CHAPTER 13

Meromorphic 1-forms on a Riemann surface

In the next chapter we will see one more application of normalization, via intersection numbers: the degree-genus formula. As more will be needed for its proof, presently we make a detour to define and study differential forms (with poles) on manifolds — how to patch them together via local coordinates, how to pull them back under a morphism, and so forth. Like meromorphic functions, 1-forms have an associated divisor. In contrast to the function case, the degree of this divisor is not zero: it tells you the genus of the Riemann surface, via the so-called Poincaré-Hopf theorem. This result will be key to proving the Riemann-Hurwitz and genus formulae.

13.1. Differential 1-forms

These are the expressions you integrate over paths in calculus and complex analysis. For example, on $\mathbb{R}^2$

$$\eta = F(x,y)dx + G(x,y)dy$$

is a 1-form. For a differentiable map

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$$

given by

$$(u,v) \mapsto (x(u,v), y(u,v)),$$

the *pullback of η by Φ* is

(13.1.1)

$$\begin{aligned}\Phi^*\eta \quad := \quad & F\left(x(u,v), y(u,v)\right) d(x(u,v)) + G(x(u,v), y(u,v)) d(y(u,v)) \\ = \quad & \left\{F(x(u,v), y(u,v)) \tfrac{\partial x}{\partial u}(u,v) + G(x(u,v), y(u,v)) \tfrac{\partial y}{\partial u}(u,v)\right\} du \\ & + \left\{F(x(u,v), y(u,v)) \tfrac{\partial x}{\partial v}(u,v) + G(x(u,v), y(u,v)) \tfrac{\partial y}{\partial v}(u,v)\right\} dv.\end{aligned}$$

A "0-form" is just a function $f(x, y)$, and

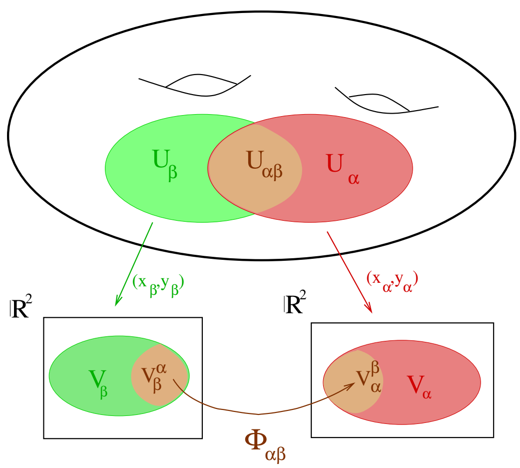
$$\Phi^* f := f \circ \Phi = f\left(x(u,v), y(u,v)\right)$$

is nothing but precomposing with Φ. (13.1.1) is simply the analogue for 1-forms of "precomposition with Φ" . This is exactly what you are doing when you change variables in an integral.

We want to generalize 1-forms from $\mathbb{R}^2$ to real 2-manifolds (and then to complex 1-manifolds), which seems to call for a bit of motivation.

Let M be a differentiable real 2-manifold, $f\colon M \to \mathbb{R}$ a differentiable function, and $p \in M$ a point. If $M \subset \mathbb{R}^3$, then the notion of "taking partial derivatives of f at p in directions tangent to M" makes immediate sense – you just precompose f with a (differentiable) path in M having a given tangent at p, and differentiate with respect to the variable parametrizing this path.

In abstract differential topology, one has no embedding in $\mathbb{R}^3$. Rather, the differentiability of M is arranged by requiring the transition functions $\Phi_{\alpha\beta}$ relative to local coordinates on an open cover, to be smooth:



(This was discussed at the beginning of §2.2.) One then defines the *tangent spaces*

$$T_pM := \text{vector space of linear differential operators (at } p \in U_\alpha)$$

$$\cong \mathbb{R}\left\langle \left.\frac{\partial}{\partial x_\alpha}\right|_p, \left.\frac{\partial}{\partial y_\alpha}\right|_p \right\rangle$$

and *tangent bundle*

$$TM := \cup_{p\in M} T_pM.$$

One has a projection map $\pi : TM \to M$ with $\pi^{-1}(p) = T_pM$. A global section of TM, that is, is a smooth[1] map $\sigma : M \to TM$ with $\pi \circ \sigma = \mathrm{id}_M$, is called a *vector field* on M. (Typically one writes $\vec{v}$, with the understanding that $\vec{v}(p) \in T_pM$.)

