2y^4 + x^5 - x^2.$$

Viewed in $\mathbb{C}\{x\}[y]$, this is not a Weierstrass polynomial (the coefficient $1 - x^2$ of y^4 is not zero at $x = 0$), so we should expect a nontrivial unit u in (10.2.3). Indeed,

$$\begin{aligned} f &= (y^4 - x^3 + 1)(y^4 + x^3 - x^2) \\ &= \underbrace{(y^4 - x^3 + 1)}_{u}\underbrace{(y^2 - x\sqrt{1-x})}_{w_1}\underbrace{(y^2 + x\sqrt{1-x})}_{w_2}, \end{aligned}$$

where u is a unit because $u(0, 0) \neq 0$.

Now w_1, w_2 are irreducible Weierstrass polynomials and so we apply Prop. 11.2.1 (with $k = 2$) to normalize their zero-sets.

Beginning with w_1, the roots are $y_{11}(x) = \sqrt{x\sqrt{1-x}}$ and $y_{12}(x) = -\sqrt{x\sqrt{1-x}}$, which are swapped as x goes around 0. So $\tilde{y}_{11}(t^2)$ is obtained by substituting t^2 for x and analytically continuing: informally, $\sqrt{t^2\sqrt{1-t^2}} = t\sqrt[4]{1-t^2}$. This gives

$$g_1(t) = (t^2, t\sqrt[4]{1-t^2}).$$

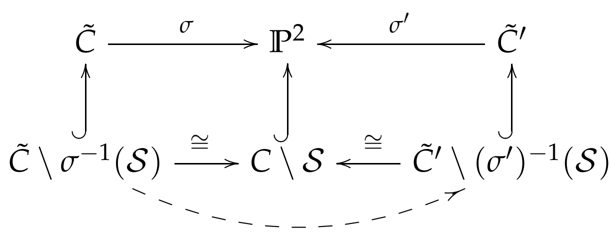
For w_2, the roots are given by $y_{21}(x) = i\sqrt{x\sqrt{1-x}}$ and $y_{22}(x) = -i\sqrt{x\sqrt{1-x}}$; and this yields

$$g_2(t) = (t^2, it\sqrt[4]{1-t^2}).$$

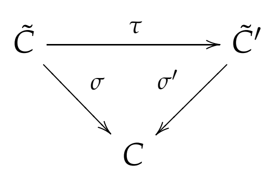
Let's check this parametrizes $w_2 = 0$: one need only write $(y(t))^2 + x(t)\sqrt{1-x(t)} = (it\sqrt[4]{1-t^2})^2 + t^2\sqrt{1-t^2} = 0$.

11.6. Uniqueness

Begin with two normalizations:



with σ, σ' holomorphic maps of complex manifolds satisfying (a)-(c) in Theorem 11.0.1. Essentially what we have to show is that neighborhoods of the points of $\sigma^{-1}(\mathcal{S})$ (in $\tilde{C}$) and $(\sigma')^{-1}(\mathcal{S})$ (in $\tilde{C}'$) are isomorphic in a way which is compatible with σ and σ'. Put together with the bottom dotted arrow[3] these isomorphisms will yield the desired map $\tau : \tilde{C} \to \tilde{C}'$ of Riemann surfaces making the diagram



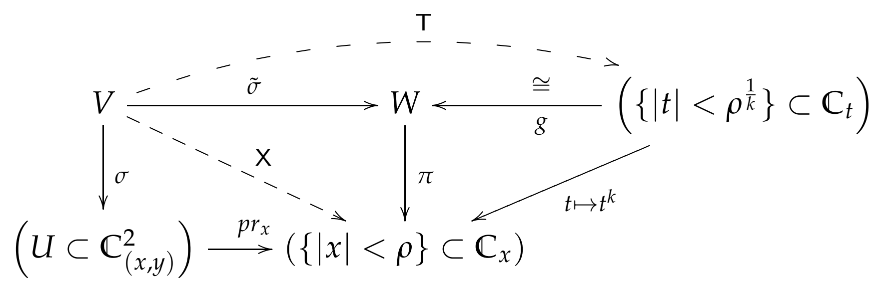
commute.

To start, let $p \in \mathcal{S} \subset C$ be a singular point and $U \subset \mathbb{P}^2$ be a small open set containing it. For simplicity assume $p = [1 : 0 : 0]$ and

[3]i.e., the obvious composition of isomorphisms in the bottom row.

choose coordinates so that $U \subset \{|x| < \rho,\ |y| < \epsilon\}$ and $C \cap U$ is given by the zero set of a Weierstrass polynomial. Write $U^* := U \backslash \{p\}$.

