CHAPTER 18

Putting a nonsingular cubic in standard form

An irreducible algebraic curve $E \subset \mathbb{P}^2$ is an *elliptic curve* if the genus of its normalization $\tilde{E}$ is 1 (topologically it looks like a donut). By the genus formula, all *smooth cubic curves* are elliptic. In the next two chapters we will show not only that such a curve is isomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ, but will get a description of Λ which shows its dependence on E. This is important, since for two different lattices $\Lambda = \mathbb{Z}\langle \alpha, \beta \rangle$ and $\Lambda' = \mathbb{Z}\langle \alpha', \beta' \rangle$, the complex 1-tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ need not be isomorphic as Riemann surfaces. (More precisely, they are isomorphic if and only if $[\alpha : \beta]$ is carried to $[\alpha' : \beta']$ by an integral projectivity, i.e. a transformation of $\mathbb{P}^1$ induced by $A \in PSL_2(\mathbb{Z})$.)

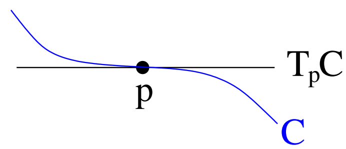
Even more significant is how we do this: by putting E in Weierstrass form, integrating a holomorphic form on it to get a map to a complex torus, and showing that the Weierstrass $\wp$-function and its derivative invert this map. To put E in this form, a choice of *flex* is required. What is that?

18.1. Flexes

Let $C = \{F(Z, X, Y) = 0\} \subset \mathbb{P}^2$ be an irreducible algebraic curve of degree $d \geq 3$. One way of thinking of the tangent line at a nonsingular point $p \in C$ is as the unique line satisfying $(C \cdot T_pC)_p \geq 2$.

18.1.1. DEFINITION. A smooth point $p \in C$ is called a *flex* if the intersection multiplicity

$$(C \cdot T_pC)_p \geq 3.$$



Intuitively these are the inflection points of C, and can be seen to correspond to cusps of the dual curve $\check{C}$ (see §4.4). Since $\check{C}$ has finitely many singularities, this gives one proof that there are finitely many flexes; we will however take a different approach.

Denoting partial derivatives by subscript, e.g. $F_{ZX} := \frac{\partial^2 F}{\partial Z \partial X}$, the *Hessian* of F is the polynomial matrix

$$Hess_F = \begin{pmatrix} F_{ZZ} & F_{ZX} & F_{ZY} \\ F_{XZ} & F_{XX} & F_{XY} \\ F_{YZ} & F_{YX} & F_{YY} \end{pmatrix}.$$

Its determinant

$$H := \det(Hess_F)$$

is clearly a homogeneous polynomial of degree $3(d-2)$. Call $\mathcal{H}_C := \{H(Z,X,Y) = 0\} \subset \mathbb{P}^2$ the *Hessian curve* associated to C.

18.1.2. LEMMA. *Let $p \in C$ be a smooth point. Then p is a flex $\iff$ $p \in \mathcal{H}_C$.*

PROOF. Since intersection numbers are invariant under projectivities, we may assume $p = [1:0:0]$, $T_pC = \{Y = 0\}$. In affine coordinates, writing $f(x,y) := F(1,x,y)$, this means that the curve $\{f(x,y) = 0\} \subset \mathbb{C}^2$ contains $(0,0)$ and is tangent to $\{y = 0\}$. So $f(0,0) = 0$ and $(f_x(0,0), f_y(0,0)) = (0,\lambda)$ where $\lambda \neq 0$, so that

$$f(x,y) = \lambda y + (ax^2 + 2bxy + cy^2) + \text{higher-order terms}.$$

Parametrizing T_pC by $t \mapsto (t,0)$, we have

$$(C \cdot T_pC)_p = \text{ord}_0\,(f(t,0)) = \text{ord}_0(at^2 + \text{h.o.t.}),$$

which is ≥ 3 (yielding a flex) if and only if $a = 0$.

