CHAPTER 14

The genus formula

We are ready to prove two formulas for the genus of a Riemann surface (RS) which are especially useful in algebraic geometry. For the first result (the Riemann-Hurwitz formula), the RS will arise as a finite branched cover of another RS whose genus is known. The proof makes essential use of Poincaré-Hopf and a *ramification divisor* which we introduce below. For the second result, which is an application of the first (and of the intersection theory from Chap. 12), the RS will arise as the normalization of an irreducible algebraic curve in $\mathbb{P}^2$ with only ordinary double point (ODP) singularities. This is a very concrete payoff for the preceding hard work: now we can compute the genus of [the desingularization of] a projective algebraic curve!

14.1. Order and multiplicity for maps of Riemann surfaces

Consider a nonconstant morphism $f : M \to M'$ of Riemann surfaces with $f(p) = q$. In Exercise 4 of Chapter 3, the following was established: there exist

- neighborhoods $U \ni p$, $V \ni q$ with $f(U) \subset V$, and
- local holomorphic coordinates $z : U \to \mathbb{C}$ and $w : V \to \mathbb{C}$ with $z(p) = 0 = w(q)$,

such that $w \circ f = z^\nu$ for some unique $\nu \in \mathbb{N}$. More informally, in these local coordinates f "takes the form" $(w =)f(z) = z^\nu$. We write $\nu_p(f) := \nu$. This is the *ramification index*, and f *ramifies* at p precisely when it exceeds 1.

14.1.1. REMARK. Note the (small) possibility of confusion if $M' = \mathbb{P}^1$, since $\nu_p(\cdot)$ already has meaning for meromorphic functions on

M. In that case, we simply have to be clear about whether we are considering f as a meromorphic function or as a morphism of Riemann surfaces, which is a good thing to do in any case.

For any $q \in M'$, consider the sum

$$d(q) \ := \sum_{\substack{p \in M \\ \text{with } f(p)=q}} \nu_p(f).$$

If a ramification point p of index d lies over q, then over a nearby point q_0, p is replaced by d points with ramification index 1. This is by virtue of the local form $w = z^\nu$, as is the fact that the ramification points are isolated hence finite in number (M is compact!). Evidently then, $d(q)$ is constant in q; we will call this constant $d \in \mathbb{N}$ the *mapping degree* $\deg(f)$ of the morphism f.[1] This generalizes Definition 13.1.12.

Here is a more "gentrified" way to define the mapping degree. We can think of a point $q \in M'$ as a divisor $[q] \in \mathrm{Div}(M')$, and "pull it back" to a divisor on M by the formula

$$f^{-1}([q]) := \sum_{f(p)=q} \nu_p(f)[p] \ \in \ \mathrm{Div}(M). \tag{14.1.2}$$

We then put (for any $q \in M'$, it doesn't matter)

$$\deg(f) := \deg\left(f^{-1}([q])\right) \ \left(= \sum_{f(p)=q} \nu_p(f) \right).$$

14.1.3. REMARK. (14.1.2) extends linearly to define a pullback divisor $f^{-1}(D) \in \mathrm{Div}(M)$ (also written f^*D) for any $D \in \mathrm{Div}(M')$.

Associated to $f : M \to M'$, finally, is the *ramification divisor*

$$R_f := \sum_{p \in M} \big(\nu_p(f) - 1\big)\,[p] \ \in \ \mathrm{Div}(M).$$

By the discussion above, the sum is clearly finite.

[1] When f is a nonconstant map from M to $\mathbb{P}^1$, you can think of it as a meromorphic function and take the degree of its divisor, $\deg((f))$, which is always 0. Or, you can think of it as a morphism of Riemann surfaces and take $\deg(f)$, which is never 0. So that extra parenthesis matters!

14.2. Riemann-Hurwitz formula

Again take $f : M \to M'$ to be a nonconstant morphism, write $d := \deg(f)$, and put

$$r := \deg(R_f).$$

In the following formula, stated by Riemann and proved by Hurwitz, g resp. g' will refer to the genus of M resp. M'.