Now integration is dual to differentiation, so differentials are dual to tangent vectors. For $\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}$ a dual basis (for the dual vector space) is dx_α, dy_α: we write

$$dx\left(\frac{\partial}{\partial x}\right) = 1\,, \quad dy\left(\frac{\partial}{\partial x}\right) = 0\,,$$
$$dx\left(\frac{\partial}{\partial y}\right) = 0\,, \quad dy\left(\frac{\partial}{\partial y}\right) = 1.$$

The *co*tangent spaces are then

$$T_p^*M \cong \mathbb{R}\left\langle dx_\alpha|_p\,, dy_\alpha|_p \right\rangle.$$

Global sections of the cotangent bundle $T^*M = \cup_{p\in M} T_p^*M$ are then the *differential* 1*-forms* on M. In local coordinates a differential 1-form η looks like:

$$\eta_\alpha = F_\alpha(x_\alpha, y_\alpha)dx_\alpha + G_\alpha(x_\alpha, y_\alpha)dy_\alpha. \tag{13.1.2}$$

Just as a function on M given locally by $\{g_\alpha : V_\alpha \to \mathbb{R}\}$ must satisfy

$$g_\beta|_{V_\beta^\alpha} = \left(g_\alpha|_{V_\alpha^\beta}\right) \circ \Phi_{\alpha\beta} \ \left(= \Phi_{\alpha\beta}^*\left(g_\alpha|_{V_\alpha^\beta}\right)\right),$$

the $\{\eta_\alpha\}$ are subject to compatibility conditions

$$\eta_\beta|_{V_\beta^\alpha} = \Phi_{\alpha\beta}^*\left(\eta_\alpha|_{V_\alpha^\beta}\right).$$

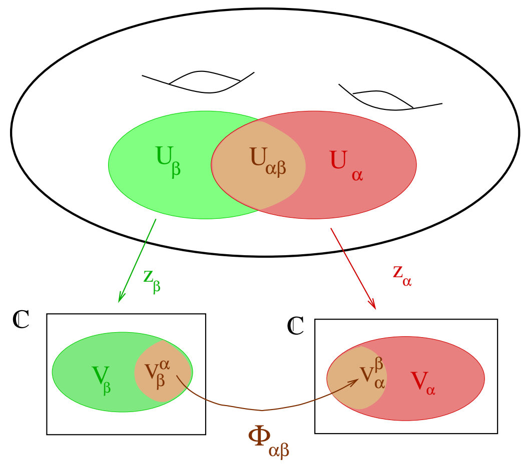
Now since M (hence each $\Phi_{\alpha\beta}$) is smooth, smoothness of η_α (i.e. of F_α and G_α in (13.1.2)) is preserved under pullback, and it makes sense

[1] To define smoothness one has to put a manifold structure on TM, which I won't do here.

to define

$$\begin{aligned} A^1_{\mathbb{R}}(M) &:= \text{smooth, real-valued 1-forms on } M \\ &= \text{collections } \{\eta_\alpha\} \text{ with } \{F_\alpha, G_\alpha\} \text{ infinitely differentiable.} \end{aligned}$$

For a complex 1-manifold, which we recall (from §2.2) is a special kind of smooth real 2-manifold (the $\Phi_{\alpha\beta}$ are conformal), the labels on the diagram change:



Omitting subscript α's for the moment, and writing a subscript $\mathbb{C}$ to indicate $\otimes_{\mathbb{R}}\mathbb{C}$, one has

$$T_{\mathbb{C},p}M = \mathbb{C}\left\langle \left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p \right\rangle \cong \mathbb{C}\left\langle \left.\frac{\partial}{\partial z}\right|_p, \left.\frac{\partial}{\partial \bar{z}}\right|_p \right\rangle$$

$$T^*_{\mathbb{C},p}M \otimes \mathbb{C} = \mathbb{C}\left\langle dx|_p, dy|_p \right\rangle \cong \mathbb{C}\left\langle dz|_p, d\bar{z}|_p \right\rangle,$$

