CHAPTER 16

Genera of singular curves

Consider an irreducible projective algebraic curve

$$C \subset \mathbb{P}^2$$

of degree d, defined over $\mathbb{C}$. We know how to piece the local normalizations about singular points together with the smooth part of C to construct a Riemann surface $\tilde{C}$, together with a map

$$\sigma : \tilde{C} \to \mathbb{P}^2$$

with $\text{image}(\sigma) = C$. The genus formula of Chapter 14, derived from a generic stereographic projection and the Riemann-Hurwitz formula, said that

$$g(\tilde{C}) = \frac{(d-1)(d-2)}{2} - \delta$$

when all singularities of C (if any) are nodes and there are δ such points.

More generally, we would like to be able to compute the genus of the normalization of an *arbitrary* irreducible curve, with singularities of any order and type. (This is called the *geometric genus* of C.) It is true that the answer is always $\frac{1}{2}(d-1)(d-2)$ minus contributions from each singularity depending only on its local "type"; and there are combinatorial formulas for those contributions in a large class of cases (going far beyond ADE). One special case is that of an ordinary k-tuple point (cf. §6.4), each one of which subtracts $\frac{1}{2}k(k-1)$ from the geometric genus; see Exercise 6 below (or Fulton's book on algebraic curves). Our preference here, however, is for methods over formulas, particularly as the methods allow you to treat other cases and use what you've just learned.

In what follows I will introduce two methods. The first one computes the divisor of the pullback of a meromorphic differential 1-form on $\mathbb{P}^2$ to $\tilde{C}$ and applies Poincaré-Hopf. The second is based on projecting C to a line and applying Riemann-Hurwitz, as in the proof of the genus formula. Rather than stating them formally, I'll use both methods to treat an example which is "sufficiently general" that you'll be able to adapt the approaches to any other curve.

So here is the ugly curve we will study: put

$$F(Z,X,Y) := X^3Z^3 + X^6 + Y^5Z,$$

and

$$C := \{F(Z,X,Y) = 0\} \subset \mathbb{P}^2,$$

with affine form $x^3 + x^6 + y^5 = 0$. This is a degree 6 (i.e. sextic) curve; a *smooth* curve of this degree has genus 10. That will not be the answer here.

One immediately obvious singularity is at $(0,0)$ (i.e. $[1{:}0{:}0]$ in projective coordinates $[Z : X : Y]$). The lowest-order homogeneous term (of the affine equation, in coordinates vanishing at this point) is x^3. So $[1{:}0{:}0]$ is a triple point, but very definitely *not* an *ordinary* triple point of C. Ugly enough? Well, this turns out to be the only singularity.

At the end I will mention briefly (without proof) a simple combinatorial method which often works when the singularities are only at $[1{:}0{:}0]$, $[0{:}1{:}0]$, and $[0{:}0{:}1]$. It is thus less general than the formulas alluded to above, but easy to apply, and gives a nice advertisement for an area called *toric geometry*.

16.1. Method I: Poincaré-Hopf

Set $\omega = \sigma^*\left(\frac{dx}{y}\right) \in \mathcal{K}^1(\tilde{C})^*$ (it will actually turn out to be in $\Omega^1(\tilde{C})$, although this is inessential for the method). Poincaré-Hopf tells us that $\deg((\omega)) = 2g-2$, where $g = g(\tilde{C})$. So we have to compute $(\omega) = \sum m_i[p_i] \in \mathrm{Div}(\tilde{C})$. Where might these $\{p_i\}$ lie in $\tilde{C}$? Or rather, where might the $\{\sigma(p_i)\}$ lie on C? There are four (not necessarily disjoint) possibilities:

(1) on the intersections of C with the x-axis, i.e. in $C \cap \{Y = 0\}$;
(2) at points where C has a vertical tangent, hence in $C \cap \{F_Y = 0\}$;
(3) at singularities of C, i.e. in $\mathrm{Sing}(C)$; and
(4) on the line at infinity, i.e. in $C \cap \{Z = 0\}$.

