CHAPTER 12

Intersections of curves

Now we come to the applications of normalization, which will occupy this chapter and Chapters 14-15. You may recall that in Chapter 2 we studied intersections of an plane algebraic curve C with a (projective) line L. The points of $L \cap C$ were each assigned a multiplicity by restricting the equation of C under a parametrization of L, and looking at the multplicities of the roots of the resulting one-variable polynomial. With this definition, the multiplicities added up to the degree of the curve (cf. Prop. 2.1.15).

If we had tried to replace L by an arbitrary curve E at that point, we would have run into the problem of no longer knowing how to locally parametrize E near the intersection points. Now that we can do this (Prop. 11.2.1), we can pull the defining equation of C back under the parametrization and look at its order of vanishing at the intersection point. This leads to the general definition of *intersection multiplicity*, and with this in hand that we can finally state (and prove!) Bézout's theorem in general. In its proof the *intersection divisor* will make an appearance, so we begin with a short bit on divisors.

12.1. Divisors on a Riemann surface

Let M be a Riemann surface. The group of *divisors* on M is the free abelian group on points of M,

$$\mathrm{Div}(M) := \left\{ \sum_{\text{finite}} m_i[p_i] \;\middle|\; m_i \in \mathbb{Z}\,,\ p_i \in M \right\}.$$

The uncountably many symbols $[p_i]$ are the generators of this (very big) abelian group. Associated to a divisor $D = \sum m_i[p_i] \in \mathrm{Div}(M)$

is a *degree*

$$\deg(D) := \sum m_i.$$

The resulting group homomorphism

$$\text{(12.1.1)} \qquad \mathrm{Div}(M) \xrightarrow{\deg} \mathbb{Z}.$$

is called the *degree map*.

The divisor of a (nontrivial) meromorphic function f is given by

$$(f) := \sum_{p \in M} \nu_p(f).[p] \ \in \ \mathrm{Div}(M),$$

where $\nu_p(f)$ is the order of f at p (Defn. 3.1.5). Note that the sum is actually finite (as required by the definition of divisor) since at all but finitely many points of M, $\nu_p(f) = 0$. Now $\mathcal{K}(M)^*$ is a multiplicative abelian group. Sending $f \to (f)$ yields a homomorphism

$$\text{(12.1.2)} \qquad \mathcal{K}(M)^* \xrightarrow{(\cdot)} \mathrm{Div}(M)$$

of abelian groups, as you will show in an exercise below, which takes multplication to addition: $(fg) = (f) + (g)$, $(f^{-1}) = -(f)$.

With these definitions, the composition of (12.1.2) with (12.1.1) takes f to $\sum_{p \in M} \nu_p(f)$, which by Exercise 2 of Ch. 3 is zero. That is, $\deg \circ (\cdot) = 0$. Note that one can define meromorphic functions and divisors more generally on complex 1-manifolds, but it is only in the compact case (Riemann surfaces) that the divisors of meromorphic functions are always of degree 0.

12.1.3. EXAMPLE. On $\mathbb{P}^1$, the easiest meromorphic function is $z = \frac{Z_1}{Z_0} \in \mathcal{K}(\mathbb{P}^1)^*$. Writing simply $0, \infty$ for the points $[1:0], [0:1]$, its divisor is $(z) = [0] - [\infty]$, obviously of degree 0.

12.2. Intersection multiplicities

For a polynomial $f(x)$ in one variable with $f(0) = 0$, $\deg(f)$ is the exponent of the highest degree term, while the order of vanishing $\mathrm{ord}_0(f) := \nu_0(f)$ is the exponent of the term of lowest degree. Order (unlike degree) also makes sense for power series in 1 variable.

