CHAPTER 19

Canonical normalization of the Weierstrass cubic

This chapter will focus on the precise relationship between the Weierstrass-form elliptic curves and complex 1-tori (or equivalently, 2-lattices in $\mathbb{C}$). We will begin by associating to a Weierstrass cubic E a "period lattice" Λ_E, and to a (full) lattice Λ a Weierstrass cubic E_Λ. These will ultimately be shown to be bijections of sets and mutual inverses. The key step is the inversion of the Weierstrass $\wp$-function and its derivative (embedding a 1-torus in $\mathbb{P}^2$) by the *Abel map* $u : E \to \mathbb{C}/\Lambda_E$. This map is closely related to the *elliptic integral*

$$\int_\infty^* \frac{dx}{\pm\sqrt{4x^3 - g_2 x - g_3}},$$

a variant of which will be studied in the exercises.

19.1. Holomorphic forms on an elliptic curve

Let E be a Weierstrass cubic, viz., the projective closure of

$$f(x,y) := y^2 - Q(x) = 0$$

in $\mathbb{P}^2$, where

$$Q(x) = 4(x-e_1)(x-e_2)(x-e_3)\ , \qquad e_1 + e_2 + e_3 = 0.$$

19.1.1. CLAIM. $\omega := \left.\frac{dx}{y}\right|_E \in \Omega^1(E)$ is nowhere vanishing.

19.1.2. REMARK. This statement perhaps requires clarification. You may interpret $\left.\frac{dx}{y}\right|_E$ in either of two equivalent ways:

(a) any algebraic differential form (such as $\frac{dx}{y}$) on $\mathbb{C}^2$ extends to a meromorphic form on $\mathbb{P}^2$, and you can think of $|_E$ as shorthand for pullback to E (rather than introducing $\sigma : E \hookrightarrow \mathbb{P}^2$ just to write $\sigma^*\frac{dx}{y}$);

(b) alternatively, writing $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$ exhibits x and y as meromorphic functions on $\mathbb{P}^2$ (and hence, via pullback, on E), and Example 13.1.4 tells us that $\frac{d(x|_E)}{y|_E}$ is a meromorphic 1-form.

Either way, we have $\omega \in \mathcal{K}^1(E)$; and part of the content of the Claim is that ω is holomorphic: $\nu_p(\omega) \geq 0$ for all $p \in E$. The "nowhere vanishing" statement says that actually $\nu_p(\omega) = 0$ for all p.

PROOF OF 19.1.1. Look at the affine part $E \backslash \mathcal{O}$. Wherever $f_y \neq 0$, so that x gives a local coordinate, $\left.\frac{dx}{y}\right|_E$ is holomorphic and nonvanishing. We have $f = 0$ and $f_y = 0$ precisely at the three points $\{(e_i, 0)\}_{i=1,2,3}$, where $f_x = Q'(e_i) \neq 0$ so that y is a local coordinate. On E we have $0 = df = 2ydy - Q'(x)dx$ so that $\left.\frac{dx}{y}\right|_E =$

$$2\left.\frac{dy}{Q'(x)}\right|_E ,$$

which is evidently nonvanishing and holomorphic in a neighborhood of each $(e_i, 0)$.

What about the (flex) point at infinity $\mathcal{O} = [0{:}0{:}1]$? By Poincaré-Hopf, $g = 1 \implies \sum_{p \in E} \nu_p(\omega) = 2g - 2 = 0$, so that if $\nu_p(\omega) = 0$ for all $p \in E \backslash \mathcal{O}$, there can be no contribution from $\mathcal{O}$ either. □

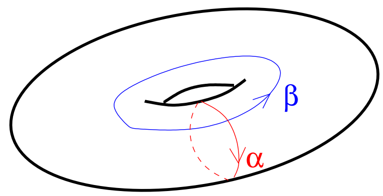
19.1.3. COROLLARY. $\Omega^1(E) = \mathbb{C}\langle \omega \rangle$. *That is, every holomorphic 1-form on E is a multiple of* ω.

