CHAPTER 2

Riemann surfaces and algebraic curves

In this chapter we will define (complex) algebraic curves (represented by "C"),[1] complex 1-manifolds (represented by "M"), and Riemann surfaces, and start to consider under what additional hypotheses they are equivalent concepts.

2.1. Algebraic curves

2.1.1. DEFINITION. Let S_2^m denote *homogeneous* polynomials of degree m in x, y (the "2" stands for "2 variables").[2] These are polynomials of the form

$$f_m(x,y) = \sum_{\substack{j,k\geq 0 \\ j+k=m}} c_{jk} x^j y^k,$$

that is, each term has total degree m. Clearly S_2^m is a subset of $\mathcal{P}_2^m$.

More generally, S_k^m is the space of degree-m homogeneous polynomials in k variables — that is, linear combinations

$$\sum_{\substack{i_1,\ldots,i_k\geq 0 \\ i_1+\cdots+i_k=m}} c_{i_1,\ldots,i_k} x_1^{i_1} \cdots x_k^{i_k}$$

of monomials with total degree m. Elements $f \in S_k^m$ have the property that $f_m(\alpha x_1, \alpha x_2, \ldots, \alpha x_k) = \alpha^m f_m(x_1, x_2, \ldots, x_k)$. (See exercise 3 below.)

[1] in the affine resp. projective plane. Later we will define algebraic curves in higher dimensional projective spaces and products thereof, but the "most intrinsic" definition of an algebraic curve as a 1-dimensional reduced scheme (some authors require this to be irreducible and over an algebraically closed field as well) is probably something to learn only once you have a first course in algebraic geometry under your belt.

[2] the field of definition, from which the c_{jk} are taken, will depend on context; S_2^m is a vector space over that field.

Given a real affine algebraic curve of degree d

$$C := \{0 = f(x,y) = f_d(x,y) + f_{d-1}(x,y) + \cdots + f_0\} \subset \mathbb{R}^2$$

with f_d not identically zero, we would like to count its *intersections* with a real line given parametrically by

$$L : t \longmapsto (\alpha t, \beta t)$$

(where α, β are real constants). These are just the solutions of

$$0 = f_d(\alpha,\beta)t^d + f_{d-1}(\alpha,\beta)t^{d-1} + \cdots f_0. \tag{2.1.2}$$

Naively, we would like to get d points:

$$\#\{C \cap L\} = d \text{ ??} \tag{2.1.3}$$

Some issues arise . . .

	Problem	Solution
(a)	$\mathbb{R}$ is not algebraically closed!	pass to $\mathbb{C}^2$
(b)	solutions “at infinity”!	add a “line at infinity” to $\mathbb{C}^2$
(c)	multiple roots!	count intersections with multiplicity
(d)	C might contain L!	Uh-oh

Each “Problem” is an obstruction to (2.1.3), and the object of each “Solution” is to remove the obstruction.

In a little more depth, (a) says that in spite of the fact that the $f_i(\alpha,\beta) \in \mathbb{R}$, roots of (2.1.2) can be non-real. So we had better consider C and L as complex algebraic curves — take $x, y \in \mathbb{C}$ in the definition of C and $t \in \mathbb{C}$ in the definition of L. Their dimensions over $\mathbb{R}$ then, of course, double, and $C \cap L$ now contains the points corresponding to non-real roots of (2.1.2).

Next, if we plug $t = s^{-1}$ into (2.1.2) and multiply by s^d, then it becomes

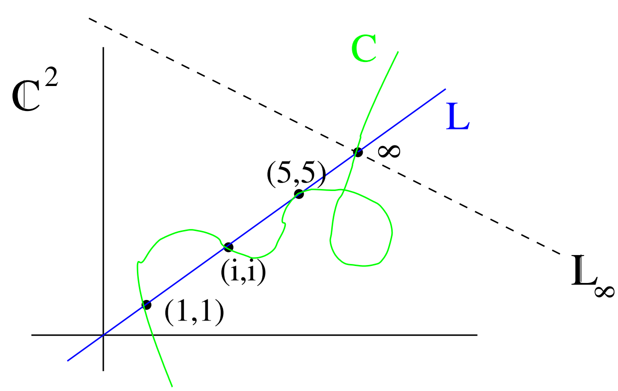
$$0 = f_0 s^d + f_1(\alpha,\beta)s^{d-1} + \cdots + f_d(\alpha,\beta). \tag{2.1.4}$$

A “solution at ∞” to (2.1.2) is a solution at 0 to (2.1.4), which exists if and only if $f_d(\alpha,\beta) = 0$. For this to be counted in $C \cap L$, we must add a line $\mathbb{L}_\infty$ to $\mathbb{C}^2$ to get $\mathbb{P}^2$,[3] the complex projective plane (which we shall discuss in a moment). This adds a point to L "at ∞" corresponding to $s = 0$, yielding a $\mathbb{P}^1$ (projective line), and adds *points* at infinity to C (yielding a compact "projective" curve).