14.2.1. THEOREM. [RIEMANN, 1857; HURWITZ, 1891]

$$r = 2\left\{g + d - dg' - 1\right\}.$$

14.2.2. REMARK. Some alternative ways to write this result are:

(i) $g = (g' - 1)d + \frac{r}{2} + 1$

(ii) $\chi_M = \deg(f)\chi_{M'} - \deg(R_f)$

These better represent the way you want to think of it: as a formula for the genus (or Euler characteristic) of M, if you know that of M' and data about how M "sits over" M'.

PROOF. For $p \in M$ with $a = \nu_p(f)$, we choose local coordinates z, w as in §14.1 so that $z \overset{f}{\mapsto} z^a (= w)$.

We shall need to assume the existence of a nonzero meromorphic 1-form $\omega \in \mathcal{K}^1(M')$. This is obvious if M' arises as the normalization of an algebraic curve in $\mathbb{P}^2$, as you can just pull back any nonconstant meromorphic function (say, Z_1/Z_0) and take its differential. Every Riemann surface arises in this way, but to see that you need the Riemann-Roch theorem. We proceed with the proof modulo this detail.

Locally writing $\omega = g(w)dw$, we have

$$f^*\omega \overset{\text{loc}}{=} g(z^a)d(z^a) = a.g(z^a)z^{a-1}dz\,,$$

hence

$$\nu_p(f^*\omega) = a.\nu_0(g) + (a-1) = \nu_p(f).\nu_{f(p)}(\omega) + (\nu_p(f) - 1).$$

In $\mathrm{Div}(M)$ we have therefore

$$(f^*\omega) := \sum_{p\in M} \nu_p(f^*\omega)[p] = \sum_p \nu_p(f).\nu_{f(p)}(\omega)[p] + \underbrace{\sum_p \left(\nu_p(f)-1\right)[p]}_{R_f}$$

$$\begin{aligned} &= \sum_{q\in M'} \nu_q(\omega) \sum_{f(p)=q} \nu_p(f)[p] \;+\; R_f \\ &= \sum_{q\in M'} \nu_q(\omega) f^{-1}([q]) \;+\; R_f \\ &= f^{-1}\left(\sum_q \nu_q(\omega)[q]\right) \;+\; R_f \\ &= f^{-1}((\omega)) + R_f, \end{aligned}$$

where $(\omega) \in \mathrm{Div}(M')$.

Now $f^*\omega \in \mathcal{K}^1(M)$, and so Poincaré-Hopf on M tells us that

$$2g-2 = \deg\left((f^*\omega)\right),$$

which by the computation just done

$$\begin{aligned} &= \deg\left(f^{-1}((\omega))\right) + \deg R_f \\ &= \sum_q \nu_q(\omega) \underbrace{\sum_{f(p)=q} \nu_p(f)}_{\deg(f)} \;+\; r \\ &= \deg(f) \underbrace{\sum \nu_q(\omega)}_{\deg((\omega))} \;+\; r. \end{aligned}$$

Applying Poincaré-Hopf once more (but on M'), we get that this

$$= d(2g'-2) + r.$$

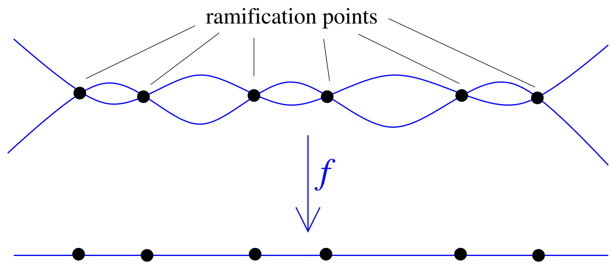
So we have shown $2-2g = d(2-2g') - r$, which is the version of R-H stated in Remark 14.2.2(ii). □

We turn to some examples.