Now the above form of f implies

$$F(Z,X,Y) = \lambda Y Z^{d-1} + (aX^2 + 2bXY + cY^2)Z^{d-2} + \cdots$$

so that

$$Hess_F(1,0,0) = \begin{pmatrix} 0 & 0 & (d-1)\lambda \\ 0 & 2a & 2b \\ (d-1)\lambda & 2b & 2c \end{pmatrix}.$$

Taking the determinant,

$$H(p) = \det(Hess_F(p)) = -2(d-1)^2\lambda^2 a.$$

This is clearly zero (i.e. $p \in \mathcal{H}_C$) if and only if $a = 0$. □

Now Bezout guarantees intersections of C and $\mathcal{H}_C$. If C is singular then these might all be at singular points, so that there may be no flexes (though this isn't typical: see the exercises). On the other hand, if C is smooth then by Lemma 18.1.2 we *do* have flexes. Refining this observation:

18.1.3. PROPOSITION. *On a nonsingular curve C of degree $d \geq 3$, there exists at least one and at most $3d(d-2)$ flexes.*

PROOF. By Bezout,

$$\sum_{p \in C \cap \mathcal{H}_C} (C \cdot \mathcal{H}_C)_p = (C \cdot \mathcal{H}_C) = \deg(C) \cdot \deg(\mathcal{H}_C) = d \cdot 3(d-2).$$

So the number of points in $C \cap \mathcal{H}_C$ is between 1 and $3d(d-2)$, all points are smooth points, and we apply Lemma 18.1.2. □

18.1.4. REMARK. Since $Hess_F$ is just the multivariable derivative (Jacobian matrix) of $\mathcal{D}_C : \mathbb{P}^2 \to \mathbb{P}^2$ (§4.4), the intersections of C and $\mathcal{H}_C$ may be viewed as degeneracies of the map $\mathcal{D}_C|_C : C \twoheadrightarrow \check{C}$. This is what gives rise to the cusps in $\check{C}$ referred to above.

18.1.5. DEFINITION. The *multiplicity* of a flex $p \in C$ is defined to be $(C \cdot \mathcal{H}_C)_p$.

Now take $C = E$ to be a smooth elliptic curve ($d = 3$). Then in the proof of Lemma 18.1.2, the precise form of the homogeneous polynomial is $(F(Z,X,Y) =)$

(18.1.6)

$$\lambda Y Z^2 + (aX^2 + 2bXY + cY^2)Z + \alpha X^3 + \beta X^2 Y + \gamma X Y^2 + \delta Y^3.$$

Assume $a = 0$ so that we have a flex at $[1:0:0]$. (Note that α must then be nonzero, in order that Y not divide F — which would make E reducible hence singular.) Then a short computation gives

$$Hess_F(1,x,y) = \begin{pmatrix} 2\lambda y & 2by & 2bx+2cy+2\lambda \\ 2by & 6\alpha x + 2\beta y & 2\beta x + 2\gamma y + 2b \\ 2bx+2cy+2\lambda & 2\beta x + 2\gamma y + 2b & 2\gamma x + 6\delta y + 2c \end{pmatrix}.$$

Pull this back to $T_pE = \{y = 0\}$ by making the substitution

$$Hess_F(1,t,0) = \begin{pmatrix} 0 & 0 & 2bt + 2\lambda \\ 0 & 6\alpha t & 2\beta t + 2b \\ 2bt+2\lambda & 2\beta t + 2b & 2\gamma t + 2c \end{pmatrix};$$

this has determinant

$$H(1,t,0) = -(2\lambda + 2bt)^2 6\alpha t,$$

and since $\alpha, \lambda \neq 0$

$$(T_pE \cdot \mathcal{H}_E)_p = \text{ord}_0(H(1,t,0)) = 1.$$

So $\mathcal{H}_E$ is smooth at p and T_pE is *not* its tangent line. But then it intersects E transversely (since they have distinct tangent lines), so that $(E \cdot \mathcal{H}_E)_p = 1$. This computation is valid at any flex of E (after a projective change of coordinates, of course), and so leads to:

18.1.7. PROPOSITION. *Any smooth cubic has 9 flexes, each of multiplicity one.*

PROOF. Since $\deg(\mathcal{H}_E) = 3(d-2) = 3$, Bezout gives us 9 intersection points of $\mathcal{H}_E$ and E, counted with multiplicity; and we have demonstrated that the multiplicities are all 1. $\square$

18.2. Weierstrass form

Consider an arbitrary smooth cubic curve

$$E = \{F(Z, X, Y) = 0\} \subset \mathbb{P}^2.$$

In this section we will show that there exists a projective transformation putting E uniquely into a convenient form. (Alternately, you can view this as the existence of new projective coordinates in terms of which the equation of E takes said form, which is actually how the proof will go.)