How does all this generalize to two variables? First, a polynomial $F(x,y)$ can be written as a sum of homogeneous terms. If this is $F = F_k + F_{k+1} + \cdots + F_{d-1} + F_d$, then $\deg(F) := d$ (highest homogeneous degree) while $\mathrm{ord}_{(0,0)}(F) := k$ (lowest homogeneous degree). From §6.4, k is also the order of singularity of the curve $C = \{F = 0\}$ at $(0,0)$, i.e. the number of tangent lines to C counted with multiplicity. When we don't want to refer to the polynomial, we will write $\mathrm{ord}_{(0,0)}C$; remember this is 1 when C is smooth at $(0,0)$, 2 when C has an ODP (normal crossing) there, and so on. Finally, $\mathrm{ord}_{(0,0)}$ also makes sense for 2-variable power series.

Now suppose $V = \{f(x,y) = 0\}$, $W = \{h(x,y) = 0\}$ are reduced affine algebraic curves that intersect properly — i.e. have no common irreducible components. Then $V \cup W$ has no repeated components, so is itself reduced. For $p \in V \cap W$,

$$\left(\frac{\partial}{\partial x}(fh)\right)(p) = f_x(p)h(p) + h_x(p)f(p) = f_x(p).0 + h_x(p).0 = 0$$

and similarly $\left(\frac{\partial}{\partial y}(fh)\right)(p) = 0$. Therefore $V \cap W \subset \mathrm{sing}(V \cup W)$, and Prop. 8.1.10 yields

$$\#\{V \cap W\} \leq \#\{\mathrm{sing}(V \cup W)\} < \infty. \tag{12.2.1}$$

12.2.2. DEFINITION. Assume V and W are irreducible (and distinct), and let $p \in V \cap W$. Let $U \subset \mathbb{C}^2$ be a neighborhood of p. Writing the *local* decomposition of V into irreducibles (uniquely)

$$V \cap U = V_1^\Delta + \cdots + V_k^\Delta,$$

with local normalizations (again, essentially unique)

$$g_i : \Delta \to V_i^\Delta \ (\subset \mathbb{C}^2)$$

$$t_i \mapsto (x_i(t), y_i(t)),$$

we define the *(local) intersection multiplicity at p*

$$(V \cdot W)_p := \sum_{i=1}^{k} \mathrm{ord}_0(h(g_i(t))).$$

The *(global) intersection number* is then defined by

$$(V \cdot W) := \sum_{p \in V \cap W} (V \cdot W)_p,$$

in which the sum is finite by (12.2.1).

12.2.3. REMARK. (a) If either V or W is smooth, the intersection number is actually the degree of a divisor,

$$V \cdot W := \sum_{p \in V \cap W} (V \cdot W)_p [p].$$

This is because we can regard the smooth one (say, W) as a Riemann surface and then $V \cdot W \in \mathrm{Div}(W)$. Alternatively, you can think of $V \cdot W$ as a formal sum of points of $\mathbb{P}^2$, known as a *zero-cycle*[1] on $\mathbb{P}^2$. The degree is defined in the same way as for divisors.

(b) The composition $h \circ g_i$ appearing in Defn. 12.2.2 will frequently be written $g_i^*(h)$ – that is, we are pulling the function h back by the local normalization g_i^*.

The local intersection multiplicities are well-defined essentially by the uniqueness of local normalizations. They also have some reasonable properties:

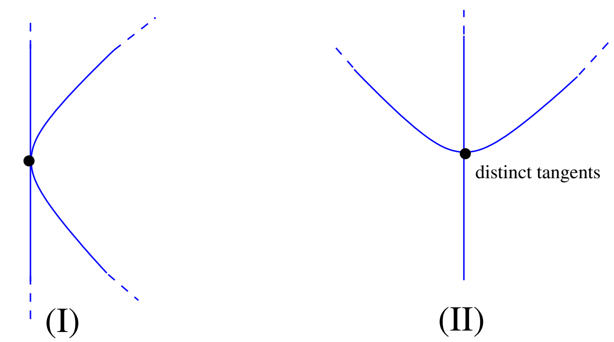
12.2.4. PROPOSITION. $(V \cdot W)_p = (W \cdot V)_p$.

12.2.5. PROPOSITION. *$(V \cdot W)_p \geq ord_p(V) \cdot ord_p(W)$, with equality precisely when none of V's tangents at p coincide with the tangents of W at p.*

We will postpone proof of these results to §§12.4-12.5, since the details are a bit tedious.