14.2.3. EXAMPLE. Let $C = \{y^2 = \prod_{i=1}^{2m}(x-\alpha_i)\} \subset \mathbb{C}^2$, and let M be the normalization of its projective closure $\bar{C} \subset \mathbb{P}^2$. The original curve had a projection map to the x-axis ($(x,y) \mapsto x$), and this extends to

$$f : M \to \mathbb{P}^1 =: M',$$

as depicted below:[2]



Clearly $g' = 0$, $d = 2$, and

$$r = \sum(\nu_p(f) - 1) = 2m$$

since $\nu_p(f) - 1 = 1$ at each of the ramification points. So by Remark 14.2.2(i)

$$g = (0-1).2 + \frac{2m}{2} + 1 = m - 1.$$

14.2.4. EXAMPLE. Let $M = M' = \mathbb{C}/\Lambda$ be a complex 1-torus; as usual $\Lambda = \{m_1\lambda_1 + m_2\lambda_2 \mid m_1, m_2 \in \mathbb{Z}\}$, where $\lambda_1, \lambda_2 \in \mathbb{C}$ are independent over $\mathbb{R}$. Now assume $\alpha\Lambda \subseteq \Lambda$ for some $\alpha \in \mathbb{C}^*$. Then we have a "complex multiplication" map

$$M \xrightarrow{f} M'$$

$$z \longmapsto \alpha z$$

with $R_f = 0$. You will treat this setting in an exercise below.

14.3. The genus of a projective algebraic curve

Let $C = \{F(Z, X, Y) = 0\} \subset \mathbb{P}^2$ be an irreducible algebraic curve of degree d with $\mathcal{S} = \text{sing}(C)$ its set of singular points. We assume that these are all ordinary double points (also called *nodes*), and that there are exactly $|\mathcal{S}| = \delta$ of these; write $\mathcal{S} = \{p_1, \ldots, p_\delta\}$. Of course, $\delta = 0 \iff \mathcal{S} = \emptyset \iff C$ is smooth.

[2]Note that $\infty \in \mathbb{P}^1$ is not a branch point of f: since $2m$ is even, $\sqrt{\prod_{i=1}^{2m}(x - \alpha_i)}$ has no monodromy about ∞.

Denoting by $\sigma : \tilde{C} \twoheadrightarrow C$ its normalization, we shall deduce from Theorem 14.2.1 the formula:

14.3.1. THEOREM. *$\tilde{C}$ has genus*

$$g = \frac{(d-1)(d-2)}{2} - \delta.$$

To get a feel for this before launching into the proof, for C smooth we have

$$\begin{aligned} d = 1 &\implies g = 0, \\ d = 2 &\implies g = 0, \\ d = 3 &\implies g = 1, \\ d = 4 &\implies g = 3, \end{aligned}$$

and so on. For degree 3 with one node, we get

$$g = \frac{(3-1)(3-2)}{2} - 1 = 0,$$

as we found using stereographic projection. Indeed, we know how to parametrize all three genus 0 cases (smooth $d = 1, 2$; singular $d = 3$) by a Riemann sphere.

The rest of this section is devoted to the proof. Begin by choosing coordinates on $\mathbb{P}^2$ so that

- $L_\infty \cap C$ consists of d distinct points,
- none of the tangents to C at its nodes are vertical (i.e. of the form $X = aZ$), and
- C does not contain $[0 : 0 : 1]$.

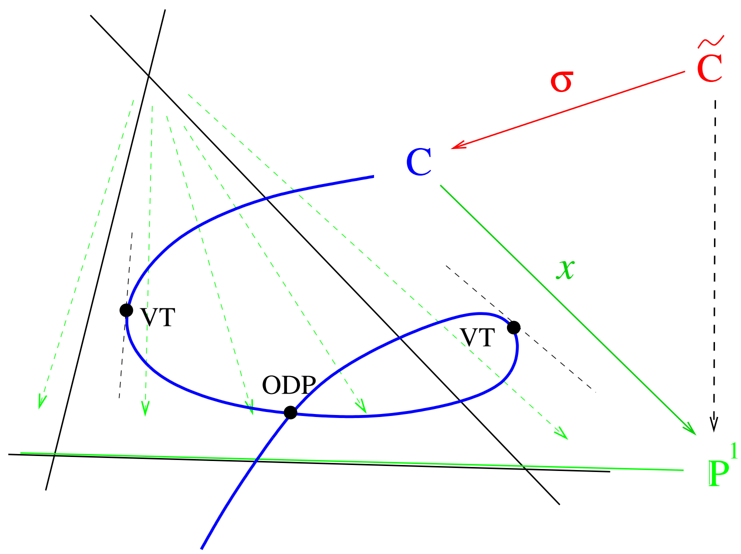
The latter requirement allows us to project from $[0 : 0 : 1]$: that is, the map