We know E has a flex, and first of all we can choose coordinates so that this is at $[0 : 0 : 1] =: \mathcal{O}$ with $T_{\mathcal{O}}E = \{Z = 0\}$. To get the general equation of such a cuve: take (18.1.6), set $a = 0$ (for a flex), swap Z and Y, and (without loss of generality since $\lambda \neq 0$) normalize λ to 1; this gives

$$F(Z, X, Y) = ZY^2 + (2bXZ + cZ^2)Y + \alpha X^3 + \beta X^2 Z + \gamma XZ^2 + \delta Z^3,$$

with affine form

$$f(x, y) := F(1, x, y) = y^2 + yf_2(x) + f_3(x).$$

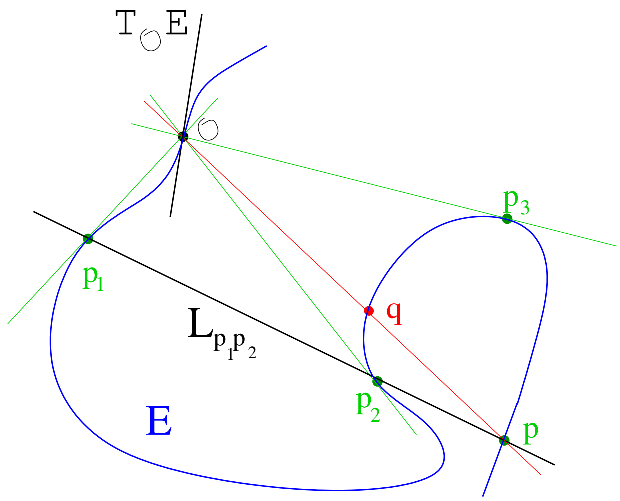
Now the discriminant

$$\begin{aligned} \mathcal{D}_y(f(x,y)) &= \mathcal{R}_y(y^2 + yf_2(x) + f_3(x), 2y + f_2(x)) \\ &= \det \begin{pmatrix} 1 & f_2 & f_3 \\ 2 & f_2 & 0 \\ 0 & 2 & f_2 \end{pmatrix} = \det \begin{pmatrix} 1 & f_2 & f_3 \\ & -f_2 & -2f_3 \\ & 2 & f_2 \end{pmatrix} \\ &= -f_2^2 + 4f_3 = -(2bx + c)^2 + 4(\alpha x^3 + \beta x^2 + \gamma x + \delta) \end{aligned}$$

is a polynomial in x of degree 3 since $\alpha \neq 0$. Roots of $(\mathcal{D}_y(f))(x)$ correspond to vertical lines $x = x_0$ which are tangent to (the affine part of) E at some point. Bezout tells us that the intersection number there can only be 2, since $\deg(E) = 3$ and $\{X = x_0 Z\}$ already meets E at $\mathcal{O}$. Such "first order" tangencies mean the roots each have multiplicity one. Therefore E has three vertical tangents (apart from $L_\infty = \{Z = 0\}$), at p_1, p_2, p_3.