where $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\frac{\partial}{\partial y}\right)$, $dz := dx + \sqrt{-1}dy$, $d\bar{z} := dx - \sqrt{-1}dy$. (This makes $dz(\frac{\partial}{\partial z}) = 1$, $dz(\frac{\partial}{\partial \bar{z}}) = 0$, $d\bar{z}(\frac{\partial}{\partial z}) = 0$, $d\bar{z}(\frac{\partial}{\partial \bar{z}}) = 1$ so that the bases are dual.) A smooth section of the complexified cotangent bundle $T^*_{\mathbb{C}}M$ thus looks locally like

$$\begin{aligned} &F(x,y)dz + G(x,y)d\bar{z} \\ &= (F+G)dx + \sqrt{-1}(F-G)dy, \end{aligned}$$

for F and G smooth (infinitely differentiable) complex-valued functions. The 1-forms we are after are substantially more restricted:

13.1.3. DEFINITION. A holomorphic [resp. meromorphic] 1-form $\omega \in \Omega^1(M)$ [resp. $\mathcal{K}^1(M)$][2] is a collection of expressions

$$\omega_\alpha = f_\alpha(z_\alpha)dz_\alpha \ ,$$

with $f_\alpha \colon V_\alpha \to \mathbb{C}$ holomorphic [resp. meromorphic], satisfying

$$\omega_\beta\big|_{V_\beta^\alpha} = \Phi_{\alpha\beta}^*\left(\omega_\alpha\big|_{V_\alpha^\beta}\right) \quad \forall \alpha, \beta. \tag{13.1.4}$$

Explicitly, (13.1.4) says that

$$\begin{aligned} f_\beta(z_\beta)dz_\beta &= f_\alpha(\Phi_{\alpha\beta}(z_\beta))d(\Phi_{\alpha\beta}(z_\beta)) \\ &= f_\alpha(\Phi_{\alpha\beta}(z_\beta))\Phi'_{\alpha\beta}(z_\beta)dz_\beta, \end{aligned}$$

and is thus equivalent to

$$f_\beta(z_\beta) = f_\alpha(\Phi_{\alpha\beta}(z_\beta))\Phi'_{\alpha\beta}(z_\beta). \tag{13.1.5}$$

Given $\omega_1, \omega_2 \in \mathcal{K}^1(M)$, we can consider their quotient as a meromorphic function $\frac{\omega_1}{\omega_2} \in \mathcal{K}(M)$. This is because in local coordinates, one can "cancel the dz's" — viz., $\frac{f_\alpha(z_\alpha)dz_\alpha}{g_\alpha(z_\alpha)dz_\alpha} = \frac{f_\alpha(z_\alpha)}{g_\alpha(z_\alpha)}$ — and the compatibility condition (13.1.5) implies that such quotients do patch together (the $\Phi'_{\alpha\beta}(z_\beta)$ factors cancel). Conversely, a meromorphic function times a meromorphic 1-form gives a new meromorphic 1-form.

13.1.6. EXAMPLE. On $M = \mathbb{P}^1$, let $\omega_1 = \omega$ be arbitrary and $\omega_2 = dz$. Here $z = \frac{Z_1}{Z_0}$ on $\mathbb{P}^1$ as usual, and dz looks as if it should be not just meromorphic but holomorphic. But in the "coordinate at ∞" $w = \frac{Z_0}{Z_1}$, dz becomes $d(\frac{1}{w}) = -\frac{dw}{w^2}$. So dz in fact has a pole of order 2 at $[0:1]$.

Now consider $F(z) := \frac{\omega_1}{\omega_2} = \frac{\omega}{dz} \in \mathcal{K}(\mathbb{P}^1)$ ($\cong \mathbb{C}(z)$ by Theorem 3.1.7(a)); we have then $\omega = F(z)dz$. Therefore

$$\mathcal{K}^1(\mathbb{P}^1) = \left\{ \frac{P(z)}{Q(z)}dz \,\middle|\, P \in \mathbb{C}[z],\ Q \in \mathbb{C}[z] \setminus \{0\} \right\}.$$

13.1.7. EXAMPLE. For $M = \mathbb{C}/\Lambda$ a complex 1-torus, write u for the coordinate on $\mathbb{C}$. Since each transition function $\Phi_{\alpha\beta}$ sends $u \mapsto u + \lambda$ (for some $\lambda \in \Lambda$), their derivatives $\Phi'_{\alpha\beta}$ are all identically 1.