$$C \xrightarrow{\mathbf{x}} \mathbb{P}^1 =: M'$$

given by

$$[Z : X : Y] \mapsto [Z : X],$$

roughly speaking the "projection of C to the x-axis", is well-defined. Writing $M := \tilde{C}$, the main idea of the proof of to apply Riemann-Hurwitz to the composition $f = \mathbf{x} \circ \sigma : M \to M'$. In a picture, where "ODP" [resp. "VT"] refers to a node [resp. point with vertical tangent]:



Now for $M' = \mathbb{P}^1$, $g' = 0$ so that Thm. 14.2.1 gives

$$r_f = 2(\text{genus}(M) + \deg(\mathbf{x}) - 1) = 2(g + d - 1). \tag{14.3.2}$$

In particular, the degree of the map $\mathbf{x}$ is d because the projection is done along vertical lines, all but finitely many such lines meet C in d points by Bézout, and σ is 1-to-1 off finitely many points. So we see that if we can compute the degree of the ramification divisor R_f then we are done.

To do this, let

$$E := \{F_Y = 0\}$$

where F_Y is the partial derivative. Obviously $\deg(E) = d - 1$, and so by Bézout,

$$(E \cdot C) = (d-1)d. \tag{14.3.3}$$

Denoting by $\sum_p'$ the sum over points where C has a vertical tangent, and by $\sum_{j=1}^{\delta}$ the sum over nodes, we have

$$(E \cdot C) = \sum_p{}' (E \cdot C)_p + \sum_{j=1}^{\delta} (E \cdot C)_{p_j}.$$

We will show

$$R_f = {\sum_p}' (E \cdot C)_p [\tilde{p}] \tag{14.3.4}$$

where $\tilde{p} = \sigma^{-1}(p) \in \tilde{C}$. (Recall that by our choice of coordinates, a point with vertical tangent cannot be a singular point, and so has a *unique* preimage point under normalization.) Taking degrees of both sides of (14.3.4) gives

$$r_f = {\sum_p}' (E \cdot C)_p = (E \cdot C) - \sum_{j=1}^{\delta} (E \cdot C)_{p_j}. \tag{14.3.5}$$

Further, we will deduce that

$$(E \cdot C)_{p_j} = 2 \quad (\forall j); \tag{14.3.6}$$

together with (14.3.3) and (14.3.4), this yields

$$r_f = d(d-1) - 2\delta.$$

Now put this together with (14.3.2) to get

$$2g + 2(d-1) = d(d-1) - 2\delta,$$

$$2g = (d-2)(d-1) - 2\delta,$$

and divide the last line by 2 to get Theorem 14.3.1. It remains only to check (14.3.4) and (14.3.6).

If C has a vertical tangent at p, then $F(p) = F_Y(p) = 0$, hence $p \in C \cap E$. By assumption, p is a smooth point, so that[3] $F_X(p) \neq 0$. By the holomorphic implicit function theorem, we can parametrize C locally by writing $x = X/Z$ as an implicit function of $y = Y/Z$, viz.

$$0 = F(1, x(y), y).$$

Now, differentiating gives

$$0 = \frac{d}{dy} F(1, x(y), y) = F_X(1, x(y), y) \cdot x'(y) + F_Y(1, x(y), y).$$

[3] If $F_X(p) = 0$ then $F_Z(p) = 0$ too by the Euler formula, contradicting smoothness at p.