Why might one expect nontrivial $\nu_p(\omega)$ at p in one of these sets? For (1), the denominator of $\frac{dx}{y}$ is zero on the line $Y = 0$; while for (2) the pullback of dx will be zero, since at such a point the curve has no "horizontal variation" to first order. You should always be suspicious of (3) and (4). Conversely: on the smooth affine part of C, dx and y never blow up, and (1) and (2) are the only ways they can develop a zero. So (1)-(4) are actually the *only* places where ω can have a zero or pole.

Now we go through these 4 sets of points for the particular curve under consideration.

(1): We look at the affine equaton and set $y = 0$, which yields $x^3 + x^6 = 0$, hence $x = 0$, ζ_6, $\overline{\zeta_6}$, or -1. (Here, $\zeta_6 = \exp(\frac{\pi\sqrt{-1}}{3})$.) While $(0,0)$ is a singular point and will be dealt with below, it is clear that $\frac{dx}{y}|_C$ will behave in the same way near the remaining three points: $(-1,0)$, $(\zeta_6,0)$, and $(\overline{\zeta_6},0)$. We look in a neighborhood of $(-1,0)$ on C. Setting $\chi = x + 1$, the equation becomes in (χ, y):

$$0 = y^5 + (\chi - 1)^3 + (\chi - 1)^6 = y^5 - 3\chi + \{\text{higher-order terms in } \chi\}$$
$$= y^5 - 3\chi h(\chi),$$

where $h(0) \neq 0$.[1] The local normalization of C at $(\chi, y) = (0,0)$ is therefore $t \mapsto \left(t^5,\, t \cdot \sqrt[5]{3h(t^5)}\right)$, under which $\frac{dx}{y} = \frac{d\chi}{y}$ pulls back to $\frac{d(t^5)}{t \cdot \sqrt[5]{3h(t^5)}} = t^3 \cdot \frac{5dt}{\sqrt[5]{3h(t^5)}}$ which has a zero of order 3 at $t = 0$. So we conclude that

$$\nu_{\sigma^{-1}[1:-1:0]}(\omega) = 3,$$

and similarly that $\nu_{\sigma^{-1}[1:\zeta_6:0]}(\omega) = \nu_{\sigma^{-1}[1:\overline{\zeta_6}:0]}(\omega) = 3$.

[1]Sometimes (though rarely) one may have to "remember" more about h in these types of problems, but not in this example.

(2): For vertical tangents or singularities we will have

$$0 = F_Y = 5Y^4 Z,$$

so that these must occur along the x-axis or along the line at ∞. The intersections with the x-axis other than $[1 : 0 : 0]$ were just dealt with. Any nonsingular intersections with $\{Z = 0\}$ will be dealt with in step (4). So vertical tangents are subsumed under the other three categories.

(3): At a singular point we must have $0 = F_Y$,

$$0 = F_X = 3X^2 Z^3 + 6X^5 = 3X^2(Z^3 + 2X^3),$$

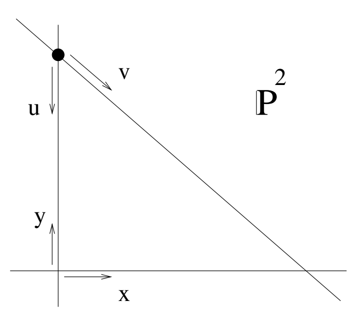
and

$$0 = F_Z = 3X^3 Z^2 + Y^5.$$

We must have $Z = 0$ or $Y = 0$. If $Z = 0$ then the last two equations imply $X = Y = 0$, a contradiction. If $Y = 0$ then the last equation gives $Z = 0$ (no!) or $X = 0$; the latter works, and so $[1 : 0 : 0]$ is the only singular point. In local coordinates about $(x, y) = (0, 0)$, our curve is $0 = y^5 + x^3 + x^6 = y^5 + x^3 h(x)$ (different $h(x)$ from above, again $h(0) \neq 0$), which is *locally* irreducible and has a singularity of order 3. Under the local normalization $t \mapsto (t^5,\, t^3 \cdot \sqrt[5]{h(t^5)})$, $\frac{dx}{y}$ pulls back to $\frac{d(t^5)}{t^3 \cdot \sqrt[5]{h(t^5)}} = t \cdot \frac{5dt}{\sqrt[5]{h(t^5)}}$, and we conclude that