[1]Here "zero" refers to the fact that we are taking a formal sum of zero-dimensional subvarieties (i.e. points) in $\mathbb{P}^2$.

12.2.6. EXAMPLE. Here are two pictures of smooth curves meeting at a point p:



In each case, $\mathrm{ord}_p V \cdot \mathrm{ord}_p W = 1$ because the curves are smooth. But in the first case, $(V \cdot W) = 2$, while in the second (which has distinct tangents) $(V \cdot W) = 1$.

12.2.7. EXAMPLE. Let $a, b, m, n \in \mathbb{N}$ with

$$\gcd(n, a) = \gcd(m, b) = 1.$$

Then by Prop. 12.2.5, we should have[2]

$$\Big(\{y^n = x^a\} \cdot \{y^m = x^b\}\Big)_{(0,0)} \geq \min(n, a) \cdot \min(m, b).$$

Let's check this by actually computing the left-hand side. The normalization of $\{y^n = x^a\}$ is just $t \overset{g}{\longmapsto} (t^n, t^a)$ by Example 11.5.1. Writing $h = y^m - x^b$, we have

$$g^*(h) = g^*(y^m - x^b) = (t^a)^m - (t^n)^b = t^{am} - t^{bn}$$

and the order of this at $(0, 0)$ is the least of am and bn:

$$\Big(\{y^n = x^a\} \cdot \{y^m = x^b\}\Big)_{(0,0)} = \min(am, bn).$$

This clearly satisfies the inequality, and it is easy to cook up an example where equality doesn't hold: with $n = 3$, $a = 4$, $m = 2$, $b = 9$ it becomes $8 \geq 6$.

[2]For instance, the polynomial $y^n - x^a$ has order given by the smallest of n and a.

To extend $(V \cdot W)_p$ to the more general setting where $V = \sum m_j V_j$ and $W = \sum n_k W_k$ with $\{V_j\}$ and $\{W_k\}$ irreducibles, we simply put

$$(V \cdot W)_p := \sum_{j,k} m_j n_k (V_j \cdot W_k)_p.$$

12.2.8. REMARK. Here are two other approaches to local intersection multiplicity which give the same numbers.

(a) The commutative algebra approach makes use of localization. Recall that $\mathbb{C}(x, y)$ denotes the fraction field of $\mathbb{C}[x, y]$. Let $p = (a, b) \in \mathbb{C}^2$. The local ring at p, denoted $\mathcal{O}_p$, is the subset of $\mathbb{C}(x, y)$ consisting of rational functions $\frac{G_1}{G_2}$ (here $G_1, G_2 \in \mathbb{C}[x, y]$) with $G_2(p) \neq 0$. You can easily check that this is a ring, and it obviously contains $\mathbb{C}[x, y]$. It has a unique maximal ideal $\mathfrak{m}_p$ consisting of functions which vanish at p.

Now let $V = \{f = 0\}$, $W = \{h = 0\}$ be as above, and assume $p \in V \cap W$. Writing $(f, h)_p$ for the ideal in $\mathcal{O}_p$ generated by f and h, we define

$$(V \cdot W)_p := \dim_{\mathbb{C}} \left(\mathcal{O}_p / (f, h)_p \right)$$

by viewing the quotient $\mathcal{O}_p / (f, h)_p$ as a vector space. (Note that from this definition, invariance of $(V \cdot W)_p$ under projectivities is immediately clear.) As a simple example, we know that the intersection multiplicity at $p = (0, 0)$ of $\{x = 0\}$ and $\{y^2 - x = 0\}$ should be 2. The quotient vector space, indeed, has basis $1, y$. See Chapter 4 of [L. Flatto, Poncelet's Theorem] for more on this approach.