For the two functions on the right-hand side to sum to zero, they must have the same order at $y(p)$:

$$\mathrm{ord}_{y(p)} F_Y(1, x(y), y) = \mathrm{ord}_{y(p)} x'(y),$$

in other words

$$\begin{aligned}(E \cdot C)_p &= \{\mathrm{ord}_{y(p)} x(y) \; - \; 1\} \\ &= \{\nu_p(\mathbf{x}) - 1\} \\ &= \{\nu_{\tilde{p}}(f) - 1\}.\end{aligned}$$

As the only ramification points of f are (σ^{-1} of) vertical tangent points,

$$R_f := \sum_{q \in \tilde{C}} (\nu_q(f) - 1)[q] = \sum_p{}' (E \cdot C)_p [\tilde{p}]$$

as claimed.

Finally, to see (14.3.6), assume for simplicity (for some j) $p_j = (0, 0)$. The local affine equation about a node is of the form

$$F(1, x, y) = ax^2 + 2bxy + cy^2 + \{\text{higher-order terms}\}.$$

To find the tangent lines, recall that one solves

$$0 = ax^2 + 2bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \underbrace{\begin{pmatrix} a & b \\ b & c \end{pmatrix}}_{B} \begin{pmatrix} x \\ y \end{pmatrix}$$

in $\mathbb{P}^1$ (for their "slopes"). That the solution Q consists of two distinct points (as p_j is a node) $\implies$ Q is "smooth" $\implies$ $\det B \neq 0 \implies$ $ac - b^2 \neq 0$. That there is no vertical tangent $\implies$ $[x : y] = [0 : 1]$ is not a solution $\implies$ $c \neq 0$. Consider the partial

$$F_Y(1, x, y) = 2bx + 2cy + \{\text{higher-order terms}\}$$

whose vanishing defines E; evidently E can be locally parametrized about p_j by

$$y = y(x) = -\frac{b}{c} x + \{\text{higher-order terms}\}.$$

To compute its intersection number against C, we pull the defining equation of C back along this parametrization and take the order at 0:

$$(E \cdot C)_{(0,0)} = \mathrm{ord}_0(F(1, x, y(x)))$$
$$= \mathrm{ord}_0 \left(ax^2 + 2bx \cdot y(x) + c(y(x))^2 + \{\text{higher-order terms}\} \right)$$
$$= \mathrm{ord}_0 \left(\frac{ac - b^2}{c} x^2 + \{\text{higher-order terms}\} \right)$$
$$= 2\,,$$

Q.E.D.

14.4. Beyond stereographic projection

The genus formula is very nice, but needs to pass a smell test: if it says that a curve $C \in \mathbb{P}^2$ has genus zero normalization, then we should be able to parametrize C by the unique genus zero Riemann surface $\mathbb{P}^1$. We know that this can be done for a smooth conic and a (singular) cubic with one node; the first new case predicted by the formula is that of an irreducible[4] quartic curve ($d = 4$) with 3 nodes ($\delta = 3$):

$$g = \frac{(4-1)(4-2)}{2} - 3 = 0.$$

Let's give this a try. Write $\{p_i\}_{i=0,1,2}$ for the nodes, and suppose another curve D passes through one of these: then by Prop. 12.2.5, $(C \cdot D)_{p_i} \geq 2$. If D is a line, then it cannot pass through all 3 p_i, as then we would have

$$4 = \deg C \cdot \deg L = (C \cdot L) \geq \sum_{i=0}^{2} (C \cdot L)_{p_i} \geq 6,$$

a contradiction. So the nodes are not collinear, and by a similar argument[5] if p_3 is any fixed smooth point of C, then no three of p_0, p_1, p_2, p_3 are collinear. We may therefore move C (and the p_i) by a projectivity of $\mathbb{P}^2$, to have $p_0 = [1 : 0 : 0]$, $p_1 = [0 : 1 : 0]$,

[4]We have to say C is irreducible explicitly, because the union of a smooth cubic and a general line is a quartic with 3 nodes.