[2]Recall the notation $\mathcal{K}(M)$ for meromorphic functions; this is short for $\mathcal{K}^0(M)$, as one can think of such functions as meromorphic 0-forms.

Hence, du gives a well-defined global holomorphic 1-form on M (i.e. belongs to $\Omega^1(\mathbb{C}/\Lambda)$).

So take $\omega_1 = \omega$ arbitrary, $\omega_2 = du$. The same argument as above, using Theorem 3.1.7(b), gives

$$\mathcal{K}^1(\mathbb{C}/\Lambda) \cong \{f(u)du \mid f = \Lambda\text{-periodic meromorphic function on } \mathbb{C}\}.$$

13.1.8. EXAMPLE. Let $f \in \mathcal{K}(M)$ be a meromorphic function. We can represent f as a collection of maps $f_\alpha : V_\alpha \to \mathbb{P}^1$. The 1-forms $df_\alpha := \frac{df_\alpha}{dz_\alpha}dz_\alpha$ are then compatible (via pullback) with the transition functions, as in (13.1.4); hence, they patch together to give a global meromorphic 1-form $df \in \mathcal{K}^1(M)$ called the *differential of* f.

Let $\omega \in \mathcal{K}^1(M)$ be given by a collection $\{f_\alpha(z_\alpha)dz_\alpha\}$; we would like to define its *order* $\nu_p(\omega)$ at a point $p \in U_\alpha \subset M$. We simply set

$$\nu_p(\omega) := \nu_{z_\alpha(p)}(f_\alpha);$$

if this is negative ω has a *pole* at p. As a well-definedness check, suppose $p \in U_\beta$ also. Then (using (13.1.5))

$$\nu_p(f_\beta) = \nu_p\left(f_\alpha \cdot \Phi'_{\alpha\beta}(z_\beta)\right) = \nu_p(f_\alpha)$$

since, as a biholomorphism, $\Phi_{\alpha\beta}$ must have nonvanishing derivative at every point. If ω has a pole at $p \in U_\alpha$, then its *residue* is

$$Res_p(\omega) := Res_{z_\alpha(p)}(f_\alpha) = \frac{1}{2\pi\sqrt{-1}} \oint_{C_\epsilon(p)} f_\alpha(z_\alpha)dz_\alpha$$

where $C_\epsilon(p)$ is a small circle (in V_α) about $z_\alpha(p)$. The well-definedness check boils down to change of variable in the integral.

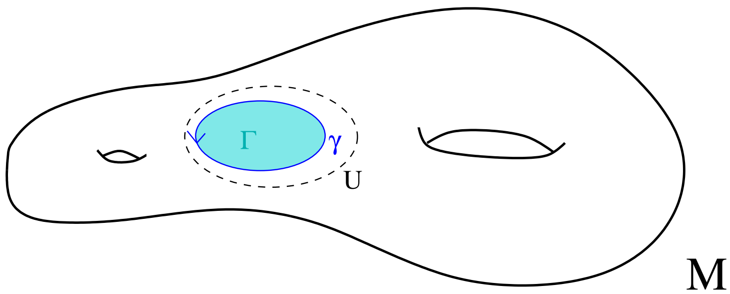
Let $\omega = \{f_\alpha(z_\alpha)dz_\alpha\} \in \mathcal{K}^1(M)$ be a form, and $\gamma = \cup\gamma_\alpha \subset M$ be a smooth real closed curve.[3] Then we define

$$\int_\gamma \omega \ := \ \sum_\alpha \int_{\gamma_\alpha} f_\alpha(z_\alpha)dz_\alpha,$$

[3]A "real curve" means something 1-dimensional over $\mathbb{R}$ (not $\mathbb{C}$), so you should think of a closed path on the Riemann surface; and $\gamma_\alpha \subset U_\alpha$ are the segments from which the path is pieced together.

where we observe that 1-forms have been set up so that the right-hand side is independent of choices of local coordinates and the partition of γ into local pieces. The following can be viewed as a version of either Stokes's theorem or Cauchy's theorem.