$$v_{\sigma^{-1}[1:0:0]}(\omega) = 1.$$

(4): $C \cap \{Z = 0\}$ is the single point $[0 : 0 : 1]$. We will need to switch to affine coordinates vanishing at this point, namely $u = \frac{1}{y} = \frac{Z}{Y}$, $v = \frac{x}{y} = \frac{X}{Y}$ (or conversely $y = \frac{1}{u}$, $x = \frac{v}{u}$):



We divide $Z^3X^3 + X^6 + Y^5Z = 0$ by Y^6, obtaining

$$\left(\frac{Z}{Y}\right)^3\left(\frac{X}{Y}\right)^3 + \left(\frac{X}{Y}\right)^6 + \frac{Z}{Y} = 0$$

$$v^6 + u^3v^3 + u = 0$$

which is a locally irreducible Weierstrass polynomial in v with (multivalued) roots of the form

$$v_*(u) := \sqrt[3]{\frac{-u^3 \pm \sqrt{u^6 - 4u}}{2}}.$$

(Use the quadratic formula to solve for v^3, then take cube root.) Substituting in t^6 gives

$$\tilde{v}(t^6) = \sqrt[3]{\frac{-t^{18} + \sqrt{t^{36} - 4t^6}}{2}} = \sqrt[3]{\frac{-t^{18} + t^3 \cdot \sqrt{t^{30} - 4}}{2}}$$

$$= t \cdot \sqrt[3]{\frac{-t^{15} + \sqrt{t^{30} - 4}}{2}}$$

which is just t times some local analytic $H(t)$ with $H(0) \neq 0$. So the normalization is $t \mapsto (t^6, t \cdot H(t))$ and $\frac{dx}{y} = \frac{d\left(\frac{v}{u}\right)}{\frac{1}{u}}$ pulls back to $\frac{d\left(\frac{tH(t)}{t^6}\right)}{\frac{1}{t^6}} = t^6 d\left(\frac{H(t)}{t^5}\right) = (tH'(t) - 5H(t))\,dt$, which has neither zero nor pole at $t = 0$. Hence,

$$\nu_{\sigma^{-1}[0:0:1]}(\omega) = 0.$$

Upshot: Putting everything together, the divisor of ω on $\tilde{C}$ is

$$(\omega) = [\sigma^{-1}[1:0:0]] + 3[\sigma^{-1}[1:\zeta_6:0]] + 3[\sigma^{-1}[1:-1:0]] + 3[\sigma^{-1}[1:\overline{\zeta_6}:0]].$$

Taking degrees on both sides (and invoking Poincaré-Hopf) gives

$$2g - 2 \;=\; \deg((\omega)) \;=\; 1 + 3 + 3 + 3 \;=\; 10,$$

whence the geometric genus g is 6.

16.2. Method II: Riemann-Hurwitz

Recall that this dealt with *maps* of Riemann surfaces

$$f : M \to N,$$

and told us that

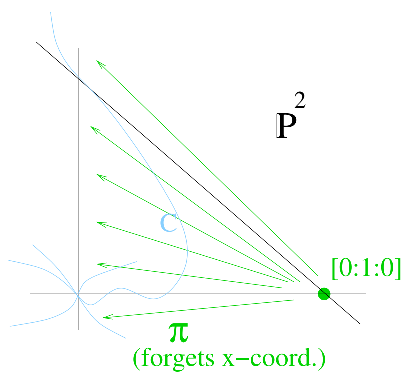
$$\chi_M = \deg(f) \cdot \chi_N - r_f.$$

Here $\deg(f)$ is the mapping degree of f (the number of points in the preimage of a general point on N) and r_f is the degree of the ramification divisor[2] $R_f := \sum_{p \in M} (\nu_p(f) - 1)[p]$.