[5]in which the 6 gets replaced by a 5 in the above inequality

$p_2 = [0:0:1]$, $p_3 = [1:1:1]$. (We'll do so for this abstract analysis but not for the concrete example that follows.)

The general conic in $\mathbb{P}^2$ is of the form

$$aXY + bYZ + cXZ + dX^2 + eY^2 + fZ^2 = 0.$$

By substitution, we find that the general conic through the above four points is of the form

$$Q_{[a:b]} = \{aXY + bYZ - (a+b)XZ = 0\}.$$

This is a 1-parameter family parametrized by $[a:b] \in \mathbb{P}^1$.

The zero-cycle (cf. Remark 12.2.3(a)) $Q_{[a:b]} \cdot C$ has degree 8 by Bézout, and is of the form $2[p_0] + 2[p_1] + 2[p_2] + [p_3] + \mathit{more}$. This "more" can only be one more point $q_{[a:b]}$ with multiplicity one, since what is already written has degree 7 (and by construction, we don't have negative intersection numbers). Naturally, q could be one of the p_i: if it is p_3, then this would say that Q is tangent to C there. Define a map

$$\sigma : \mathbb{P}^1 \to C$$

by

$$[a:b] \mapsto q_{[a:b]} \ (:= Q_{[a:b]} \cdot C - \{2[p_0] + 2[p_1] + 2[p_2] + [p_3]\}).$$

In fact, this is a morphism of complex manifolds from $\mathbb{P}^1 \to \mathbb{P}^2$ (I won't prove this carefully). Also, since C is irreducible, that σ is onto essentially follows from the open mapping theorem and compactness of $\mathbb{P}^1$.

We claim that σ is 1-to-1 off the singular points of C. Take $q \in C$ distinct from the p_i; since no three of the p_i are collinear, no four of q, p_0, p_1, p_2, p_3 are collinear, so there exists a *unique* conic Q through all five. (The uniqueness when $q = p_3$ then essentially follows from continuity of σ.)

14.4.1. EXAMPLE. So what does such a normalization look like? Take the very concrete quartic curve

$$C = \{X^2Z^2 + Y^2Z^2 + 2X^2Y^2 = 0\}.$$

Irreducibility can be checked by putting the polynomial in affine form $y^2(1+2x^2)+x^2$ and showing it doesn't factor into terms of lower degree in y. As you may check, the only singularities are $p_0=[1:0:0]$, $p_1=[0:1:0]$, and $p_2=[0:0:1]$.

Now pick $p_3:=[i:1:1]$ ($i=\sqrt{-1}$). The general conic through the 4 points $\{p_i\}_{i=0}^3$ is

$$Q_{[\alpha:\beta]}:=\{\alpha XZ+\beta YZ=i(\alpha+\beta)XY\}.$$

Substituting this into α^2 times the equation of C gives

$$\begin{aligned}0&=(i(\alpha+\beta)XY-\beta YZ)^2+\alpha^2Y^2Z^2+2\alpha^2X^2Y^2=\cdots\\&=(\beta^2+\alpha^2)Y^2(Z-iX)\left(Z-i\frac{2\alpha\beta+\beta^2-\alpha^2}{\beta^2+\alpha^2}X\right),\end{aligned}$$

in which the last factor gives us the x $\left(=\frac{X}{Z}\right)$-coordinate of the point $q_{[\alpha:\beta]}$. The y-coordinate is obtained by substituting into the equation of Q, and we find $\sigma([\alpha:\beta])=$

$$\Big[i(2\alpha\beta+\alpha^2-\beta^2)(2\alpha\beta+\beta^2-\alpha^2):(\alpha^2+\beta^2)(2\alpha\beta+\alpha^2-\beta^2)$$
$$:(\alpha^2+\beta^2)(2\alpha\beta+\beta^2-\alpha^2)\Big].$$

Or, in affine coordinates ($t=\frac{\beta}{\alpha}$ in particular),

$$\sigma(t)=\left(-i\frac{1+t^2}{t^2+2t-1},i\frac{1+t^2}{t^2-2t-1}\right).$$

Exercises

(1) Recall the setup of Riemann-Hurwitz: C, C' compact RS with $g=genus(C)$, $g'=genus(C')$, $f\colon C\to C'$ nonconstant holomorphic map of degree d. Show that for any $d\geq 1$, $g\geq g'$. (The covering surface "has at least as many handles".)

