CHAPTER 13

Meromorphic 1-forms on a Riemann surface

In the next chapter we will see one more application of normalization, via intersection numbers: the degree-genus formula. As more will be needed for its proof, presently we make a detour to define and study differential forms (with poles) on manifolds — how to patch them together via local coordinates, how to pull them back under a morphism, and so forth. Like meromorphic functions, 1-forms have an associated divisor. In contrast to the function case, the degree of this divisor is not zero: it tells you the genus of the Riemann surface, via the so-called Poincaré-Hopf theorem. This result will be key to proving the Riemann-Hurwitz and genus formulae.

13.1. Differential 1-forms

These are the expressions you integrate over paths in calculus and complex analysis. For example, on \( \mathbb{R}^2 \)

\[
\eta = F(x, y)dx + G(x, y)dy
\]

is a 1-form. For a differentiable map

\[
\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2
\]

given by

\[(u, v) \mapsto (x(u, v), y(u, v)),\]

the pullback of \( \eta \) by \( \Phi \) is

(13.1.1)

\[
\Phi^* \eta := \left\{ F(x(u, v), y(u, v)) \frac{\partial x}{\partial u}(u, v) \, du + G(x(u, v), y(u, v)) \frac{\partial y}{\partial u}(u, v) \right\} \, du
\]

\[
+ \left\{ F(x(u, v), y(u, v)) \frac{\partial x}{\partial v}(u, v) \, dv + G(x(u, v), y(u, v)) \frac{\partial y}{\partial v}(u, v) \right\} \, dv.
\]
A “0-form” is just a function $f(x, y)$, and

$$
\Phi^* f := f \circ \Phi = f(x(u, v), y(u, v))
$$

is nothing but precomposing with $\Phi$. (13.1.1) is simply the analogue for 1-forms of “precomposition with $\Phi$”. This is exactly what you are doing when you change variables in an integral.

We want to generalize 1-forms from $\mathbb{R}^2$ to real 2-manifolds (and then to complex 1-manifolds), which seems to call for a bit of motivation.

Let $M$ be a differentiable real 2-manifold, $f: M \to \mathbb{R}$ a differentiable function, and $p \in M$ a point. If $M \subset \mathbb{R}^3$, then the notion of “taking partial derivatives of $f$ at $p$ in directions tangent to $M$” makes immediate sense – you just precompose $f$ with a (differentiable) path in $M$ having a given tangent at $p$, and differentiate with respect to the variable parametrizing this path.

In abstract differential topology, one has no embedding in $\mathbb{R}^3$. Rather, the differentiability of $M$ is arranged by requiring the transition functions $\Phi_{\alpha\beta}$ relative to local coordinates on an open cover, to be smooth:

$$
T_pM := \text{vector space of linear differential operators (at } p \in U_\alpha\)
$$
and tangent bundle
\[ TM := \cup_{p \in M} T_p M. \]
One has a projection map \( \pi : TM \to M \) with \( \pi^{-1}(p) = T_p M \). A global section of \( TM \), that is, is a smooth map \( \sigma : M \to TM \) with \( \pi \circ \sigma = \text{id}_M \), is called a vector field on \( M \). (Typically one writes \( \vec{v} \), with the understanding that \( \vec{v}(p) \in T_p M \).)