Now let $q \in \mathbb{P}^2 \backslash C$, $M = \tilde{C}$, $N = \mathbb{P}^1$, $\pi =$stereographic projection $(\mathbb{P}^2 \backslash \{q\}) \to \mathbb{P}^1$ through q; and take

$$f : \tilde{C} \to \mathbb{P}^1$$

to be given by $f := \pi \circ \sigma$. Usually it is easiest to take $[1 : 0 : 0]$, $[0 : 1 : 0]$, or $[0 : 0 : 1]$ as q. In our case the only one of these *not* on C is $[0 : 1 : 0]$. So our projection looks like



and the mapping degree is the number of intersection points of $\{y = y_0\}$ and $\{x^3 + x^6 + y^5 = 0\}$ for general y_0 — i.e. $\deg(f) = 6$. Obviously $\chi_{\mathbb{P}^1} = 2 - 2 \cdot 0 = 2$, so we have

$$2 - 2g = \chi_{\tilde{C}} = 6 \cdot 2 - r_f = 12 - r_f$$

$$\implies \quad g = \tfrac{1}{2} r_f - 5.$$

[2]Remember that about any point $p \in M$ and its image $f(p) \in N$, one has local holomorphic coodinates z resp. w (with $z(p) = 0$ resp. $w(f(p)) = 0$), in which f takes the form $z \mapsto z^{\nu_p(f)} = w$.

So we will have to compute $R_f = \sum m_i[p_i]$ (or at least r_f) and the first issue to resolve is where the $\sigma(p_i)$ can lie on C:

(1) points having horizontal tangents (subset of $\{F = 0\} \cap \{F_X = 0\}$);
(2) singular points ($F_X = F_Y = F_Z = 0$) — i.e. $[1 : 0 : 0]$ for our example; and
(3) $L_\infty \cap C$ — i.e. $[0 : 0 : 1]$ in our example.

(1): $0 = F_X = 3X^2(Z^3 + 2X^3)$ has solutions other than $X = 0$, which corresponds to the singular point. Namely, writing $x = \frac{X}{Z}$ we get $x^3 + \frac{1}{2} = 0$ hence $x = \frac{-1}{\sqrt[3]{2}}, \frac{\zeta_6}{\sqrt[3]{2}}, \frac{\overline{\zeta_6}}{\sqrt[3]{2}}$. Plugging this into the affine equation of C yields $y^5 = \frac{1}{4}$ hence $y = \frac{1}{\sqrt[5]{4}}, \frac{\zeta_5}{\sqrt[5]{4}}, \frac{\zeta_5^2}{\sqrt[5]{4}}, \frac{\zeta_5^3}{\sqrt[5]{4}}, \frac{\zeta_5^4}{\sqrt[5]{4}}$. These are independent of the choice amongst the 3 values for x, and so we get $5 \cdot 3 = 15$ ramification points. As you may check, the intersections between $F = 0$ and $F_X = 0$ at these points are all of first order, hence correspond to ramifications of order 2 and so make a contribution of $\nu_p(f) - 1 = 2 - 1 = 1$ each to r_f.

(2): Near $(x, y) = (0, 0)$, the composition

$$t \overset{\sigma}{\longmapsto} (t^5, t^3 h(t)) \overset{\pi}{\longmapsto} t^3 h(t)$$

has $\nu_p(f) = 3$ hence contributes 2 to r_f.

(3): Near $(u, v) = (0, 0)$,

$$t \overset{\sigma}{\longmapsto} \underbrace{(t^6, tH(t))}_{u,v} \overset{\pi}{\longmapsto} t^6.$$

This is because π is supposed to take the y-coordinate, which is $\frac{1}{u}$ here; but we have to compute the image in a holomorphic coordinate vanishing at the image of $p = [0 : 0 : 1]$. So in fact u is the correct variable, and the map indeed has $\nu_p(f) = 6$ and contributes 5 to r_f.