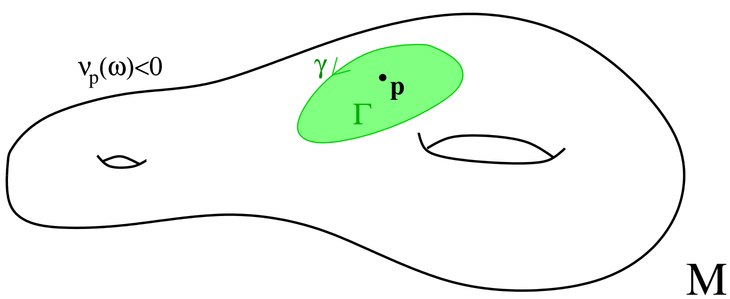
13.1.9. PROPOSITION. *Let* $\Gamma \subset M$ *be a closed region*[4] *with piecewise smooth boundary* $\partial\Gamma = \gamma$.



Assume that the meromorphic form ω *is holomorphic on some open set* U *containing* Γ. *Then*

$$\int_\gamma \omega = 0.$$

13.1.10. PROPOSITION. *Again let* $\partial\Gamma = \gamma$, *but assume that* ω *is only holomorphic on an open set containing* γ *(so that* Γ *may contain poles of* ω*).*



(a) *Then we have the residue formula*

$$\frac{1}{2\pi\sqrt{-1}}\int_\gamma \omega \quad = \sum_{\substack{p\in\Gamma \\ \nu_p(\omega)<0}} Res_p(\omega).$$

(b) *In general for* $\omega \in \mathcal{K}^1(M)$, $\sum_{p\in M} Res_p(\omega) = 0$.

[4]The technical term here is 2-chain, though we won't get into this here.

PROOF. For the residue formula (a), let $\Gamma_0 \subset \Gamma$ be a union of disks about those $p \in \Gamma$ where ω has poles, and $\gamma_0 = \partial\Gamma_0$ the sum of circular paths. Apply Prop. 13.1.9 to the pair $\Gamma - \Gamma_0$, $\gamma - \gamma_0$.

Applying the residue formula to the case $\Gamma = M$, $\gamma = \emptyset$ gives (b). □

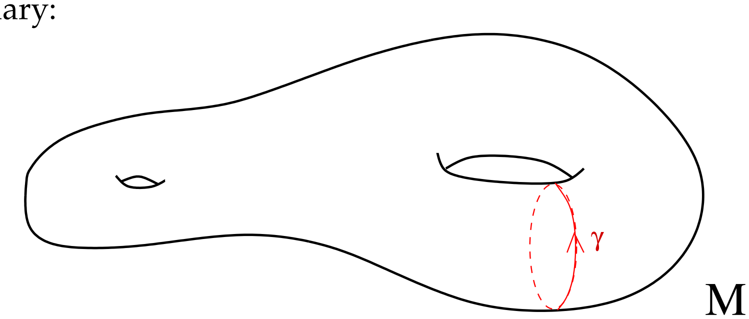
13.1.11. COROLLARY. *Consider a nonconstant meromorphic function $f \in \mathcal{K}(M)$. Then*

(a) $\sum_{p \in M} \nu_p(f) = 0$, *i.e. the number of zeroes (counted with multiplicity) equals the number of poles (counted with multiplicity); and*
(b) $\#\{f^{-1}(\alpha)\}$ *(counted with multiplicity) is independent of $\alpha \in \mathbb{P}^1$.*

PROOF. (a) is Prop. 13.1.10(b) applied to $\omega = \frac{df}{f}$. Replacing f by $f - \alpha$, and noting that the number of poles doesn't change, by (a) the number of zeroes can't change either, giving (b). □

13.1.12. DEFINITION. The *degree of f*, $\deg(f)$, is defined to be the number in Cor. 13.1.11(b). Thinking of f as a covering map from $M \to \mathbb{P}^1$, $\deg(f)$ can be visualized as the number of branches (or "sheets").[5]

13.1.13. REMARK. We have said nothing about $\int_\gamma \omega$ when γ is not a boundary:



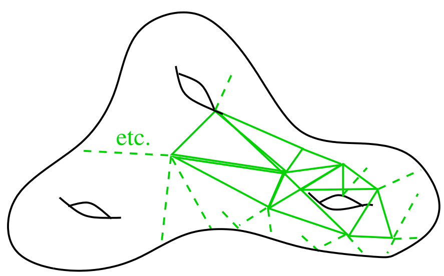
Indeed, there is nothing we can say yet — this is the study of *periods*, which depend on the complex analytic structure of M. We will be able to compute some periods of holomorphic forms on algebraic curves later in the course.