(b) For an approach via resultants, it is convenient to work with homogeneous polynomials. Write $\bar{V} = \{F = 0\}$, $\bar{W} = \{H = 0\}$, $P = [P_0 : P_1 : P_2] \in \bar{V} \cap \bar{W}$ (in homogeneous coordinates $[Z : X : Y]$ on $\mathbb{P}^2$). Assume that $[0 : 0 : 1]$ neither belongs to (i) $C \cup D$, nor (ii) any line joining points of $C \cap D$, nor (iii) any line tangent to C or D at a point of $C \cap D$. Then we may define

$$(\bar{V} \cdot \bar{W})_P := \mathrm{ord}_{[P_0:P_1]}(\mathcal{R}_Y(F, H)).$$

Here we are thinking of F, H as elements of $\mathbb{C}[Z,X][Y]$; so the resultant $\mathcal{R}_Y(F,H)$, which eliminates Y,[3] is a polynomial in Z and X. It is in fact homogeneous and of degree $\deg(F)\cdot\deg(H)$. Its order at $[P_0 : P_1]$ is just the highest power of $(P_0X - P_1Z)$ dividing it.

Justifying this definition takes a bit of work, but it leads immediately to a proof of Bezout since the intersection multiplicities have to add up to $\deg\mathcal{R}_Y(F,H) = \deg\bar{V}\cdot\deg\bar{W}$ by construction. This is the point of view taken in [F. Kirwan, Complex Algebraic Curves].

12.3. Bézout's theorem

We first do a quick recap of Prop. 2.1.15:

12.3.1. PROPOSITION. *Let $C = \{F(Z,X,Y) = 0\} \subset \mathbb{P}^2$ be a degree d curve, $L\,(\cong \mathbb{P}^1) \subset \mathbb{P}^2$ a line not contained in C. Then $(L\cdot C) = d$.*

PROOF. By a change of coordinates, we may assume $L = \{Y = 0\}$ and $[0:1:0] \notin C$. Then by the Fundamental Theorem of Algebra,

$$F(Z,X,0) = \prod_{i=1}^{k}(X - \alpha_i Z)^{d_i},$$

where $\sum_{i=1}^{k} d_i = d$ since F is homogeneous of degree d. Hence $C\cap L = \{[1:\alpha_i:0]\}_{i=1}^{k}$.

Passing to affine coordiantes ($f = \prod(x-\alpha_i)^{d_i}$) and locally normalizing L at $(\alpha_i, 0)$ by $t \overset{g_i}{\longmapsto} \alpha_i + t$, we have

$$(L\cdot C)_{(\alpha_i,0)} := \mathrm{ord}_0(g_i^* f) = d_i.$$

We conclude that $(L\cdot C) = \sum d_i = d$. □

12.3.2. THEOREM. [E. BÉZOUT, 1779] *Let $C, E \subset \mathbb{P}^2$ be properly intersecting projective algebraic curves. Then $(C\cdot E) = \deg C\cdot\deg E$.*

PROOF. Assume C is irreducible. Let $k = \deg E$, and choose lines $L_1,\ldots,L_k$ avoiding the points of $C\cap E$. Write $E = \{H(Z,X,Y) = 0\}$,

[3]As usual you can think about this resultant in terms of a projection onto the x- (or rather, $[Z:X]$-) axis.

$L_j = \{\Lambda_j(Z, X, Y) = 0\}$. Then by Propositions 12.2.4 and 12.3.1,

$$(C \cdot L_j) = (L_j \cdot C) = \deg C,$$

and

$$(C \cdot (\cup_{j=1}^k L_j)) = \sum_{j=1}^{k} (C \cdot L_j) = \deg C \cdot \deg E.$$

Now by Example 7.3.5, the quotient of two homogeneous polynomials *of the same degree* gives a meromorphic function on projective space. H is of degree k and each Λ_j is of degree 1, so we may define

$$\varphi := \frac{H}{\Lambda_1 \cdot \cdots \cdot \Lambda_k} \in \mathcal{K}(\mathbb{P}^2).$$

