CHAPTER 18

Putting a nonsingular cubic in standard form

An irreducible algebraic curve $E \subset \mathbb{P}^2$ is an elliptic curve if the genus of its normalization $\bar{E}$ is 1 (topologically it looks like a donut). By the genus formula, all smooth cubic curves are elliptic. In the next two chapters we will show not only that such a curve is isomorphic to $C/\Lambda$ for some lattice $\Lambda$, but will get a description of $\Lambda$ which shows its dependence on $E$. This is important, since for two different lattices $\Lambda = \mathbb{Z} \langle \alpha, \beta \rangle$ and $\Lambda' = \mathbb{Z} \langle \alpha', \beta' \rangle$, the complex 1-tori $C/\Lambda$ and $C/\Lambda'$ need not be isomorphic as Riemann surfaces. (More precisely, they are isomorphic if and only if $[\alpha : \beta]$ is carried to $[\alpha' : \beta']$ by an integral projectivity, i.e. a transformation of $\mathbb{P}^1$ induced by $A \in PSL_2(\mathbb{Z})$.)

Even more significant is how we do this: by putting $E$ in Weierstrass form, integrating a holomorphic form on it to get a map to a complex torus, and showing that the Weierstrass $\wp$-function and its derivative invert this map. To put $E$ in this form, a choice of flex is required. What is that?

18.1. Flexes

Let $C = \{F(Z, X, Y) = 0\} \subset \mathbb{P}^2$ be an irreducible algebraic curve of degree $d \geq 3$. One way of thinking of the tangent line at a nonsingular point $p \in C$ is as the unique line satisfying $(C \cdot T_p C)_p \geq 2$.

18.1.1. Definition. A smooth point $p \in C$ is called a flex if the intersection multiplicity

$$(C \cdot T_p C)_p \geq 3.$$
Intuitively these are the inflection points of $C$, and can be seen to correspond to cusps of the dual curve $\tilde{C}$ (see §4.4). Since $\tilde{C}$ has finitely many singularities, this gives one proof that there are finitely many flexes; we will however take a different approach.

Denoting partial derivatives by subscript, e.g. $F_{ZX} := \frac{\partial^2 F}{\partial Z \partial X}$, the Hessian of $F$ is the polynomial matrix

$$\text{Hess}_F = \begin{pmatrix} F_{ZZ} & F_{ZX} & F_{ZY} \\ F_{XZ} & F_{XX} & F_{XY} \\ F_{YZ} & F_{YX} & F_{YY} \end{pmatrix}.$$ 

Its determinant

$$H := \det(\text{Hess}_F)$$

is clearly a homogeneous polynomial of degree $3(d-2)$. Call $\mathcal{H}_C := \{H(Z, X, Y) = 0\} \subset \mathbb{P}^2$ the Hessian curve associated to $C$.

**18.1.2. Lemma.** Let $p \in C$ be a smooth point. Then $p$ is a flex $\iff p \in \mathcal{H}_C$.

**Proof.** Since intersection numbers are invariant under projectivities, we may assume $p = [1 : 0 : 0]$, $T_p C = \{Y = 0\}$. In affine coordinates, writing $f(x, y) := F(1, x, y)$, this means that the curve $\{f(x, y) = 0\} \subset \mathbb{C}^2$ contains $(0, 0)$ and is tangent to $\{y = 0\}$. So $f(0, 0) = 0$ and $(f_x(0, 0), f_y(0, 0)) = (0, \lambda)$ where $\lambda \neq 0$, so that

$$f(x, y) = \lambda y + (ax^2 + 2bxy + cy^2) + \text{higher-order terms}.$$ 

Parametrizing $T_p C$ by $t \mapsto (t, 0)$, we have

$$(C \cdot T_p C)_p = \text{ord}_0 (f(t, 0)) = \text{ord}_0 (at^2 + \text{h.o.t.}),$$

which is $\geq 3$ (yielding a flex) if and only if $a = 0$.

Now the above form of $f$ implies

$$F(Z, X, Y) = \lambda Y Z^{d-1} + (aX^2 + 2bXY + cY^2) Z^{d-2} + \cdots$$
so that

\[
Hess_F(1, 0, 0) = \begin{pmatrix}
0 & 0 & (d - 1)\lambda \\
0 & 2a & 2b \\
(d - 1)\lambda & 2b & 2c
\end{pmatrix}.
\]

Taking the determinant,

\[
H(p) = \det(\text{Hess}_F(p)) = -2(d - 1)^2\lambda^2 a.
\]

This is clearly zero (i.e. \( p \in \mathcal{H}_C \)) if and only if \( a = 0 \). \qed

Now Bezout guarantees intersections of \( C \) and \( \mathcal{H}_C \). If \( C \) is singular then these might all be at singular points, so that there may be no flexes (though this isn’t typical: see the exercises). On the other hand, if \( C \) is smooth then by Lemma 18.1.2 we do have flexes. Refining this observation:

18.1.3. **Proposition.** On a nonsingular curve \( C \) of degree \( d \geq 3 \), there exists at least one and at most \( 3d(d - 2) \) flexes.