We know that to get d solutions out of a degree d polynomial equation you have to count a twice repeated root as two solutions. So to get d intersection points you will certainly have to count intersections of C with L however many times the corresponding root of (2.1.2) is repeated. The curious case is an m-times repeated root at infinity: via (2.1.4), this corresponds to $f_d(\alpha, \beta) = \cdots = f_{d-m+1}(\alpha, \beta) = 0$. In that case, (2.1.2) is only in fact of degree $d - m$. One *does* have to worry about these degenerate cases, but they will look completely natural in projective coordinates.

Finally, if all $f_\ell(\alpha, \beta) = 0$, then $C \supset L$ and we are in trouble — there is no way around (d). We shall have to demand that plane curves intersect "properly" (in points only), disallowing this possibility, in order to make any statement about the number of intersection points.

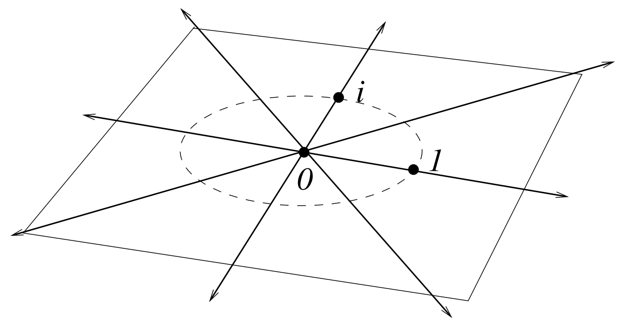
2.1.5. EXAMPLE. Given a quintic curve C as in the following picture



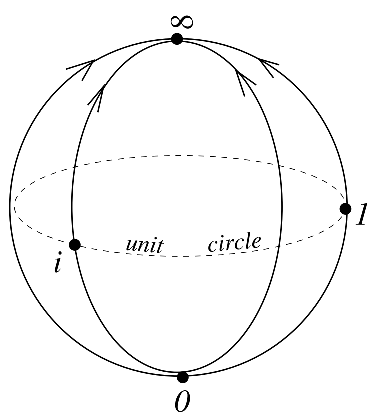
the number of intersection points in $\mathbb{R}^2$ is only 2. But the number of complex intersection points, counting multiplicities and intersections at infinity, is 5.

[3] called $\mathbb{CP}^2$ or $\mathbb{P}^2_{\mathbb{C}}$ in some books

So . . . how does one go about adding a line (resp. point) at infinity to $\mathbb{C}^2$ (resp. L)? First, visualize L as $\mathbb{C}$



and think of all arrows as going off to the same point. Adding this point gives the "1-point" compactification $\mathbb{C} \cup \{\infty\}$, resulting in a sphere



This is an informal way of thinking of $\mathbb{P}^1$ in the following:

2.1.6. DEFINITION. Projective space $\mathbb{P}^n$ is the set of complex lines through the origin $\underline{0} \in \mathbb{C}^{n+1}$.[4] More precisely,

$$\mathbb{P}^n := \frac{\left(\mathbb{C}^{n+1} \setminus \{\underline{0}\}\right)}{\left\langle \substack{(z_0, z_1, \ldots, z_n) \sim (\alpha z_0, \alpha z_1, \ldots, \alpha z_n) \\ \forall \alpha \in \mathbb{C}^*} \right\rangle}$$

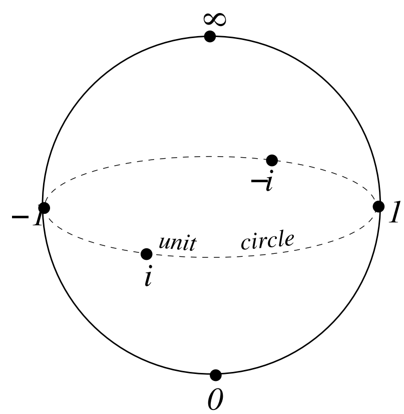
consists of nonzero vectors in $\mathbb{C}^{n+1}$, modulo the equivalence relation equating all vectors lying on a complex line. Elements are written $[z_0 : z_1 : \cdots : z_n]$.

[4]as will be proved in Chapter 5, one should really think of $\mathbb{P}^n$ as a complex manifold (but we haven't defined these yet). $\underline{0}$ denotes $(0, 0, \ldots, 0)$.