18.2.1. LEMMA. *The* $\{p_i\}_{i=1}^3$ *are collinear.*

PROOF. In the picture



define p to be the third intersection point of $L_{p_1p_2}$ and E, and q the third intersection point of $L_{\mathcal{O}p}$ with E. Consider (in addition to E) the cubic curves $C_1 = L_{\mathcal{O}p_1} + L_{\mathcal{O}p_2} + L_{\mathcal{O}p}$ and $C_2 = T_{\mathcal{O}}E + 2L_{p_1p_2}$. We have

$$E \cdot C_1 = 3\mathcal{O} + 2p_1 + 2p_2 + p + q$$

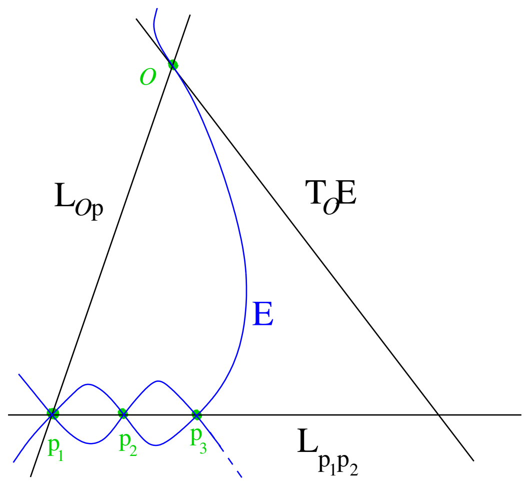
and

$$E \cdot C_2 = 3\mathcal{O} + 2p_1 + 2p_2 + 2p.$$

Arguing as in §15.2, the ratio of the homogeneous polynomials defining C_1 and C_2 gives a degree 1 map $E \to \mathbb{P}^1$ (which is impossible) if $p \neq q$. So $p = q$, and $L_{\mathcal{O}p}$ is tangent to E at p. It follows that p is p_1, p_2, or p_3. The first two are impossible since the tangent to p_1 doesn't pass through p_2 and vice versa; so $p = p_3$. Hence $p_1, p_2, p_3 \in L_{p_1p_2}$. □

Now stereographic projection from $\mathcal{O}$ to $L_{p_1p_2}(\cong \mathbb{P}^1)$ presents E as a $2:1$ cover of $\mathbb{P}^1$ branched over p_1, p_2, p_3, and the image $T_{\mathcal{O}}E \cap L_{p_1p_2}$ of $\mathcal{O}$. Furthermore $L_{p_1p_2}$, $L_{\mathcal{O}p_1}$, $T_{\mathcal{O}}E$ form a triangle, and so we can choose new projective coordinates X', Y', Z' in order that $L_{p_1p_2} = \{Y' = 0\}$, $L_{\mathcal{O}p_1} = \{X' = 0\}$, and $T_{\mathcal{O}}E = \{Z' = 0\}$. For simplicity I'll drop the primes and just write X, Y, Z for this *new* coordinate system.

The following picture summarizes what we know:



where (on $Y = 0$) p_1 is at $\frac{X}{Z} = 0$. Write α_1 (resp. α_2) for the value of $\frac{X}{Z}$ at p_2 (resp. p_3).

We would like an equation corresponding to this picture. Now, in the new coordinate system, the equation of E is still of the form

$$F(Z, X, Y) = ZY^2 + (2bXZ + cZ^2)Y + \alpha X^3 + \beta X^2 Z + \gamma XZ^2 + \delta Z^3,$$

because we still have a flex at $[0 : 0 : 1]$ with tangent line $Z = 0$. But now (referring to the picture) also $[1 : 0 : 0] \in E$, which implies $\delta = 0$. Moreover, $F_Y (= 2YZ + 2bXZ + cZ^2) = 0$ at $p_1 = [1 : 0 : 0]$, $p_2 = [1 : \alpha_1 : 0]$, and $p_3 = [1 : \alpha_2 : 0]$ since the tangents are vertical there. This yields $c = 0$, then $2b\alpha_1 = 2b\alpha_2 = 0$. As the $\{p_i\}$ are distinct (so $\alpha_i \neq 0$), we have $b = 0$, and

$$\begin{aligned} F(Z, X, Y) &= Y^2 Z + X(\alpha X^2 + \beta XZ + \gamma Z^2) \\ &= Y^2 Z + \alpha X(X - \alpha_1 Z)(X - \alpha_2 Z). \end{aligned}$$