Conclusion: $r_f = 15 \cdot 1 + 2 + 5 = 22$

$$\implies \quad g \;=\; \frac{22}{2} - 5 \;=\; 6,$$

confirming the previous computation.

16.3. A teaser on toric geometry

Back in §2.1, we discussed how $\mathbb{P}^n$ compactifies $\mathbb{C}^n$ for each n; a bit more generally, we can ask about compactifications of the "algebraic torus" $(\mathbb{C}^*)^n$.[3] For $n = 1$, $\mathbb{P}^1$ is still the only answer, but already for $n = 2$, infinitely many possibilities arise simply from drawing compact, convex polygons $\Delta \subset \mathbb{R}^2$ with vertices in $\mathbb{Z}^2$. I won't carefully define the resulting (compact) *toric surfaces* $\mathbb{P}_\Delta$, but here are some of their properties:

- $\mathbb{P}_\Delta$ contains $\mathbb{C}^* \times \mathbb{C}^*$ (with coords. (x, y)) as a dense open subset;
- the action (by coordinatewise multiplication) of $\mathbb{C}^* \times \mathbb{C}^*$ on itself extends to an action on $\mathbb{P}_\Delta$;
- the "boundary" $\mathbb{P}_\Delta \setminus (\mathbb{C}^* \times \mathbb{C}^*)$ is a chain of $\mathbb{P}^1$'s, connected "head to tail", in 1-to-1 correspondence with the edges of Δ;
- if the shortest integer vector *along* an edge is (a, b), then $x^a y^b$ is a coordinate on the corresponding $\mathbb{P}^1$;
- there is a bijection between vertices of the polygon and fixed points of the torus action (which is also where the $\mathbb{P}^1$'s meet, and are the only possible singularities[4] of $\mathbb{P}_\Delta$);

. . . and here are some examples:

- if Δ is any rectangle with vertical/horizontal edges, then $\mathbb{P}_\Delta \cong \mathbb{P}^1 \times \mathbb{P}^1$; and
- if Δ is a triangle with vertices $(0, 0)$, $(d, 0)$, and $(0, d)$ (for some $d \in \mathbb{Z}_{>0}$), then $\mathbb{P}_\Delta \cong \mathbb{P}^2$.

The main point is that, given a "toric" curve $C^* \subset \mathbb{C}^* \times \mathbb{C}^*$ defined by a polynomial equation $F(x, y) = 0$, we can consider its closure not just in $\mathbb{P}^2$ but in any $\mathbb{P}_\Delta$. Why would we want to do this, and which Δ should we choose?

---

[3] We require that they be *normal*, i.e. have local coordinate rings integrally closed in their fraction fields, but don't insist on smoothness.

[4] whatever "singular" means, since I haven't presented $\mathbb{P}_\Delta$ via equations — though this can be done, inside some larger $\mathbb{P}^N$. If the two integral "edge vectors" emanating from a vertex are (a_i, b_i), then the point is smooth iff $|a_1 b_2 - a_2 b_1| = 1$.

Suppose that the closure $C \subset \mathbb{P}^2$ is smooth outside of [1:0:0], [0:1:0], and [0:0:1]. Then it may be that singularities at these points are a result of $\mathbb{P}^2$ being a "non-optimal choice" for compactifying C^*. In that case, we should let the polynomial $F = \sum_{(m,n)\in\mathfrak{M}} c_{m,n}x^m y^n$ guide our choice of Δ. Here[5] $\mathfrak{M} \subset \mathbb{Z}^2$ is some finite subset (on which $c_{m,n} \neq 0$), and we take Δ to be its convex hull, called the *Newton polygon*[6] of F. With this choice, let C' denote the closure of C^* in $\mathbb{P}_\Delta$.

Now decorate the integer points (m, n) of the Newton polygon Δ with the coefficients of $x^m y^n$ in F (some of which *may* be zero). For each edge of Δ, define an *edge polynomial* by writing $a_0 + a_1 z + \cdots + a_m z^m$ with $a_0, a_1, \ldots, a_m$ the coefficients along that edge, from vertex to vertex (in either order).