Now pick $q \in \sigma^{-1}(p)$; by continuity of σ, $\sigma^{-1}(U)$ is open in $\tilde{C}$. So there exists an open set V, which we may assume to be connected, with $q \in V \subset \sigma^{-1}(U)$ (and $\sigma^{-1}(\mathcal{S}) \cap V = \{q\}$). Since $\sigma(V \backslash \{q\}) \subset C \cap U^*$ must then be connected, and $C \cap U^*$ is homeomorphic to a disjoint union of punctured disks, σ maps $V \backslash \{q\}$ into one of these punctured disks. Consequently, V is mapped into only one (local) irreducible component[4] W of $C \cap U$. This yields the following diagram:



in which pr_x and σ are morphisms of complex manifolds, so that their composition X is evidently a holomorphic (obviously bounded) function on V.

The composition T is also evidently a bounded, well-defined function on V. By the holomorphic IFT (and holomorphicity of X), it is holomorphic on $V \backslash \{q\}$; hence by the removable singularity theorem, $\mathsf{T} \in \mathcal{O}(V)$. It is also clear that $\mathsf{T}(0) = 0$. So by the open mapping theorem, T maps V onto a neighborhood $\mathcal{N}$ of 0 in $\mathbb{C}_t$ (which we may assume is a disk). Shrinking U (and thus W) if necessary, we may conclude that $\tilde{\sigma}$ — the restriction of σ to a neighborhood of q — maps V onto W in 1-to-1 fashion. From the diagram, this $\tilde{\sigma}$ is just $g \circ \mathsf{T}$.

Since σ is 1-to-1 off $\sigma^{-1}(\mathcal{S})$, no neighborhood of any other point $q_0 \in \sigma^{-1}(\mathcal{S})$ can be sent to W. Repeating the argument above by

[4]Remember that these components are homeomorphic to disks; take out p and that is where the punctured disks came from.

varying q, sets up a 1-to-1 correspondence between "V's" (i.e. neighborhoods of points in $\sigma^{-1}(\mathcal{S})$ in $\tilde{C}$) and "W's" (irreducible local components of C at points of $\mathcal{S}$). We can play the same game for the normalization $\tilde{C}'$, and find that for a unique $q' \in (\sigma')^{-1}(\mathcal{S})$ we have a neighborhood V' and an isomorphism $\mathsf{T}' : V' \to \mathcal{N}$ whose composition with g gives $\tilde{\sigma}' : V' \to W$.

The piece of τ carrying (V, q) to (V', q') is now defined simply by $(\mathsf{T}')^{-1} \circ \mathsf{T}$. This is automatically holomorphic, and its composition with σ' is $g \circ \mathsf{T} = \sigma$ as desired.

Exercises

(1) Locally normalize the zero-set of $f(x,y) = y^4 - (x+1)^7$ at $(-1,0)$.

(2) Locally normalize the zero-set of $g(x,y) = y^4 - x^6 + x^7$ at $(0,0)$.

(3) Example 11.0.2 described the "algebraic normalization" of V by replacing $R = \mathbb{C}[V] = \frac{\mathbb{C}[x,y]}{(x^3-x^2+y^2)}$ by $\hat{R}$, its integral closure in its fraction field $\mathbb{C}(V)$. (This comprises all elements of $\mathbb{C}(V)$ solving monic equations with coefficients in R.). The element $\frac{y}{x} \in \mathbb{C}(V)$ satisfies $(\frac{y}{x})^2 + x - 1 = 0$ and $(\frac{y}{x})^3 - \frac{y}{x} + y = 0$. (a) Show that $S := \frac{R[\xi]}{(\xi^2+x-1, \xi^3-\xi+y)}$ (the result of adjoining $\frac{y}{x}$ to R) is isomorphic to $\mathbb{C}[\xi]$, the coordinate ring of $\mathbb{P}^1$. (b) Show that $\mathbb{C}[\xi]$ is integrally closed (in $\mathbb{C}(\xi)$). Conclude that $\hat{R} = S \cong \mathbb{C}[\xi]$ and interpret this geometrically.

(4) This problem considers the effect of blowing up at the origin on the curve of Example 11.0.2. (a) Check that the preimage of $V = \{x^3 - x^2 + y^2 = 0\}$ in $\widetilde{\mathbb{C}^2}$ is singular. (b) Show that if you throw out the extraneous copy of $\mathbb{P}^1$ (i.e. $\{x = 0\}$ in U_0, $\{y = 0\}$ in U_1), the resulting *proper transform* of V is smooth.

(5) (a) Check that the proper transform of $\{f = 0\}$ from Exercise (1) under blow-up at $(-1,0)$ is not smooth. (b) Show that blowing up one more time (where?) yields a smooth curve. (c) Explain why this corresponds to adjoining $\frac{y}{x+1}$ and $\frac{(x+1)^2}{y}$ to the coordinate ring of the original curve.

(6) Normalize the curve $\{g = 0\}$ of Exercise (2) by blowing up three times. What elements does this correspond to adjoining to the coordinate ring? Try to sketch what happens (under proper transform) to the curve at each stage.