PROOF. For any $\omega_0 \in \Omega^1(E)$, the discussion preceding Example 13.1.6 tells us $\frac{\omega_0}{\omega} \in \mathcal{K}(E)$. But since ω is nowhere vanishing, $\frac{\omega_0}{\omega}$ is actually a *holomorphic* function. Now use Liouville's theorem ($\mathcal{O}(E) \cong \mathbb{C}$). □

Among the standard *topological* invariants of a 1-manifold M is its first homology group. An *ad hoc* definition is

$$H_1(M, \mathbb{Z}) := \frac{\left\{ \begin{array}{c} \text{free abelian group generated by} \\ \text{closed piecewise-}C^\infty \text{ paths on } M \end{array} \right\}}{\left\{ \begin{array}{c} \text{subgroup generated by} \\ \text{boundaries of finitely triangulable regions} \end{array} \right\}},$$

or simply "cycles modulo boundaries". From the picture



it isn't hard to convince yourself that

$$H_1(E, \mathbb{Z}) \cong \mathbb{Z}\langle \alpha, \beta \rangle .$$

That is, for any closed C^∞ path $\gamma \subset E$, there exists a closed set $\Gamma \subset E$ (with boundary $\partial\Gamma$) such that

$$\gamma = m\alpha + n\beta + \partial\Gamma.$$

The integers m, n are uniquely determined by γ. One then has

$$\begin{aligned} \int_\gamma \omega &= \int_{\partial\Gamma} \omega + m\int_\alpha \omega + n\int_\beta \omega \\ &= m\int_\alpha \omega + n\int_\beta \omega \end{aligned}$$

by Cauchy's theorem (Prop. 13.1.9). The values of the integrals $\int_\gamma \omega$ over cycles are called the *periods* of ω, and we define the *period lattice*

$$\Lambda_E := \mathbb{Z}\left\langle \int_\alpha \omega, \int_\beta \omega \right\rangle \subset \mathbb{C}.$$

This furnishes an invariant of the complex structure[1] on E which, unlike the topological invariant, actually distinguishes elliptic curves which are non-isomorphic as complex manifolds (or algebraic curves).

19.1.4. REMARK. Given a lattice of the form $\mathbb{Z}\langle \lambda_1, \lambda_2 \rangle =: \Lambda \subset \mathbb{C}$ (with λ_1, λ_2 $\mathbb{R}$-linearly independent), we have a Weierstrass $\mathcal{P}$-map

$$\begin{aligned} \mathbb{C}/\Lambda &\xrightarrow{\mathcal{P}} \mathbb{P}^2 \\ u &\longmapsto [1 : \wp(u) : \wp'(u)] \end{aligned}$$

[1]Algebraic geometers call this a *transcendental* invariant, to distinguish it from algebraic ones.

whose image (by Exercise 5 of Chap. 7) is a Weierstrass cubic! Define

$$E_\Lambda := \mathcal{P}(\mathbb{C}/\Lambda),$$

which we henceforth consider to be the range of the map $\mathcal{P}$. Obviously it is of interest to find out whether *all* Weierstrass cubics arise in this fashion (as E_Λ's).

Before moving on we should note that $\mathcal{P}$ is injective. Its composition with (the x-coordinate projection) $x\colon E_\Lambda \to \mathbb{P}^1$ has degree 2 since $\wp$ has a unique pole on $\mathbb{C}/\Lambda$ (at 0), which is a double pole. But mapping degrees of Riemann surfaces multiply under composition, and the degree of x itself is 2; so that of $\mathcal{P}\colon \mathbb{C}/\Lambda \to E_\Lambda$ must be 1.

19.2. The Abel map

Let

$$E = \overline{\{y^2 = \underbrace{4x^3 - g_2 x - g_3}_{Q(x)}\}} \subset \mathbb{P}^2$$

be a Weierstrass cubic with $\omega = \left.\frac{dx}{y}\right|_E \in \Omega^1(E)$. Integrating this gives a (holomorphic) map of Riemann surfaces

$$\begin{aligned} u \,:\, E &\longrightarrow \mathbb{C}/\Lambda_E \\ p &\longmapsto \int_{\mathcal{O}}^{p} \omega \end{aligned}$$

where the integration is over any C^∞ path from $\mathcal{O}$ to p. This *Abel map* is well-defined: if γ', γ'' are two such paths, then their difference is closed and so

$$\gamma' - \gamma'' = \partial\Gamma + m\alpha + n\beta.$$

Integrating, we have

$$\int_{\gamma'} \omega - \int_{\gamma''} \omega \;=\; m\int_\alpha \omega + n\int_\beta \omega \;\in\; \Lambda_E.$$

A "baby" version of Abel's theorem for elliptic curves[2] is then:

19.2.1. THEOREM. *The Abel map is injective.*

[2]We will meet the grownup version, which is phrased in the language of divisors, in Chapter 21.