Now integration is dual to differentiation, so differentials are dual to tangent vectors. For \( \frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^a} \) a dual basis (for the dual vector space) is \( dx^a, dy^a \): we write
\[
\begin{align*}
\left. dx \left( \frac{\partial}{\partial x} \right) \right|_p &= 1, & \left. dy \left( \frac{\partial}{\partial x} \right) \right|_p &= 0, \\
\left. dx \left( \frac{\partial}{\partial y} \right) \right|_p &= 0, & \left. dy \left( \frac{\partial}{\partial y} \right) \right|_p &= 1.
\end{align*}
\]
The cotangent spaces are then
\[ T^*_p M \cong \mathbb{R} \left\langle \left. dx^a \right|_p, \left. dy^a \right|_p \right\rangle. \]
Global sections of the cotangent bundle \( T^* M = \cup_{p \in M} T^*_p M \) are then the differential 1-forms on \( M \). In local coordinates a differential 1-form \( \eta \) looks like:
\[
(13.1.2) \quad \eta_a = F_a(x^a, y^a) dx^a + G_a(x^a, y^a) dy^a.
\]
Just as a function on \( M \) given locally by \( \{g_a : V_a \to \mathbb{R}\} \) must satisfy
\[ g_{\beta} \ restriced_{V_{\beta}} = (g_{\alpha} \ restriced_{V_{\alpha}}) \circ \Phi_{\alpha \beta} \left( \Phi_{\alpha \beta}^* \left( g_{\alpha} \ restriced_{V_{\alpha}} \right) \right), \]
the \( \{\eta_a\} \) are subject to compatibility conditions
\[ \eta_{\beta} \ restriced_{V_{\beta}} = \Phi_{\alpha \beta}^* \left( \eta_{\alpha} \ restriced_{V_{\alpha}} \right). \]
Now since \( M \) (hence each \( \Phi_{\alpha \beta} \)) is smooth, smoothness of \( \eta_a \) (i.e. of \( F_a \) and \( G_a \) in (13.1.2)) is preserved under pullback, and it makes sense

\[ ^1 \text{To define smoothness one has to put a manifold structure on } TM, \text{ which I won’t do here.} \]
to define

\[ A^1_R(M) := \text{smooth, real-valued 1-forms on } M \]

\[ = \text{collections } \{ \eta_a \} \text{ with } \{ F_\alpha, G_\alpha \} \text{ infinitely differentiable.} \]

For a complex 1-manifold, which we recall (from §2.2) is a special kind of smooth real 2-manifold (the \( \Phi_{\alpha\beta} \) are conformal), the labels on the diagram change:

\[ \text{Omitting subscript } \alpha's \text{ for the moment, and writing a subscript } C \text{ to indicate } \otimes_R C, \text{ one has} \]

\[ T_{C,p}M = C \left\langle \frac{\partial}{\partial x} \bigg|_p, \frac{\partial}{\partial y} \bigg|_p \right\rangle \cong C \left\langle \frac{\partial}{\partial \bar{z}} \bigg|_p, \frac{\partial}{\partial z} \bigg|_p \right\rangle \]

\[ T^*_{C,p}M \otimes C = C \left\langle dx \bigg|_p, dy \bigg|_p \right\rangle \cong C \left\langle dz \bigg|_p, d\bar{z} \bigg|_p \right\rangle, \]

where \( \frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - \sqrt{-1} \frac{\partial}{\partial y} \right), \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} \right), \]

\[ dz := dx + \sqrt{-1}dy, \quad d\bar{z} := dx - \sqrt{-1}dy. \] (This makes \( dz(\frac{\partial}{\partial z}) = 1, \quad d\bar{z}(\frac{\partial}{\partial \bar{z}}) = 0, \]

\[ d\bar{z}(\frac{\partial}{\partial z}) = 0, \quad dz(\frac{\partial}{\partial \bar{z}}) = 1 \text{ so that the bases are dual.} \]

A smooth section of the complexified cotangent bundle \( T^*_C M \) thus looks locally like

\[ F(x, y)dz + G(x, y)d\bar{z} \]

\[ = (F + G)dx + \sqrt{-1}(F - G)dy, \]

for \( F \) and \( G \) smooth (infinitely differentiable) complex-valued functions. The 1-forms we are after are substantially more restricted:
13.1.3. DEFINITION. A holomorphic [resp. meromorphic] 1-form \( \omega \in \Omega^1(M) \) [resp. \( K^1(M) \)]\(^2\) is a collection of expressions
\[
\omega_\alpha = f_\alpha(z_\alpha)dz_\alpha,
\]
with \( f_\alpha : V_\alpha \to \mathbb{C} \) holomorphic [resp. meromorphic], satisfying
\[
(13.1.4) \quad \omega_\beta|_{V_\alpha^\beta} = \Phi_\alpha^\beta \left( \omega_\alpha|_{V_\alpha^\beta} \right) \quad \forall \alpha, \beta.
\]