**Proof.** By Bezout,

\[
\sum_{p \in C \cap \mathcal{H}_C} (C \cdot \mathcal{H}_C)_p = (C \cdot \mathcal{H}_C) = \deg(C) \cdot \deg(\mathcal{H}_C) = d \cdot 3(d - 2).
\]

So the number of points in \( C \cap \mathcal{H}_C \) is between 1 and \( 3d(d - 2) \), all points are smooth points, and we apply Lemma 18.1.2. \qed

18.1.4. **Remark.** Since \( \text{Hess}_F \) is just the multivariable derivative (Jacobian matrix) of \( D_C : \mathbb{P}^2 \to \mathbb{P}^2 \) (§4.4), the intersections of \( C \) and \( \mathcal{H}_C \) may be viewed as degeneracies of the map \( D_C|_C : C \to \tilde{\mathbb{C}} \). This is what gives rise to the cusps in \( \tilde{\mathbb{C}} \) referred to above.

18.1.5. **Definition.** The multiplicity of a flex \( p \in C \) is defined to be \( (C \cdot \mathcal{H}_C)_p \).

Now take \( C = E \) to be a smooth elliptic curve \( (d = 3) \). Then in the proof of Lemma 18.1.2, the precise form of the homogeneous polynomial is \( (F(Z, X, Y) =) \)

\[
\lambda Y Z^2 + (a X^2 + 2b XY + c Y^2) Z + aX^3 + \beta X^2 Y + \gamma XY^2 + \delta Y^3.
\]
Assume \( a = 0 \) so that we have a flex at \([1 : 0 : 0]\). (Note that \( \alpha \) must then be nonzero, in order that \( Y \) not divide \( F \) — which would make \( E \) reducible hence singular.) Then a short computation gives

\[
Hess_F(1, x, y) = \begin{pmatrix}
2\lambda y & 2by & 2bx + 2cy + 2\lambda \\
2by & 6ax + 2\beta y & 2\beta x + 2\gamma y + 2b \\
2bx + 2cy + 2\lambda & 2\beta x + 2\gamma y + 2b & 2\gamma x + 6\delta y + 2c
\end{pmatrix}.
\]

Pull this back to \( T_pE = \{ y = 0 \} \) by making the substitution

\[
Hess_F(1, t, 0) = \begin{pmatrix}
0 & 0 & 2bt + 2\lambda \\
0 & 6\alpha t & 2\beta t + 2b \\
2bt + 2\lambda & 2\beta t + 2b & 2\gamma t + 2c
\end{pmatrix};
\]

this has determinant

\[
H(1, t, 0) = -(2\lambda + 2bt)^2 6\alpha t,
\]

and since \( \alpha, \lambda \neq 0 \)

\[
(T_pE \cdot \mathcal{H}_E)_p = \text{ord}_0(H(1, t, 0)) = 1.
\]

So \( \mathcal{H}_E \) is smooth at \( p \) and \( T_pE \) is not its tangent line. But then it intersects \( E \) transversely (since they have distinct tangent lines), so that \( (E \cdot \mathcal{H}_E)_p = 1 \). This computation is valid at any flex of \( E \) (after a projective change of coordinates, of course), and so leads to:

18.1.7. Proposition. Any smooth cubic has 9 flexes, each of multiplicity one.

Proof. Since \( \text{deg}(\mathcal{H}_E) = 3(d - 2) = 3 \), Bezout gives us 9 intersection points of \( \mathcal{H}_E \) and \( E \), counted with multiplicity; and we have demonstrated that the multiplicities are all 1. \( \square \)
18.2. Weierstrass form

Consider an arbitrary smooth cubic curve
\[ E = \{ F(Z, X, Y) = 0 \} \subset \mathbb{P}^2. \]

In this section we will show that there exists a projective transformation putting \( E \) uniquely into a convenient form. (Alternately, you can view this as the existence of new projective coordinates in terms of which the equation of \( E \) takes said form, which is actually how the proof will go.)