Abel's theorem is usually paired with something called "Jacobi inversion", the baby version of which is:

19.2.2. PROPOSITION. *The Abel map u is surjective* (*and thus an isomorphism*).

PROOF. Since $\omega \neq 0$, u is nonconstant; since u is also holomorphic, $u(E)$ is open in $\mathbb{C}/\Lambda_E$. Moreover, as u is continuous and E compact, $u(E)$ is compact (hence closed). Since $\mathbb{C}/\Lambda_E$ is connected, this forces u to be onto (and also $\mathbb{C}/\Lambda_E$ to be compact).[3] □

19.2.3. REMARK. Since $\mathbb{C}/\Lambda_E$ is compact, Λ_E must be a lattice of rank 2: that is, $\int_\alpha \omega$ and $\int_\beta \omega$ are linearly independent over $\mathbb{R}$.[4]

SKETCH OF PROOF FOR 19.2.1. Suppose $u(p) \equiv u(q) \bmod \Lambda_E$ for $p \neq q$ points of E; then

$$\int_q^p \omega = \int_{\mathcal{O}}^p \omega - \int_{\mathcal{O}}^q \omega = u(p) - u(q) \in \Lambda_E.$$

Modifying the path from q to p by $m\alpha + n\beta$ (for some $m, n \in \mathbb{Z}$), we get

$$\int_p^q \omega = 0.$$

Dirichlet's existence theorem (which we won't prove, but follows from the theory of Green's functions on Riemann surfaces in complex analysis) guarantees the existence of $\eta_0 \in \mathcal{K}^1(E)$ with only simple poles, only at p and q, with

$$Res_p(\eta_0) = -Res_q(\eta_0) = 1.$$

This is true for any two (distinct) points p and q, and has nothing to do with our assumption (that $u(p) = u(q)$). Now referring to the

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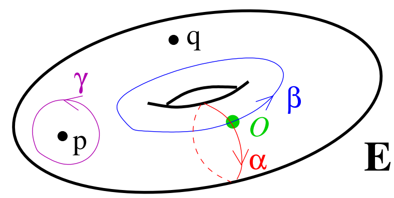
[3]One should say something about the ugly possibility of $\int_\alpha \omega, \int_\beta \omega \in \mathbb{C}$ being linearly dependent over $\mathbb{R}$ but not $\mathbb{Q}$, in which case $\mathbb{C}/\Lambda_E$ would not be Hausdorff. Here one can just quotient by the real span of the periods to get a map to $\mathbb{R}$ and apply the same argument; since $\mathbb{R}$ is noncompact, this case doesn't occur.

[4]If $\Lambda_E = \{0\}$ (rank 0), then $\mathbb{C}/\Lambda_E = \mathbb{C}$. If $\Lambda_E \cong \mathbb{Z}$ (rank 1), then $\mathbb{C}/\Lambda_E \cong \mathbb{C}/2\pi\sqrt{-1}\mathbb{Z} \overset{\cong}{\to} \mathbb{C}^*$ (by taking exp). Both are noncompact so can't occur. The other possibility was ruled out in the previous footnote.

picture



we have

$$H_1\left(E\backslash\{p,q\},\,\mathbb{Z}\right)\cong\mathbb{Z}\left\langle \alpha,\beta,\gamma\right\rangle \tag{19.2.4}$$

where

$$\int_\gamma \eta_0 = 2\pi i.$$

Next, "normalize" η_0, putting

$$\eta := \eta_0 - \left(\frac{\int_\alpha \eta_0}{\int_\alpha \omega}\right)\omega,$$