Explicitly, (13.1.4) says that
\[
f_\beta(z_\beta)dz_\beta = f_\alpha(\Phi_\alpha^\beta(z_\beta))d(\Phi_\alpha^\beta(z_\beta))
= f_\alpha(\Phi_\alpha^\beta(z_\beta))\Phi'_\alpha^\beta(z_\beta)dz_\beta,
\]
and is thus equivalent to
\[
(13.1.5) \quad f_\beta(z_\beta) = f_\alpha(\Phi_\alpha^\beta(z_\beta))\Phi'_\alpha^\beta(z_\beta).
\]

Given \( \omega_1, \omega_2 \in K^1(M) \), we can consider their quotient as a meromorphic function \( \frac{\omega_1}{\omega_2} \in K(M) \). This is because in local coordinates, one can “cancel the \( dz \)'s” — viz., \( \frac{f_\alpha(z_\alpha)dz_\alpha}{g_\alpha(z_\alpha)dz_\alpha} = \frac{f_\alpha(z_\alpha)}{g_\alpha(z_\alpha)} \) — and the compatibility condition (13.1.5) implies that such quotients do patch together (the \( \Phi'_\alpha^\beta(z_\beta) \) factors cancel). Conversely, a meromorphic function times a meromorphic 1-form gives a new meromorphic 1-form.

13.1.6. EXAMPLE. On \( M = \mathbb{P}^1 \), let \( \omega_1 = \omega \) be arbitrary and \( \omega_2 = dz \). Here \( z = \frac{Z_1}{Z_0} \) on \( \mathbb{P}^1 \) as usual, and \( dz \) looks as if it should be not just meromorphic but holomorphic. But in the “coordinate at \( \infty \)” \( w = \frac{Z_0}{Z_1} \), \( dz \) becomes \( d\left( \frac{1}{i} \right) = -\frac{dw}{w} \). So \( dz \) in fact has a pole of order 2 at \([0 : 1]\).

Now consider \( F(z) := \frac{\omega_1}{\omega_2} = \frac{\omega}{dz} \in K(\mathbb{P}^1) \) (\( \cong \mathbb{C}(z) \) by Theorem 3.1.7(a)); we have then \( \omega = F(z)dz \). Therefore
\[
K^1(\mathbb{P}^1) = \left\{ \frac{P(z)}{Q(z)}dz \mid P \in \mathbb{C}[z], \; Q \in \mathbb{C}[z] \setminus \{0\} \right\}.
\]

13.1.7. EXAMPLE. For \( M = \mathbb{C}/\Lambda \) a complex 1-torus, write \( u \) for the coordinate on \( \mathbb{C} \). Since each transition function \( \Phi_{\alpha\beta} \) sends \( u \mapsto u + \lambda \) (for some \( \lambda \in \Lambda \), their derivatives \( \Phi'_{\alpha\beta} \) are all identically 1.

\(^2\)Recall the notation \( K(M) \) for meromorphic functions; this is short for \( K^0(M) \), as one can think of such functions as meromorphic 0-forms.
Hence, $du$ gives a well-defined global holomorphic 1-form on $M$ (i.e. belongs to $\Omega^1(\mathbb{C}/\Lambda)$).

So take $\omega_1 = \omega$ arbitrary, $\omega_2 = du$. The same argument as above, using Theorem 3.1.7(b), gives

$$\mathcal{K}^1(\mathbb{C}/\Lambda) \cong \{ f(u)du \mid f = \Lambda\text{-periodic meromorphic function on } \mathbb{C} \}.$$ 

13.1.8. Example. Let $f \in \mathcal{K}(M)$ be a meromorphic function. We can represent $f$ as a collection of maps $f_\alpha : V_\alpha \to \mathbb{P}^1$. The 1-forms $df_\alpha := \frac{df}{dz_\alpha}dz_\alpha$ are then compatible (via pullback) with the transition functions, as in (13.1.4); hence, they patch together to give a global meromorphic 1-form $df \in \mathcal{K}^1(M)$ called the differential of $f$.