We know \( E \) has a flex, and first of all we can choose coordinates so that this is at \( [0 : 0 : 1] =: \mathcal{O} \) with \( T_{\mathcal{O}} E = \{ Z = 0 \} \). To get the general equation of such a curve: take (18.1.6), set \( a = 0 \) (for a flex), swap \( Z \) and \( Y \), and (without loss of generality since \( \lambda \neq 0 \)) normalize \( \lambda \) to 1; this gives
\[ F(Z, X, Y) = ZY^2 + (2bXZ + cZ^2)Y + aX^3 + \beta X^2 Z + \gamma XZ^2 + \delta Z^3, \]
with affine form
\[ f(x, y) := F(1, x, y) = y^2 + yf_2(x) + f_3(x). \]

Now the discriminant
\[
\mathcal{D}_y(f(x, y)) = \mathcal{R}_y(y^2 + yf_2(x) + f_3(x), 2y + f_2(x))
\]
\[ = \det \begin{pmatrix} 1 & f_2 & f_3 \\ 2 & f_2 & 0 \\ 0 & 2 & f_2 \end{pmatrix} = \det \begin{pmatrix} 1 & f_2 & f_3 \\ -f_2 & -2f_3 \\ 2 & f_2 \end{pmatrix}
\]
\[ = -f_2^2 + 4f_3 = -(2bx + c)^2 + 4(\alpha x^3 + \beta x^2 + \gamma x + \delta) \]
is a polynomial in \( x \) of degree 3 since \( a \neq 0 \). Roots of \( (\mathcal{D}_y(f))(x) \) correspond to vertical lines \( x = x_0 \) which are tangent to (the affine part of) \( E \) at some point. Bezout tells us that the intersection number there can only be 2, since \( \deg(E) = 3 \) and \( \{ X = x_0Z \} \) already meets \( E \) at \( \mathcal{O} \). Such “first order” tangencies mean the roots each have multiplicity one. Therefore \( E \) has three vertical tangents (apart from \( L_\infty = \{ Z = 0 \} \)), at \( p_1, p_2, p_3. \)
18.2.1. LEMMA. The \( \{p_i\}_{i=1}^3 \) are collinear.

PROOF. In the picture

\[
\begin{align*}
E \cap T &= \{p_3\} \quad \text{and} \\
L_{p_1 p_2} &= \{q\} \\
\end{align*}
\]

define \( p \) to be the third intersection point of \( L_{p_1 p_2} \) and \( E \), and \( q \) the third intersection point of \( L_{O \cdot p} \) with \( E \). Consider (in addition to \( E \)) the cubic curves \( C_1 = L_{O \cdot p_1} + L_{O \cdot p_2} + L_{O \cdot p} \) and \( C_2 = T_{O \cdot E} + 2L_{p_1 p_2} \). We have

\[
E \cdot C_1 = 3O + 2p_1 + 2p_2 + p + q
\]

and

\[
E \cdot C_2 = 3O + 2p_1 + 2p_2 + 2p.
\]

Arguing as in §15.2, the ratio of the homogeneous polynomials defining \( C_1 \) and \( C_2 \) gives a degree 1 map \( E \to \mathbb{P}^1 \) (which is impossible) if \( p \neq q \). So \( p = q \), and \( L_{O \cdot p} \) is tangent to \( E \) at \( p \). It follows that \( p \) is \( p_1 \), \( p_2 \), or \( p_3 \). The first two are impossible since the tangent to \( p_1 \) doesn’t pass through \( p_2 \) and vice versa; so \( p = p_3 \). Hence \( p_1, p_2, p_3 \in L_{p_1 p_2} \).

Now stereographic projection from \( O \) to \( L_{p_1 p_2} (\cong \mathbb{P}^1) \) presents \( E \) as a \( 2 : 1 \) cover of \( \mathbb{P}^1 \) branched over \( p_1, p_2, p_3 \), and the image \( T_{O \cdot E} \cap L_{p_1 p_2} \) of \( O \). Furthermore \( L_{p_1 p_2}, L_{O \cdot p_1}, T_{O \cdot E} \) form a triangle, and so we can choose new projective coordinates \( X', Y', Z' \) in order that \( L_{p_1 p_2} = \{Y' = 0\}, L_{O \cdot p_1} = \{X' = 0\}, \) and \( T_{O \cdot E} = \{Z' = 0\} \). For simplicity I’ll drop the primes and just write \( X, Y, Z \) for this new coordinate system.
The following picture summarizes what we know:

where (on $Y = 0$) $p_1$ is at $\frac{X}{Z} = 0$. Write $\alpha_1$ (resp. $\alpha_2$) for the value of $\frac{X}{Z}$ at $p_2$ (resp. $p_3$).