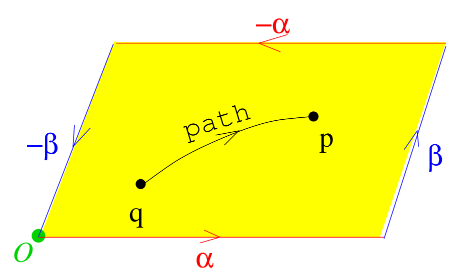
which has the same residues as η_0. Observe that

$$\int_\gamma \eta = 2\pi i,$$

while

$$\int_\alpha \eta = 0.$$

Cutting open the above figure along α and β yields the fundamental domain $\mathfrak{F}$ (the yellow region):[5]



[5]In order to accomodate the path from q to p, it may be necessary to "dilate" $\mathfrak{F}$ by an integer factor M. (You can think of this as the fundamental domain of a finite unbranched covering of E.) This doesn't really affect the proof, except for replacing $\partial\mathfrak{F}$ by $M\alpha + M\beta - M\alpha - M\beta$.

On the interior of $\mathfrak{F}$, $\mathfrak{U} := \int_{\mathcal{O}}^{*} \omega$ gives a holomorphic function which is continuous on the boundary. Now

$$0 = \int_p^q \omega = \mathfrak{U}(p) - \mathfrak{U}(q)$$

which by the Residue theorem

$$= \frac{1}{2\pi i} \int_{\partial \mathfrak{F}} \mathfrak{U} \cdot \eta.$$

Noting that $\int_\alpha \omega$ (resp. $\int_\beta \omega$) is the change in $\mathfrak{U}$ from "$-\beta$" to "β" (resp. "$-\alpha$" to "α"), this

$$= \frac{1}{2\pi i} \left\{ \int_\beta \eta \int_\alpha \omega - \int_\alpha \eta \int_\beta \omega \right\}$$

$$= \frac{1}{2\pi i} \left(\int_\beta \eta \right) \left(\int_\alpha \omega \right),$$

where $\int_\alpha \omega \neq 0$. Hence,

$$\int_\beta \eta = 0.$$

By (19.2.4), any closed path on $E \backslash \{p, q\}$ is, up to boundaries, of the form $n\alpha + m\beta + \ell\gamma$; and so the integral of η over such a path is $\ell \int_\gamma \eta = 2\pi i \ell$. Consequently,

$$F := \exp \left(\int_{\mathcal{O}}^{*} \eta \right)$$

is a well-defined function on E which is holomorphic off $\{p, q\}$. Let z (resp. w) be a local coordinate about p (resp. q) with $z(p) = 0$ (resp. $w(q) = 0$). We know that the leading term of η at p is $\frac{dz}{z}$, and at q is $-\frac{dw}{w}$. This makes F locally at p (resp. q) the product of a nonvanishing holomorphic function by $e^{\int \frac{dz}{z}} = e^{\log z} = z$ (resp. $e^{-\int \frac{dw}{w}} = \frac{1}{w}$), so that F is meromorphic on E with divisor

$$(F) = [p] - [q].$$

Therefore $\deg(F) = 1$, making $F : E \to \mathbb{P}^1$ an isomorphism, which is impossible.

We conclude from this contradiction that p and q cannot have been distinct. □

Essentially all of the foregoing (with the exception of Remark 19.1.4) works for any nonsingular cubic. There is a unique holomorphic 1-form up to scale; it vanishes nowhere; and integrating it from a base point gives an isomorphism from the cubic to a complex 1-torus. This follows from the last 2 sections by applying the projective transformation of Chapter 18 to put the cubic in Weierstrass form (which has just been slightly more convenient for writing down ω). For the next section, however, the Weierstrass form will be crucial.

19.3. Abel inverts Weierstrass

We now make the BIG CLAIM that

(19.3.1)

a Weierstrass cubic is always the image E_Λ of a Weierstrass $\mathcal{P}$-map

(cf. Remark 19.1.4), and we have

$$u \circ \mathcal{P} = \mathrm{id}_{\mathbb{C}/\Lambda} \tag{19.3.2}$$

and

$$\mathcal{P} \circ u = \mathrm{id}_E. \tag{19.3.3}$$

The next two Propositions will establish (19.3.1)–(19.3.3). First we study the case where E is (by assumption) the image of a $\mathcal{P}$-map.