Writing $\sigma : \tilde{C} \to \mathbb{P}^2$ (with $\sigma(\tilde{C}) = C$) for the normalization, we have by Example 7.3.6 $\sigma^*\varphi \in \mathcal{K}(\tilde{C})$. We can compute the divisor of this meromorphic function if we notice that locally about each point of $C \cap E$ [resp. $C \cap (\cup L_j)$], φ [resp. $\frac{1}{\varphi}$] gives a defining equation for E [resp. $\cup L_j$]. So by Defn. 12.2.2,

$$\begin{aligned}(\sigma^*\varphi) &= \sum_{p \in C \cap E} \nu_p(\sigma^*\varphi)[p] \quad + \sum_{q \in C \cap (\cup L_j)} \nu_q(\sigma^*\varphi)[q] \\ &= \sum_{p \in C \cap E} \mathrm{ord}_p(\sigma^*\varphi)[p] \quad - \sum_{q \in C \cap (\cup L_j)} \mathrm{ord}_q(\sigma^* \tfrac{1}{\varphi})[q] \\ &= \sum_{p \in C \cap E} (C \cdot E)_p[p] \quad - \sum_{q \in C \cap (\cup L_j)} (C \cdot (\cup L_j))_q[q].\end{aligned}$$

But as divisors of meromorphic functions on Riemann surfaces have degree 0,

$$\begin{aligned}0 = \deg((\sigma^*\varphi)) &= \sum_{p \in C \cap E} (C \cdot E)_p \quad - \sum_{q \in C \cap (\cup L_j)} (C \cdot (\cup L_j))_q \\ &= (C \cdot E) - \deg C \cdot \deg E.\end{aligned}$$

Finally, if C is reducible, break it into irreducible components and sum the results! □

12.3.3. REMARK. In terms of zero-cycles (cf. Remark 12.2.3(a)), Bézout is saying that $C \cdot E$ has degree $\deg C \cdot \deg E$.

12.4. Proof of Proposition 12.2.4

We now show the symmetry of intersection numbers. Write $V = \{f = 0\}$, $W = \{h = 0\}$, $p \in V \cap W$. For simplicity assume that $p = (0,0)$, V and W are irreducible, and the defining (polynomial) equations are in the form

$$f = y^m + B_1(x)y^{m-1} + \cdots + B_m(x)\ , \quad h = y^n + b_1(x)y^{n-1} + \cdots + b_n(x).$$

We decompose these according to (10.2.3): viz.,

$$f = u_1 \cdot v_1 \cdots v_r\ , \quad h = u_2 \cdot w_1 \cdots w_s$$

where the v_j, w_k are irreducible Weierstrass polynomials. For the roots of v_j [resp. w_k] on a slit disk $\{|x| < \rho,\ x \notin \mathbb{R}_{>0}\}$ we shall write $y_\mu^{(j)}(x)$ ($\mu = 1, \ldots, m_j$) [resp. $z_\nu^{(k)}(x)$ ($\nu = 1, \ldots, n_k$)]. On the non-slit x-disk these become multivalued, and we will assume that counterclockwise analytic continuation sends $y_\mu \mapsto y_{\mu+1}$ to keep the numbering simple. As in §11.2, the $\tilde{y}_\mu^{(j)}(t^{m_j})$ [resp. $\tilde{z}_\nu^{(k)}(t^{n_k})$] are well-defined on a small t-disk $\{|t| < \rho_0\}$, and we have[4]

$$\tilde{y}_{\mu+\mu_0}^{(j)}(t^{m_j}) = \tilde{y}_\mu^{(j)}((\zeta_{m_j}^{\mu_0} t)^{m_j})$$

for some primitive m_j^{th} root of unity ζ_{m_j}. (This changes the branch you start at when $\arg(t) = 0$.) Parametrize $\{v_j = 0\}$ and $\{w_k = 0\}$ by $g_j(t) := (t^{m_j}, \tilde{y}_\mu^{(j)}(t^{m_j}))$ resp. $G_k(t) := (t^{n_k}, \tilde{z}_\nu^{(k)}(t^{n_k}))$.