16.3.1. DEFINITION. F is *nondegenerate* if no edge polynomial has a repeated root and $F_x = F_y = F = 0$ has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$.

The main point, which we can now state (but not prove), is:

16.3.2. PROPOSITION. *If C^* is defined by a nondegenerate polynomial, then C' is smooth, and its genus is the number of integer points in the* interior *of Δ:*

$$g = |\mathbb{Z}^2 \cap (\Delta \setminus \partial\Delta)|.$$

In this case, C' normalizes C, and so g is the geometric genus of C. Why should you believe this?

16.3.3. EXAMPLE. Suppose F is nondegenerate of degree d, with Newton polytope Δ the triangle with vertices $(0, 0)$, $(d, 0)$, $(0, d)$. Then C is smooth of genus $\frac{1}{2}(d-1)(d-2)$ by the genus formula, which also happens to be the number of integer points in the interior of Δ. We will later use this fact to produce an explicit basis for $\Omega^1(C)$, and it is this construction which generalizes to prove Prop. 16.3.2.

[5]Of course, we have $\mathfrak{M} \subset \mathbb{Z}^2_{\geq 0}$ for F a *polynomial*; but in the toric world one tends to work with *Laurent polynomials* (allowing negative exponents), as C^* is unaffected by multiplying its equation by negative (or positive) powers of x and y.

[6]This is different from the version considered in Exercise (3) of Chapter 10, which was noncompact.

If if you like this combinatorial side of algebraic geometry, I highly recommend Fulton's book on toric varieties.

Exercises

(1) Find the geometric genus of the curve $C \subset \mathbb{P}^2$ with affine equation $x^3 + y^2 + y^2x^3 - \frac{2}{5}y^5 = 0$. Do this in 2 different ways: (a) by using an appropriate projection, computing the degree of the ramification divisor, and applying Riemann-Hurwitz; (b) by computing the divisor of the pull-back of a meromorphic 1-form on the normalization and applying Poincaré-Hopf (try it with $\frac{dy}{x}$).[7]
(2) Consider the curve $C = \{YX^2Z^2 - Y^3X^2 + Z^5 = 0\} \subset \mathbb{P}^2$. (a) Find all singularities of C. (b) What are their orders and types? (c) Compute the tangent lines and their multiplicities. (d) Use Method I or II to find the geometric genus (you may check the answer with a Newton polygon).
(3) Find the geometric genus of $C = \{Y^7 - X^5Z^2 + X^5Y^2 = 0\} \subset \mathbb{P}^2$ by computing the divisor of the pullback of $\frac{dy}{x}$ (i.e. by Method I).
(4) Use the Newton polygon to give a third "proof" that the geometric genus of the curve defined by $x^3 + x^6 + y^5$ is 6.
(5) Find the genus of the curve $y^2 - \prod_{i=1}^{d}(x - \alpha_i)$ for each $d > 0$ (already seen in Example 14.2.3 for even d) using Newton polygons.
(6) Show that if p is an ordinary k-tuple point (with no vertical tangents) on an irreducible curve $C = \{F = 0\}$, and $E = \{F_Y = 0\}$, then we have $(E \cdot C)_p = k(k-1)$. [Hint: if a polynomial f in one variable has only simple roots, then it shares no roots with f'.] Modify the proof of the genus formula to verify that each such point subtracts $\frac{1}{2}k(k-1)$ from the geometric genus of C.

[7]Hint: to do the local normalizations, first make sure you are dealing with an irreducible Weierstass polynomial $f(x,y) = 0$ – you may have to change variable, swap coordinates, divide out by a unit (which can reduce the degree of the equation!), factor into irreducibles, whatever. If you can't find a multivalued solution $y(x)$ by taking roots, using quadratic equation, and so on, you can always use power series. If $f(x,y)$ is an irreducible Weierstrass polynomial of degree k in y, then try substituting in t^k for x: write $0 = f(t^k, y)$ and solve for y as a power-series in t, call this $G(t)$. Then the local normalization is $t \mapsto (t^k, G(t))$.