Let $\omega \in \mathcal{K}^1(M)$ be given by a collection $\{ f_\alpha(z_\alpha)dz_\alpha \}$; we would like to define its order $\nu_p(\omega)$ at a point $p \in U_\alpha \subset M$. We simply set

$$\nu_p(\omega) := \nu_{z_\alpha(p)}(f_\alpha);$$

if this is negative $\omega$ has a pole at $p$. As a well-definedness check, suppose $p \in U_\beta$ also. Then (using (13.1.5))

$$\nu_p(f_\beta) = \nu_p(f_\alpha \cdot \Phi_{\alpha\beta}'(z_\beta)) = \nu_p(f_\alpha)$$

since, as a biholomorphism, $\Phi_{\alpha\beta}$ must have nonvanishing derivative at every point. If $\omega$ has a pole at $p \in U_\alpha$, then its residue is

$$Res_p(\omega) := Res_{z_\alpha(p)}(f_\alpha) = \frac{1}{2\pi i} \int_{C_\epsilon(p)} f_\alpha(z_\alpha)dz_\alpha$$

where $C_\epsilon(p)$ is a small circle (in $V_\alpha$) about $z_\alpha(p)$. The well-definedness check boils down to change of variable in the integral.

Let $\omega = \{ f_\alpha(z_\alpha)dz_\alpha \} \in \mathcal{K}^1(M)$ be a form, and $\gamma = \bigcup \gamma_\alpha \subset M$ be a smooth real closed curve. Then we define

$$\int_\gamma \omega := \sum_{\alpha} \int_{\gamma_\alpha} f_\alpha(z_\alpha)dz_\alpha,$$

$^3$A “real curve” means something 1-dimensional over $\mathbb{R}$ (not $\mathbb{C}$), so you should think of a closed path on the Riemann surface; and $\gamma_\alpha \subset U_\alpha$ are the segments from which the path is pieced together.
where we observe that 1-forms have been set up so that the right-hand side is independent of choices of local coordinates and the partition of $\gamma$ into local pieces. The following can be viewed as a version of either Stokes’s theorem or Cauchy’s theorem.

13.1.9. Proposition. Let $\Gamma \subset M$ be a closed region\(^4\) with piecewise smooth boundary $\partial \Gamma = \gamma$.

Assume that the meromorphic form $\omega$ is holomorphic on some open set $U$ containing $\Gamma$. Then

$$\int_{\gamma} \omega = 0.$$  

13.1.10. Proposition. Again let $\partial \Gamma = \gamma$, but assume that $\omega$ is only holomorphic on an open set containing $\gamma$ (so that $\Gamma$ may contain poles of $\omega$).

(a) Then we have the residue formula

$$\frac{1}{2\pi \sqrt{-1}} \int_{\gamma} \omega = \sum_{\overset{p \in \Gamma}{\nu_p(\omega) < 0}} \text{Res}_p(\omega).$$

(b) In general for $\omega \in \mathcal{K}^1(M)$, $\sum_{p \in \mathcal{M}} \text{Res}_p(\omega) = 0$.

\(^4\)The technical term here is 2-chain, though we won’t get into this here.
PROOF. For the residue formula (a), let $\Gamma_0 \subset \Gamma$ be a union of disks about those $p \in \Gamma$ where $\omega$ has poles, and $\gamma_0 = \partial \Gamma_0$ the sum of circular paths. Apply Prop. 13.1.9 to the pair $\Gamma - \Gamma_0, \gamma - \gamma_0$.

Applying the residue formula to the case $\Gamma = M, \gamma = \emptyset$ gives (b). \qed

13.1.11. COROLLARY. Consider a nonconstant meromorphic function $f \in \mathcal{K}(M)$. Then

(a) $\sum_{p \in M} v_p(f) = 0$, i.e. the number of zeroes (counted with multiplicity) equals the number of poles (counted with multiplicity); and

(b) $\# \{ f^{-1}(\alpha) \}$ (counted with multiplicity) is independent of $\alpha \in \mathbb{P}^1$.