(2) Let $z=\frac{Z_1}{Z_0}$ (where $[Z_0:Z_1]$ are the homogeneous coordinates) be the "canonical coordinate" on $\mathbb{P}^1$. If a holomorphic map $f\colon\mathbb{P}^1\to\mathbb{P}^1$ takes the form $f(z)=z^n+a_1z^{n-1}+\ldots+a_n$, then (a) what is $\deg(f)$? (b) What can you say about the ramification divisor R_f?

(at least, what is its degree?) (c) Use Riemann-Hurwitz to check your answers.

(3) Let $C = \mathbb{C}/\Lambda$ where $\Lambda = \{m\lambda_1 + n\lambda_2 \,|\, m, n \in \mathbb{Z}\}$ is a lattice in $\mathbb{C}$. (In particular, λ_1 and λ_2 are independent over $\mathbb{R}$.) Suppose that $\alpha \in \mathbb{C}^*$ satisfies $\alpha\Lambda \subseteq \Lambda$. [Remark: if $\alpha \notin \mathbb{Z}$ this places a strong condition on Λ; we say Λ, or C, "has *complex multiplication* (or *CM*)."] The multiplication by α induces a holomorphic map $\mu_\alpha : C \to C$, i.e. an automorphism of the RS C. (a) Show that the ramification divisor $R \in Div(C)$ for this map is zero. (b) Prove that the degree of μ_α equals the index $[\Lambda : \alpha\Lambda]$ of the image lattice $\alpha \cdot \Lambda \subseteq \Lambda$.

(4) Find the genus of the normalization $\widetilde{C}$ of the irreducible curve C given by taking the closure of $x^2 + x^2y^2 + y^2 = 0$ in $\mathbb{P}^2$. (First convert to homogeneous coordinates and check for singularities. Then apply the genus formula. This is very similar to something above...)

(5) This problem complements (3) above, but you won't use anything from this chapter in doing it. A (holomorphic) map $f : \frac{\mathbb{C}}{\Lambda} \to \frac{\mathbb{C}}{\Lambda}$ of Riemann surfaces is nothing but an analytic map $\tilde{f} : \mathbb{C} \to \mathbb{C}$ (i.e. an entire function) such that for all $\lambda \in \Lambda$, $z \in \mathbb{C}$,

$$(*) \quad \tilde{f}(z+\lambda) - \tilde{f}(z) \ \in \ \Lambda,$$

i.e. $z_1 \equiv z_2 \bmod \Lambda \implies \tilde{f}(z_1) \equiv \tilde{f}(z_2) \bmod \Lambda$ (this is just the well-definedness condition for f). Show that such a map is necessarily affine, i.e. of the form

$$\tilde{f}(z) = \alpha z + \beta.$$

[If α other than $\alpha \in \mathbb{Z}$ works, then we are of course in the CM case described above. So a non-CM complex 1-torus, which is the "generic" case, has endomorphisms of the form $z \mapsto nz + \beta$, $n \in \mathbb{Z}$, and that's all.] Hint: consider $\tilde{f}'(z)$, and use $(*)$.

(6) Let M denote a Riemann surface of genus 4. (a) Can M be embedded as a *smooth* curve in $\mathbb{P}^2$? (b) Compute the genus of a smooth double-cover $\tilde{M}$ of M branched over 6 points.

(7) Let C be a quintic curve ($d = 5$) with three nodes, and no other singularities. (a) Show that C is irreducible and find its genus. [Hint: suppose it was reducible and apply Bézout to the various possibilities.] (b) Show that the nodes cannot be collinear.