[5]You may wish to refer back to Remark 3.1.9.

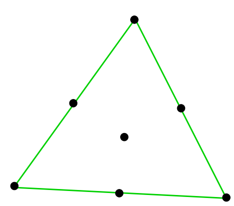
13.2. Poincaré-Hopf theorem

The usual statement of this theorem is that *the sum of indices of any*[6] *vector field* $\vec{v}$ *on a compact oriented smooth manifold* M *is equal to the Euler characteristic* χ_M; we'll only worry about the case where the real dimension of M is 2. In that case, the *index* $\mathrm{Ind}_p(\vec{v})$ of $\vec{v}$ at $p \in M$ is the number of counterclockwise rotations done by (the head of) $\vec{v}$ as one goes once counterclockwise on a small circle about p. It can only be nonzero if $\vec{v}(p) = 0$.

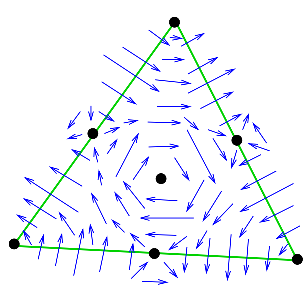
I'll give a heuristic proof of the italicized statement, which is probably more illuminating than a formal one. Subdivide a given compact smooth oriented real 2-manifold M into triangles:



Then put one marked point on each edge, vertex, and face of the triangulation:



Next draw the following vector field on each triangle:



[6]technical point: $\vec{v}$ should have only finitely many zeroes

These match up to give a global vector field on M. Evidently the index of this $\vec{v}$ is -1 at the marked points on the edges, and $+1$ at the marked points on the faces and vertices. Hence,

$$\sum_{p\in M} \mathrm{Ind}_p(\vec{v}) = \#F - \#E + \#V = \chi_M = 2-2g \tag{13.2.1}$$

where g is the genus of M. That (13.2.1) holds for *any* vector field $\vec{v}$ on M is the version of the theorem proved by Poincaré. It still holds if we allow $\vec{v}$ to have singularities at a finite set of points $\{p_1,\ldots,p_n\}$ (i.e. it is just a section over $M\backslash\{p_1,\ldots,p_n\}$), provided one adds the indices of $\vec{v}$ at the p_i to the sum.

In fact, (13.2.1) even holds if $\vec{v}$ is replaced by a smooth 1-form $\eta \in A^1_{\mathbb{R}}(M\backslash\{p_1,\ldots,p_n\})$. The idea is to use a metric on M, i.e. a nonvanishing section of $\mathrm{Sym}^2(T^*M)$, to smoothly identify TM with T^*M. The corresponding notion of index, if (in local coordinates at p) η takes the form $Fdx + Gdy$, is

$$\mathrm{Ind}_p\eta := \frac{1}{2\pi}\oint d\arctan\left(\frac{G}{F}\right), \tag{13.2.2}$$

and once again the sum in (13.2.1) must be over all zeroes of η and the $\{p_i\}$.

Now let M be a compact complex 1-manifold, and write $\omega \in \mathcal{K}^1(M)$ locally in the form $f.dx + g.dy$ where f,g are complex-valued. To get into the above setting, we may of course view M as a smooth real 2-manifold, and take the real part of ω:

$$\eta := \Re(\omega) \overset{\text{loc}}{=} \Re(f)dx + \Re(g)dy.$$

Let p be a zero or pole of ω, and put $\nu = \nu_p(\omega)$. Of course, in a local holomorphic coordinate z about p with $z(p) = 0$, we have[7]

$$\omega \overset{\text{loc}}{\approx} z^\nu dz \;=\; r^\nu\left(\cos(\nu\theta) + \sqrt{-1}\sin(\nu\theta)\right)(dx + \sqrt{-1}dy)$$

$$=\; r^\nu\left(\cos(-\nu\theta) - \sqrt{-1}\sin(-\nu\theta)\right)dx + r^\nu\left(\sin(-\nu\theta) + \sqrt{-1}\cos(-\nu\theta)\right)dy.$$