PROOF. (a) is Prop. 13.1.10(b) applied to $\omega = \frac{df}{f}$. Replacing $f$ by $f - \alpha$, and noting that the number of poles doesn’t change, by (a) the number of zeroes can’t change either, giving (b). \qed

13.1.12. DEFINITION. The degree of $f$, $\deg(f)$, is defined to be the number in Cor. 13.1.11(b). Thinking of $f$ as a covering map from $M \to \mathbb{P}^1$, $\deg(f)$ can be visualized as the number of branches (or “sheets”).

13.1.13. REMARK. We have said nothing about $\int_\gamma \omega$ when $\gamma$ is not a boundary:

Indeed, there is nothing we can say yet — this is the study of periods, which depend on the complex analytic structure of $M$. We will be able to compute some periods of holomorphic forms on algebraic curves later in the course.

\footnote{You may wish to refer back to Remark 3.1.9.}
The usual statement of this theorem is that the sum of indices of any\textsuperscript{6} vector field $\vec{v}$ on a compact oriented smooth manifold $M$ is equal to the Euler characteristic $\chi_M$; we’ll only worry about the case where the real dimension of $M$ is 2. In that case, the index $\text{Ind}_p(\vec{v})$ of $\vec{v}$ at $p \in M$ is the number of counterclockwise rotations done by (the head of) $\vec{v}$ as one goes once counterclockwise on a small circle about $p$. It can only be nonzero if $\vec{v}(p) = 0$.

I’ll give a heuristic proof of the italicized statement, which is probably more illuminating than a formal one. Subdivide a given compact smooth oriented real 2-manifold $M$ into triangles:

Then put one marked point on each edge, vertex, and face of the triangulation:

Next draw the following vector field on each triangle:

\textsuperscript{6}technical point: $\vec{v}$ should have only finitely many zeroes
These match up to give a global vector field on $M$. Evidently the index of this $\bar{\nu}$ is $-1$ at the marked points on the edges, and $+1$ at the marked points on the faces and vertices. Hence,

$$\sum_{p \in M} \text{Ind}_p(\bar{\nu}) = \#F - \#E + \#V = \chi_M = 2 - 2g$$

where $g$ is the genus of $M$. That (13.2.1) holds for any vector field $\bar{\nu}$ on $M$ is the version of the theorem proved by Poincaré. It still holds if we allow $\bar{\nu}$ to have singularities at a finite set of points $\{p_1, \ldots, p_n\}$ (i.e. it is just a section over $M \setminus \{p_1, \ldots, p_n\}$), provided one adds the indices of $\bar{\nu}$ at the $p_i$ to the sum.

In fact, (13.2.1) even holds if $\bar{\nu}$ is replaced by a smooth 1-form $\eta \in A^1_R(M \setminus \{p_1, \ldots, p_n\})$. The idea is to use a metric on $M$, i.e. a nonvanishing section of $\text{Sym}^2(T^*M)$, to smoothly identify $TM$ with $T^*M$. The corresponding notion of index, if (in local coordinates at $p$) $\eta$ takes the form $Fdx + Gdy$, is

$$\text{Ind}_p \eta := \frac{1}{2\pi} \oint d \arctan \left( \frac{G}{F} \right),$$

and once again the sum in (13.2.1) must be over all zeroes of $\eta$ and the $\{p_i\}$.

Now let $M$ be a compact complex 1-manifold, and write $\omega \in \mathcal{K}^1(M)$ locally in the form $f.dx + g.dy$ where $f, g$ are complex-valued. To get into the above setting, we may of course view $M$ as a smooth real 2-manifold, and take the real part of $\omega$:

$$\eta := \Re(\omega) = \Re(f)dx + \Re(g)dy.$$

Let $p$ be a zero or pole of $\omega$, and put $\nu = \nu_p(\omega)$. Of course, in a local holomorphic coordinate $z$ about $p$ with $z(p) = 0$, we have\footnote{up to multiplication by a locally nonvanishing holomorphic function (which will not affect index)}