19.3.4. PROPOSITION. *Let $\Lambda = \mathbb{Z}\langle \lambda_1, \lambda_2 \rangle \subset \mathbb{C}$ be a lattice. The composition*

$$\mathbb{C}/\Lambda \xrightarrow[\mathcal{P}]{(\cong)} E_\Lambda \xrightarrow[u]{(\cong)} \mathbb{C}/\Lambda_{E_\Lambda}$$

is the identity.

PROOF. Obviously part of the claim is that

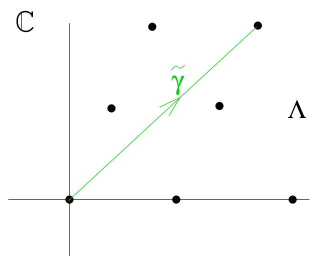
$$\Lambda_{E_\Lambda} \left(:= \left\{ \int_\gamma \omega \;\middle|\; \gamma \in H_1(E_\Lambda, \mathbb{Z}) \right\} \right) \;=\; \Lambda. \tag{19.3.5}$$

For E_Λ, not a lot is lost by working in affine (x, y) coordinates, since there is only $\mathcal{O}$ at ∞ and we know that corresponds to $u \equiv 0$ on the complex 1-tori. (Note also that "u" is used both as the Abel map and as the coordinate on $\mathbb{C}$; which one will be clear from the context.)

Since $\mathcal{P}(u) = (\wp(u), \wp'(u))$,

$$\mathcal{P}^*\omega = \mathcal{P}^*\left(\left.\frac{dx}{y}\right|_{E_\Lambda}\right) = \frac{d(\wp(u))}{\wp'(u)} = \frac{\wp'(u)du}{\wp'(u)} = du.$$

Moreover, that $\mathcal{P}$ is an isomorphism means any cycle γ on E_Λ is the image of some $\tilde{\gamma} \in H_1(\mathbb{C}/\Lambda, \mathbb{Z})$



so that

$$\int_\gamma \omega = \int_{\mathcal{P}_*(\tilde{\gamma})} \omega = \int_{\tilde{\gamma}} \mathcal{P}^*\omega = \int_{\tilde{\gamma}} du$$

gives a bijection between Λ_{E_Λ} and Λ, hence (19.3.5). So then taking $u_0 \in \mathbb{C}/\Lambda$,

$$u(\mathcal{P}(u_0)) = \int_{\mathcal{O}}^{\mathcal{P}(u_0)} \frac{dx}{y} = \int_{\mathcal{P}(0)}^{\mathcal{P}(u_0)} \omega = \int_0^{u_0} \mathcal{P}^*\omega = \int_0^{u_0} du = u_0$$

proves the Proposition. □

Now let E be any Weierstrass cubic.

19.3.6. PROPOSITION. *The composition*

$$E \xrightarrow[u]{(\cong)} \mathbb{C}/\Lambda_E \xrightarrow[\mathcal{P}]{(\cong)} E_{\Lambda_E} \; (\subset \mathbb{P}^2)$$

is the identity. In fact,

(19.3.7) $$E = E_{\Lambda_E}$$

exactly as subsets of $\mathbb{P}^2$.

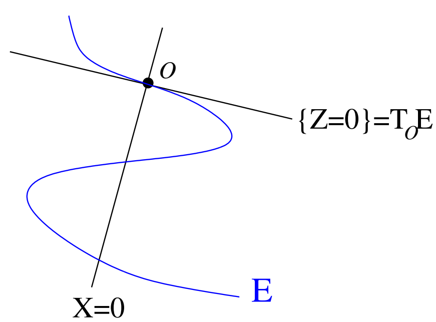
PROOF. On $(x, y) \in E$, the composition takes the form

$$(x,y) \overset{u}{\longmapsto} \int_\infty^x \frac{dw}{\pm\sqrt{Q(w)}} \overset{\mathcal{P}}{\longmapsto} \left(\wp\left(\int_\infty^x \frac{dw}{\pm\sqrt{Q(w)}}\right), \wp'\left(\int_\infty^x \frac{dw}{\pm\sqrt{Q(w)}}\right)\right),$$

with the $\pm$ determined by y. We must show that the RHS recovers (x, y), or equivalently that the inverse $(x(u), y(u)) \colon \mathbb{C}/\Lambda_E \to E$ of the Abel map u identifies with $(\wp(u), \wp'(u))$.