We then have the key identity

$$\prod_{\mu_0=1}^{m_j} w_k\left(t^{n_k m_j},\ \tilde{y}_{\mu+\mu_0}^{(j)}(t^{n_k m_j})\right) \tag{12.4.1}$$

$$= \prod_{\mu_0=1}^{m_j} \prod_{\nu_0=1}^{n_k} \left\{\tilde{y}_{\mu+\mu_0}^{(j)}(t^{n_k m_j}) - \tilde{z}_{\nu+\nu_0}^{(k)}(t^{n_k m_j})\right\}$$

$$= \pm \prod_{\nu_0=1}^{n_k} v_j\left(t^{n_k m_j},\ \tilde{z}_{\nu+\nu_0}^{(k)}(t^{n_k m_j})\right), \tag{12.4.2}$$

[4]Warning: you cannot write $(\zeta_{m_j}^{\mu_0} t)^{m_j} = (\zeta_{m_j})^{\mu_0 m_j} t^{m_j} = t^{m_j}$ inside the argument of $\tilde{y}_\mu^{(j)}$, since this assumes $\tilde{y}_\mu^{(j)}$ is well-defined on an entire disk (whereas only its composition with the m_j^{th}-power map is!).

which uses the factorization of each Weierstrass polynomial (at each fixed x) into a product of linear factors. Bearing in mind that rotation of a disk by $2\pi/m_j$ does not change the order of a function at 0, we compute

$$\begin{aligned}\mathrm{ord}_0((12.4.1)) &= n_k \sum_{\mu_0=1}^{m_j} \mathrm{ord}_0\left(w_k(t^{m_j}, \tilde{y}^{(j)}_{\mu+\mu_0}(t^{m_j}))\right)\\ &= n_k m_j \,\mathrm{ord}_0\left(w_k(t^{m_j}, \tilde{y}^{(j)}_{\mu}(t^{m_j}))\right)\\ &= n_k m_j \,\mathrm{ord}_0(g_j^* w_k).\end{aligned}$$

Dividing this by $m_j n_k$ and applying $\sum_{j=1}^{r}\sum_{k=1}^{s}$ gives

$$\sum_j \mathrm{ord}_0\left(g_j^* \prod_k w_k\right) = \sum_j \mathrm{ord}_0(g_j^* h) = (V\cdot W)_p.$$

Similarly

$$\mathrm{ord}_0((12.4.2)) = n_k m_j \,\mathrm{ord}_0(G_k^* v_j),)$$

and dividing out $m_j n_k$ and summing yields $(W\cdot V)_p$. Q.E.D.

12.5. Proof of Proposition 12.2.5

With the same notation as in the last section, we also write out the irreducible Weierstrass polynomials

$$v_j = y^{m_j} + a^{(j)}_{m_j-1}(x) y^{m_j-1} + \cdots + a^{(j)}_0(x).$$

Note that $a_0^{(j)}(x)$ is the product of the multivalued roots $y_\mu^{(j)}(x)$. We have $\mathrm{ord}_{(0,0)} v_j \le m_j$, $\sum_j \mathrm{ord}_{(0,0)} v_j = \mathrm{ord}_{(0,0)} f$, and

$$\begin{aligned}\mathrm{ord}_{(0,0)}(v_j(x,y)) \le \mathrm{ord}_0(a_0^{(j)}(x)) &= \frac{1}{m_j}\mathrm{ord}_0(a_0^{(j)}(t^{m_j}))\\ &= \frac{1}{m_j}\mathrm{ord}_0\left(\prod_{\mu_0=1}^{m_j} \tilde{y}^{(j)}_{\mu+\mu_0}(t^{m_j})\right)\\ &= \mathrm{ord}_0(\tilde{y}^{(j)}_{\mu}(t^{m_j})).\end{aligned}$$

Therefore

$$
\begin{aligned}
(V \cdot W)_p &= \sum_{j=1}^{r} \mathrm{ord}_0 \left(h(t^{m_j}, \tilde{y}_\mu^{(j)}(t^{m_j})) \right) \\
&\geq \sum_{j=1}^{r} (\mathrm{ord}_{(0,0)} h) \cdot \min \left\{ \mathrm{ord}_0(t^{m_j}), \mathrm{ord}_0(\tilde{y}_\mu^{(j)}(t^{m_j})) \right\} \\
&\geq (\mathrm{ord}_{(0,0)} h) \cdot \sum_{j=1}^{r} \min \left\{ \mathrm{ord}_0(t^{m_j}), \mathrm{ord}_{(0,0)}(v_j(x,y)) \right\} \\
&= \mathrm{ord}_{(0,0)} h \cdot \sum_{j=1}^{r} \mathrm{ord}_{(0,0)} v_j \\
&= \mathrm{ord}_{(0,0)} h \cdot \mathrm{ord}_{(0,0)} f \\
&= \mathrm{ord}_p V \cdot \mathrm{ord}_p W \ ,
\end{aligned}
$$

Q.E.D.