$$\omega \approx z^{\nu}dz = r^{\nu} \left( \cos(\nu \theta) + \sqrt{-1} \sin(\nu \theta) \right) (dx + \sqrt{-1}dy)
= r^{\nu} \left( \cos(-\nu \theta) - \sqrt{-1} \sin(-\nu \theta) \right) dx + r^{\nu} \left( \sin(-\nu \theta) + \sqrt{-1} \cos(-\nu \theta) \right) dy.$$
So locally we have for the real part
\[ \frac{\eta}{r^\nu} \approx \cos(-\nu \theta)dx + \sin(-\nu \theta)dy, \]
and thus by (13.2.2)
\[ \text{Ind}_p(\eta) = \frac{1}{2\pi} \oint d[-\nu \theta] = -\nu = -\nu_p(\omega) \]
\[ \implies \sum_p \nu_p(\omega) = 2g - 2. \]
We have arrived at the following corollary of (13.2.1), which will henceforth be the meaning of "Poincaré-Hopf" for us:

13.2.3. Theorem. Let \( \omega \in \mathcal{K}^1(M)^* \) be a nonvanishing meromorphic 1-form on a Riemann surface of genus \( g \). Then
\[ \left( \text{# of zeroes} - \text{# of poles} \right) \text{ of } \omega = 2g - 2. \]

13.2.4. Remark. Just as for meromorphic functions we can consider the divisor
\[ (\omega) := \sum_{p \in M} \nu_p(\omega)[p] \]
of a meromorphic 1-form. In this context, the Theorem says that
\[ \deg((\omega)) = 2g - 2. \]

Exercises
1. Let \( E = \{ y^2 - 4x^3 - 4x = 0 \} \), \( \omega = \frac{dx}{y} \bigg|_E \in \Omega^1(E) \). (We can talk about holomorphic 1-forms on a smooth algebraic curve now, because they are Riemann surfaces by the "smooth normalization" Theorem 7.0.1.) Consider the complex analytic automorphism \( A : E \to E \) sending \( (x, y) \mapsto (-x, iy) \), and "apply" this to the 1-form: compute the pullback \( A^*(\omega) \).

2. (a) In Example 13.1.6, \( dz \) defines a meromorphic differential 1-form on \( \mathbb{P}^1 \). Compute its divisor \( (dz) \). Explain why \( \Omega^1(\mathbb{P}^1) = \{ 0 \} \). (b) What is the divisor of \( du \) on \( \mathbb{C}/\Lambda \), from Example 13.1.7?
Explain why it is the unique holomorphic 1-form on \( \mathbb{C}/\Lambda \) up to scale.

(3) Practice with pullbacks: for the map \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) that sends \( (x, y) \mapsto (u(x, y), v(x, y)) := (x^2 - 3xy, y^3 + x) \), compute \( \Phi^* \omega \) where \( \omega = udv + vdu \). Write it in the form \( f(x, y)dx + g(x, y)dy \).

(4) Continuing from Exercise 5 of Chapter 3, compute the pullback of \( \frac{dx}{y} \) under \( \varphi : \mathbb{P}^1 \to \mathbb{C} \). [Hint: simply plug in your final \( x(z) \) and \( y(z) \) from that exercise. After simplification, your answer should be very simple indeed.]

(5) Consider a double cover \( M \to \mathbb{P}^1 \) branched over 8 points (i.e. \( \varphi \) is 2:1 except at these points, where it is locally of the form \( w \mapsto w^2 \)). Compute the genus of \( M \) by considering the divisor of \( \varphi^* \frac{dz}{z} \) and applying Poincaré-Hopf in the form of Remark 13.2.4. (For simplicity, assume none of the 8 points are 0 or \( \infty \).)

(6) Two loops are linked (inside some space) if we cannot contract one to a point (in that space) without passing through the other. In the neighborhood of a node (ODP), a curve is locally approximated by \( xy = 0 \); let \( c_1 \) and \( c_2 \) be the circles \( \{ y = 0 \} \cap \{|x| = \epsilon \} \) and \( \{ x = 0 \} \cap \{|y| = \epsilon \} \), respectively. Certainly, these are not linked in \( \mathbb{C}^2 \). Show that, however, they are linked in the 3-sphere \( S_\epsilon^3 = \{|x|^2 + |y|^2 = \epsilon^2\} \), by considering the integral \( \frac{1}{2\pi i} \int_{c_1} \frac{dx}{x} \). [Hint: what is \( \{ x = 0 \} \cap S_\epsilon^3 \)?]