Let's start with x, and compare the elliptic functions $x(u)$ and $\mathcal{P}(u)$ on $\mathbb{C}/\Lambda_E$. First I claim that both have double poles at $u = 0$: you already know that $\wp(u) = \frac{1}{u^2} +$ higher-order terms. For x, it suffices to check this on E, using

$$x = \left.\frac{X}{Z}\right|_E \in \mathcal{K}(E)^*$$



. . . which is easy:

$$\nu_{\mathcal{O}}(x) = (E \cdot \{X = 0\})_{\mathcal{O}} - (E \cdot \{Z = 0\})_{\mathcal{O}} = 1 - 3 = -2.$$

Now $x(u) = \frac{A}{u^2} +$ h.o.t., with $A =$

$$\lim_{u \to 0} x(u) \cdot u^2 = \left(\lim_{x\to\infty} \sqrt{x} \cdot u(x)\right)^2 = \left(\lim_{x\to\infty} \frac{\int_\infty^x \frac{dw}{\sqrt{Q(w)}}}{1/\sqrt{x}}\right)^2$$

$$= \left(\lim_{x\to\infty} \frac{1/\sqrt{Q(x)}}{-1/2x^{\frac{3}{2}}}\right)^2 = \lim_{x\to\infty} \frac{4x^3}{Q(x)} = 1.$$

Define an involution

$$\jmath : E \to E$$

by

$$(x, y) \mapsto (x, -y);$$

this fixes $\mathcal{O}$. For $p \in E$,

$$u(\jmath(p)) = \int_{\mathcal{O}=\jmath(\mathcal{O})}^{\jmath(p)} \frac{dx}{y} = \int_{\mathcal{O}}^{p} \jmath^* \frac{dx}{y} = -\int_{\mathcal{O}}^{p} \frac{dx}{y} = -u(p),$$

and so

$$x(-u) = x(u)\ , \quad y(-u) = -y(u).$$

All told, we now have that $x(u)$ and $\wp(u)$ are both even Λ_E-periodic functions locally of the form $\frac{1}{u^2} + \text{h.o.t.}$, and so their difference has no poles and must (by Liouville) be constant: $x - \wp = c$.

Next, differentiating $u = \int_\infty^x \frac{dx}{y}$ gives $\frac{du}{dx} = \frac{1}{y}$, or

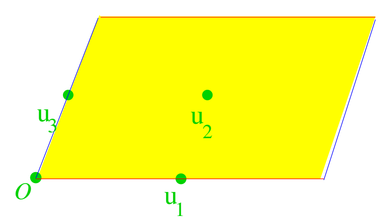
$$x'(u) = \frac{dx}{du} = y(u);$$

and then

$$0 = \frac{d}{du}(c) = x'(u) - \wp'(u) = y(u) - \wp'(u).$$

All that is left is to check that $c = 0$.

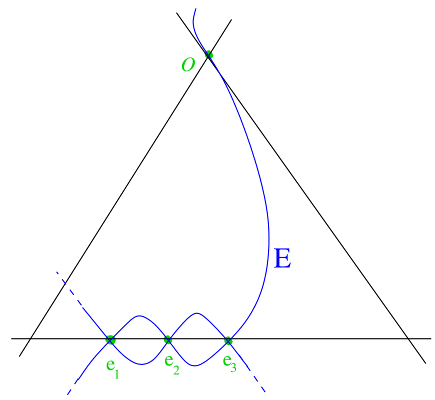
The fixed points of the involution $u \mapsto -u$ are the *2-torsion points*, i.e. those $u \in \mathbb{C}/\Lambda_E$ with $2u \equiv 0$