Now define new coordinates by the projective transformation

$$X = \sqrt[3]{\frac{4}{\alpha}} X_0 + \frac{\alpha_1 + \alpha_2}{3} Z_0 \, , \quad Y = iY_0 \, , \quad Z = Z_0 \, ,$$

which makes the equation

$$\begin{aligned}\tilde{F}(Z_0, X_0, Y_0) &= F\left(\sqrt[3]{\frac{4}{\alpha}}X_0 + \frac{\alpha_1+\alpha_2}{3}Z_0,\ iY_0,\ Z_0\right)\\ &= -Y_0^2Z_0 + 4X_0^3 - g_2X_0Z_0^2 - g_3Z_0^3.\end{aligned}$$

Dropping the subscript 0's and taking the affine equation, we have put E in *Weierstrass form*:

18.2.2. PROPOSITION. (a) *Any smooth cubic* $E \subset \mathbb{P}^2$ *is projectively equivalent to a curve with affine equation of the form*

$$y^2 = 4x^3 - g_2x - g_3. \tag{18.2.3}$$

(b) *For a given* E*, this form is unique up to a change of the form* $(g_2, g_3) \mapsto (\xi^4 g_2, \xi^6 g_3)$ *where* $\xi \in \mathbb{C}^*$*; in particular,*

$$j := \frac{g_2^3}{g_2^3 - 27g_3^2} \in \mathbb{C}$$

is an invariant of E*.*

PROOF. We have just seen (a). To show (b), write the projective equation $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$. It is not difficult to check that any projective linear transformation fixing $\mathcal{O}$ and preserving the form of this equation (up to rescaling) takes the form $X = \varepsilon X_0$, $Y = \eta Y_0$, $Z = \frac{\varepsilon^3}{\eta^2}Z_0$. Taking $\xi := \frac{\varepsilon}{\eta}$ gives exactly the claimed effect on (g_2, g_3), and j is unchanged by this transformation.

What about a projectivity which sends a different flex $\mathcal{O}'$ to [0:0:1], "replacing" $\mathcal{O}$? (As the equation says $(L_\infty \cdot E)_{[0:0:1]} = 3$, we must have a flex there.) As with $\mathcal{O}$, we have again four tangent lines ($\{L_{\mathcal{O}'p_i'}(= T_{p_i'}E)\}_{i=1}^3$ and $T_{\mathcal{O}'}E$) through $\mathcal{O}'$, which can be regarded as 4 points in a $\mathbb{P}^1$. The Weierstrass forms will be equivalent in the sense just described if (for some ordering of the p_i resp. p_i') the cross-ratios of these point-configurations are the same for $\mathcal{O}$ and $\mathcal{O}'$.

In fact, there are 4 tangent lines to E through any non-flex as well (use a discriminant as above), and so we get a continuous algebraic map from E to unordered 4-tuples of distinct points on $\mathbb{P}^1$. By passing to a finite unbranched cover of E, we get a map to ordered 4-tuples. Since the cross-ratio of 4 distinct points lies in $\mathbb{C}^*$, this gives a nonvanishing holomorphic function on E, which is constant by Liouville. In particular, it takes equal values on all 9 flexes. □

Note that the vanishing of the x^2 term on the right-hand side of (18.2.3) indicates that its roots sum to zero.

Exercises

(1) Show that the cubic curve

$$C = \{0 = X^3 + Y^3 - XY(X + Y + Z)\} \subset \mathbb{P}^2$$

has one singular point (a node) and exactly three collinear flexes. [Hint: start by computing the Hessian, then find the Hessian curve and determine its intersections with C.]

(2) (i) Fill in the computational details in the first paragraph of the proof of Prop. 18.2.2. (ii) Check that the coordinate change just before Prop. 18.2.2 eliminates the X^2Z term as claimed.

(3) Prove that through every non-flex of a smooth cubic E there are 4 distinct tangent lines to E.

(4) Put the Fermat curve $X^3 + Y^3 = Z^3$ in Weierstrass form and calculate its j-invariant.

(5) Consider the irreducible quintic curve $C = \{X^5 + YZ^4 = 0\} \subset \mathbb{P}^2$. (a) Without doing any computation, put an upper bound on the number of flexes. (b) Find all singularities of C. (c) Find all flexes of C and their multiplicities.