[7] up to multiplication by a locally nonvanishing holomorphic function (which will not affect index)

So locally we have for the real part

$$\frac{\eta}{r^\nu} \approx \cos(-\nu\theta)dx + \sin(-\nu\theta)dy,$$

and thus by (13.2.2)

$$\mathrm{Ind}_p(\eta) = \frac{1}{2\pi}\oint d[-\nu\theta] = -\nu = -\nu_p(\omega)$$
$$\implies \sum_p \nu_p(\omega) = 2g-2.$$

We have arrived at the following corollary of (13.2.1), which will henceforth be the meaning of "Poincaré-Hopf" for us:

13.2.3. THEOREM. *Let $\omega \in \mathcal{K}^1(M)^*$ be a nonvanishing meromorphic 1-form on a Riemann surface of genus g. Then*

$$\underbrace{(\#\textit{ of zeroes} - \#\textit{ of poles})}_{\textit{counted with multiplicity}} \textit{ of } \omega = 2g-2.$$

13.2.4. REMARK. Just as for meromorphic functions we can consider the divisor

$$(\omega) := \sum_{p\in M} \nu_p(\omega)[p]$$

of a meromorphic 1-form. In this context, the Theorem says that

$$\deg((\omega)) = 2g-2.$$

Exercises

(1) Let $E = \{y^2 - 4x^3 - 4x = 0\}$, $\omega = \frac{dx}{y}\Big|_E \in \Omega^1(E)$. (We can talk about holomorphic 1-forms on a smooth algebraic curve now, because they are Riemann surfaces by the "smooth normalization" Theorem 7.0.1.) Consider the complex analytic automorphism $A : E \to E$ sending $(x,y) \mapsto (-x, iy)$, and "apply" this to the 1-form: compute the pullback $A^*(\omega)$.

(2) (a) In Example 13.1.6, dz defines a meromorphic differential 1-form on $\mathbb{P}^1$. Compute its divisor (dz). Explain why $\Omega^1(\mathbb{P}^1) = \{0\}$. (b) What is the divisor of du on $\mathbb{C}/\Lambda$, from Example 13.1.7?

Explain why it is the *unique* holomorphic 1-form on $\mathbb{C}/\Lambda$ up to scale.

(3) Practice with pullbacks: for the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $(x,y) \mapsto (u(x,y), v(x,y)) := (x^2 - 3xy, y^3 + x)$, compute $\Phi^*\omega$ where $\omega = udv + vdu$. Write it in the form $f(x,y)dx + g(x,y)dy$.

(4) Continuing from Exercise 5 of Chapter 3, compute the pullback of $\frac{dx}{y}$ under $\varphi : \mathbb{P}^1 \to C$. [Hint: simply plug in your final $x(z)$ and $y(z)$ from that exercise. After simplification, your answer should be very simple indeed.]

(5) Consider a double cover $M \overset{\varphi}{\twoheadrightarrow} \mathbb{P}^1$ branched over 8 points (i.e. φ is 2:1 except at these points, where it is locally of the form $w \mapsto w^2$). Compute the genus of M by considering the divisor of $\varphi^* \frac{dz}{z}$ and applying Poincaré-Hopf in the form of Remark 13.2.4. (For simplicity, assume none of the 8 points are 0 or ∞.)

(6) Two loops are *linked* (inside some space) if we cannot contract one to a point (in that space) without passing through the other. In the neighborhood of a node (ODP), a curve is locally approximated by $xy = 0$; let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be the circles $\{y = 0\} \cap \{|x| = \epsilon\}$ and $\{x = 0\} \cap \{|y| = \epsilon\}$, respectively. Certainly, these are not linked in $\mathbb{C}^2$. Show that, however, they *are* linked in the 3-sphere $\mathcal{S}^3_\epsilon = \{|x|^2 + |y|^2 = \epsilon^2\}$, by considering the integral $\frac{1}{2\pi i}\int_{\mathfrak{c}_1} \frac{dx}{x}$. [Hint: what is $\{x = 0\} \cap \mathcal{S}^3_\epsilon$?]