since we must have $u \equiv -u \bmod \Lambda_E$. These are, of course, the images (by u) of the fixed points of $\jmath$ in E, since $u \circ \jmath = -u$. They also must map (by $\mathcal{P}$) to the fixed points of $(x, y) \mapsto (x, -y)$ in E_{Λ_E}, since $(\wp(-u), \wp'(-u)) = (\wp(u), -\wp'(u))$. Writing

$$y^2 = 4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$$

for the equation of E,



$$x\,(\text{2-torsion points}) = e_1, e_2, e_3, \infty.$$

Similarly, if $E_{\Lambda_E} = \{y^2 = 4x^3 - \tilde{g}_2 x - \tilde{g}_3 = 4(x - \tilde{e}_1)(x - \tilde{e}_2)(x - \tilde{e}_3)\}$ then

$$\wp(\text{2-torsion points}) = \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \infty;$$

and clearly

$$e_1 + e_2 + e_3 = \tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 = 0.$$

Since $\wp(u) = x(u) - c$,

$$\wp(u_1) + \wp(u_2) + \wp(u_3) = x(u_1) + x(u_2) + x(u_3) - 3c$$

which becomes

$$0 = 0 - 3c$$

so $c = 0$.

We conclude that $\wp(u(x, y)) = x$ and $\wp'(u(x, y)) = y$. □

Exercises

(1) Show that the curve E with affine form $y^2 = (1 - x^2)(1 - k^2x^2)$ (for $k \neq \pm 1$) is elliptic, i.e. has normalization of genus 1. Find an isomorphic Weierstrass cubic and use this to calculate the j-invariant. [Hint: consider the projection (of the normalization) to the x-axis ($\cong \mathbb{P}^1$) and apply a projectivity of this $\mathbb{P}^1$ to the 4 branch points.]

(2) [Adapted from Silverman-Tate.]
Let $0 < \beta \leq \alpha$, and consider the ellipse C defined by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1.$$

(a) Show that the arc-length of C is given by the integral[6]

$$4\alpha \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} d\theta$$

for an appropriate choice of constant k depending on α and β.
(b) Prove that this is equal to

$$4\alpha \int_0^1 \frac{1 - k^2 x^2}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} dx,$$

and deduce that the arc-length of C is computed by the integral of the (meromorphic) 1-form $\alpha \frac{1-k^2x^2}{y} dx|_E \in \mathcal{K}^1(\tilde{E})$ around a closed loop on the elliptic curve E from Exercise (1). [This demonstrates why such integrals (and hence curves such as E) came to be called "elliptic".]

(3) Show that the "complete elliptic integral of the first kind",

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

corresponds to the integral of a *holomorphic* 1-form around the same loop on the same E (from Exercise (1)), and so is more directly related to the Abel map above (giving a generator of the lattice Λ_E if we pretend E is a Weierstrass cubic).

(4) Let $E = \{F = 0\} \subset \mathbb{P}^2$ be an arbitrary smooth cubic, and write $f(x, y) = F(1, x, y)$. Show that $\frac{dx}{f_y}|_E$ defines a nowhere vanishing holomorphic 1-form on E. [Hint: make use of $df|_E = 0$, the homogeneity property of F, and coordinate changes to rewrite the form in neighborhoods of the (finitely many) points where "$\frac{dx}{f_y}$" is unsuitable. (This includes points "at infinity", as you need to

[6]Without the 4α, this is the so-called "complete elliptic integral of the second kind."

show the form is holomorphic on the full projective curve.) Don't try to explicitly write out a local expression for f.]

(5) Continuing Exercise (4), suppose that E is a nodal cubic. Show that the pullback of $\frac{dx}{f_y}|_E$ to the normalization $\tilde{E}$ has simple poles at the two preimage points of the node. [Hint: apply a projectivity to move the node to the origin and fix the two tangent lines there.] Why does this make sense in light of Poincaré-Hopf and the result of Exercise (4)?

(6) Let $\Lambda \subset \mathbb{C}$ be a lattice of rank 2 as above, and write the image E_Λ of the corresponding $\mathcal{P}$-map in the form $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$. Show that $g_2(\Lambda) = 60s_4(\Lambda)$ and $g_3(\Lambda) = 140s_6(\Lambda)$, where $s_k(\Lambda) := \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{\lambda^k}$. [Hint: look back at Exercise (5) of Chapter 7.]