Exercises

(1) Let M be a Riemann surface. Show that the divisor map $(\cdot) : \mathcal{K}(M)^* \to Div(M)$ is a homomorphism of (abelian) groups. [Hint: use local coordinates.]

(2) Compute the intersection multiplicity $(V \cdot W)_{(0,0)}$ for $V = \{y - \lambda x = 0\}$ and $W = \{y^2 - x^3 = 0\}$. (This will depend on $\lambda \in \mathbb{C}$.)

(3) Let $C \subset \mathbb{P}^2$ be an algebraic curve of degree $n > 1$ and L a (projective) line containing $\lfloor \frac{n}{2} \rfloor + 1$ singular points of C. (Note: $\lfloor \cdot \rfloor$ is the "greatest integer" function, which takes the greatest integer less than a given real number.) Use Bezout's theorem to prove that $C \supset L$ hence cannot be irreducible. [Hint: prove first that the intersection multiplicity of L and C at each singular point through which L passes, is at least 2.]

(4) Let $C \subset \mathbb{P}^2$ be an algebraic curve of degree 4 with 4 singular points. Using Bezout's theorem and Prop. 12.2.5, prove that C cannot be irreducible. [Hint: use the Hint from (3) together with a conic Q through the following 5 points: the 4 singularities of C plus one more point of C.]

(5) A degree d algebraic curve $C \subset \mathbb{P}^2$ can be taken to go through any $\frac{(d+1)(d+2)}{2} - 1$ distinct points. (This is just because $\dim(S_3^d) = \frac{(d+1)(d+2)}{2}$.) Prove that if all of these points are taken to lie in a single curve E of degree $e < \frac{d}{2} + 1$, then C is reducible.

(6) Compute $(V \cdot W)_{(0,0)}$ for $V = \{y^3 - x^5 + x^6 = 0\}$ and $W = \{y^3 + x^3y^2 + \lambda x^5 = 0\}$ by locally normalizing V. (As in (2), the answer will depend on λ.)

(7) Let $C \subset \mathbb{P}^2$ be the quartic curve defined by $X^2YZ + Y^2XZ + X^4 + Y^4 = 0$. (a) Prove that C has one singularity, at $[1{:}0{:}0]$ $(= [Z{:}X{:}Y])$, and say what type it is. (b) Using Bézout's Theorem and stereographic projection, construct a normalization $\mathbb{P}^1 \overset{\varphi}{\twoheadrightarrow} C$. (c) Compute the divisor (in the coordinate t on $\mathbb{P}^1$) of the meromorphic function $\varphi^* y$ (where $y = \frac{Y}{Z}$).

(8) To understand the equivalence of our definition of intersection multiplicity with those in Remark 12.2.8, proceed as follows. (a) [Relating our definition to resultants]. Compute ord_0 of the resultant $\mathcal{R}(v, w) \in \mathbb{C}\{x\}$ of two Weierstrass polynomials of degrees n, m: pulling back via $t \mapsto t^{mn} (= x)$ and dividing by mn does not change this order, but breaks v, w into linear factors whose resultants are of the form $\tilde{y}_{\mu+i} - \tilde{z}_{\nu+j}$ as in §12.4; then apply Prop. 9.1.4. (b) [Relating resultants to local rings]. Show that the determinant of multiplication by g on $\mathbb{C}\{x\}[y]/(f)$ is $\mathcal{R}(f, g)$ (by using (9.1.2)), then consult §1.6 of [Fulton, Introduction to Intersection Theory in Algebraic Geometry].