We would like an equation corresponding to this picture. Now, in the new coordinate system, the equation of $E$ is still of the form

$$F(Z, X, Y) = ZY^2 + (2bXZ + cZ^2)Y + aX^3 + \beta X^2Z + \gamma XZ^2 + \delta Z^3,$$

because we still have a flex at $[0 : 0 : 1]$ with tangent line $Z = 0$. But now (referring to the picture) also $[1 : 0 : 0] \in E$, which implies $\delta = 0$. Moreover, $F_Y(= 2YZ + 2bXZ + cZ^2) = 0$ at $p_1 = [1 : 0 : 0]$, $p_2 = [1 : \alpha_1 : 0]$, and $p_3 = [1 : \alpha_2 : 0]$ since the tangents are vertical there. This yields $c = 0$, then $2b\alpha_1 = 2b\alpha_2 = 0$. As the $\{p_i\}$ are distinct (so $\alpha_i \neq 0$), we have $b = 0$, and

$$F(Z, X, Y) = Y^2Z + X(\alpha X^2 + \beta XZ + \gamma Z^2)$$

$$= Y^2Z + \alpha X(X - \alpha_1 Z)(X - \alpha_2 Z).$$

Now define new coordinates by the projective transformation

$$X = \sqrt[3]{\frac{4}{\alpha}} X_0 + \frac{\alpha_1 + \alpha_2}{3} Z_0, \quad Y = iY_0, \quad Z = Z_0,$$
which makes the equation
\[
\tilde{F}(Z_0, X_0, Y_0) = F \left( \sqrt[3]{\frac{4}{3} X_0 + \frac{\alpha_1 + \alpha_2}{3} Z_0}, iY_0, Z_0 \right)
\]
\[= -Y_0^2 Z_0 + 4X_0^3 - g_2 X_0 Z_0^2 - g_3 Z_0^3.\]

Dropping the subscript 0’s and taking the affine equation, we have put \(E\) in Weierstrass form:

18.2.2. Proposition. (a) Any smooth cubic \(E \subset \mathbb{P}^2\) is projectively equivalent to a curve with affine equation of the form

\[
y^2 = 4x^3 - g_2 x - g_3.
\]

(18.2.3)

(b) For a given \(E\), this form is unique up to a change of the form \((g_2, g_3) \mapsto (\xi^4 g_2, \xi^6 g_3)\) where \(\xi \in \mathbb{C}^*\); in particular,

\[j := \frac{g_3^3}{g_2^3 - 27g_3^2} \in \mathbb{C}\]

is an invariant of \(E\).

Proof. We have just seen (a). To show (b), write the projective equation \(Y^2Z = 4X^3 - g_2 XZ^2 - g_3 Z^3\). It is not difficult to check that any projective linear transformation fixing \(O\) and preserving the form of this equation (up to rescaling) takes the form \(X = \varepsilon X_0, Y = \eta Y_0, Z = \xi^3 Z_0\). Taking \(\xi := \frac{\xi}{\eta}\) gives exactly the claimed effect on \((g_2, g_3)\), and \(j\) is unchanged by this transformation.

What about a projectivity which sends a different flex \(O'\) to \([0:0:1]\), “replacing” \(O\)? (As the equation says \((L_\infty \cdot E)[0:0:1] = 3\, \text{we must have a flex there.}\) As with \(O\), we have again four tangent lines \((\{L_{O'} p_i(= T_{p_i} E)\})_{i=1}^3\) and \(T_{O'} E\) through \(O'\), which can be regarded as 4 points in \(\mathbb{P}^1\). The Weierstrass forms will be equivalent in the sense just described if (for some ordering of the \(p_i\) resp. \(p'_i\)) the cross-ratios of these point-configurations are the same for \(O\) and \(O'\).

In fact, there are 4 tangent lines to \(E\) through any non-flex as well (use a discriminant as above), and so we get a continuous algebraic
map from \( E \) to unordered 4-tuples of distinct points on \( \mathbb{P}^1 \). By passing to a finite unbranched cover of \( E \), we get a map to ordered 4-tuples. Since the cross-ratio of 4 distinct points lies in \( \mathbb{C}^* \), this gives a nonvanishing holomorphic function on \( E \), which is constant by Liouville. In particular, it takes equal values on all 9 flexes.

Note that the vanishing of the \( x^2 \) term on the right-hand side of (18.2.3) indicates that its roots sum to zero.

**Exercises**

1. Show that the cubic curve
   \[
   C = \{ 0 = X^3 + Y^3 - XY(X + Y + Z) \} \subset \mathbb{P}^2
   \]
   has one singular point (a node) and exactly three collinear flexes. [Hint: start by computing the Hessian, then find the Hessian curve and determine its intersections with \( C \).]

2. (i) Fill in the computational details in the first paragraph of the proof of Prop. 18.2.2. (ii) Check that the coordinate change just before Prop. 18.2.2 eliminates the \( X^2Z \) term as claimed.

3. Prove that through every non-flex of a smooth cubic \( E \) there are 4 distinct tangent lines to \( E \).

4. Put the Fermat curve \( X^3 + Y^3 = Z^3 \) in Weierstrass form and calculate its \( j \)-invariant.

5. Consider the irreducible quintic curve \( C = \{ X^5 + YZ^4 = 0 \} \subset \mathbb{P}^2 \). (a) Without doing any computation, put an upper bound on the number of flexes. (b) Find all singularities of \( C \). (c) Find all flexes of \( C \) and their multiplicities.