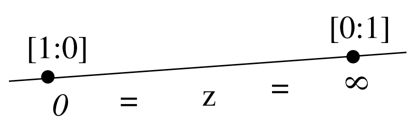
For $n = 1$ this yields the projective line $\mathbb{P}^1$, which has the isomorphism

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\cong} & \mathbb{C} \cup \{\infty\} \\ [z_0 : z_1] & \longmapsto & \frac{z_1}{z_0} =: z \end{array}$$

given by taking slope (of the line represented by $[z_0 : z_1]$). This map is well-defined as $[\alpha z_0 : \alpha z_1] \mapsto \frac{\alpha z_1}{\alpha z_0} = \frac{z_1}{z_0}$. The "honest topological picture" of $\mathbb{P}^1$ is



while the "schematic real picture" is

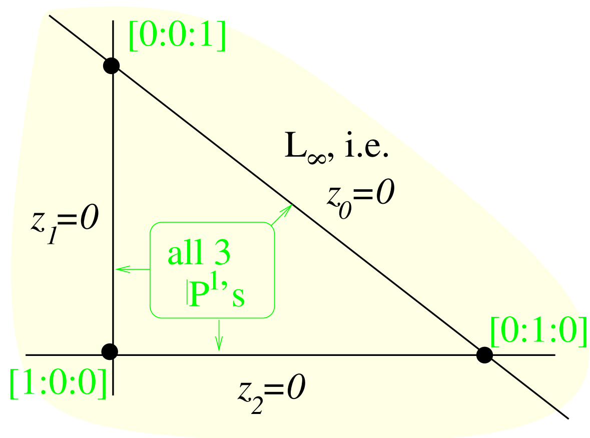


Next, setting $n = 2$ we have the projective plane $\mathbb{P}^2$, and the isomorphism

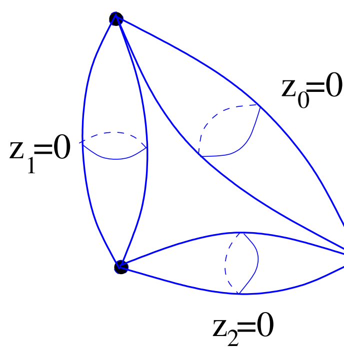
$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\cong} & (\mathbb{C} \times \mathbb{C}) \cup \mathbb{P}^1 \\ [z_0 : z_1 : z_2] & \overset{\text{if } z_0 \neq 0}{\longmapsto} & \left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right) \in \mathbb{C}^2 \\ & \overset{\text{if } z_0 = 0}{\longmapsto} & [z_1 : z_2] \in \mathbb{P}^1 \end{array} \tag{2.1.7}$$

expresses how $\mathbb{P}^2$ adds a line at infinity (the $\mathbb{P}^1$) to $\mathbb{C}^2$.

For $\mathbb{P}^2$, the (rather bad, but standard) "schematic picture" is



While I'm not going to try to represent 4 real dimensions on paper, here is a mostly honest topological depiction of the 3 $\mathbb{P}^1$'s:



2.1.8. REMARK. Equation (2.1.7) relates affine coordinates (on $\mathbb{C}^2$) and projective coordinates (on $\mathbb{P}^2$). Instead of the $\{z_i\}$, I will frequently use $[Z : X : Y]$ for a point in $\mathbb{P}^2$ and (asuming $Z \neq 0$) $\left(\frac{X}{Z}, \frac{Y}{Z}\right) =: (x, y)$ for the corresponding point in $\mathbb{C}^2$.

Also, a **warning** is in order: $[0 : 0 : 0]$ is not a point in $\mathbb{P}^2$. With homogeneous coordinates, some entry must be nonzero.

Returning to our degree d (and now complex) algebraic curve $C = \{f(x, y) = 0\} \in \mathbb{C}^2$, what happens to it as we compactify $\mathbb{C}^2$ to $\mathbb{P}^2$ as described above? To treat this, we first need to introduce the main object of study of this course.

Since $[Z : X : Y] = [\alpha Z : \alpha X : \alpha Y]$, in order for a polynomial equation $F(Z, X, Y) = 0$ to make sense projectively (i.e. in $\mathbb{P}^2$), we must have

$$F(Z, X, Y) = 0 \implies F(\alpha Z, \alpha X, \alpha Y) = 0 \quad (\forall \alpha \in \mathbb{C}^*). \tag{2.1.9}$$

This condition is guaranteed by homogeneity of F (cf. the property in Definition 2.1). (In fact, as we shall see later it is *equivalent* to homogeneity of F.)

2.1.10. DEFINITION. A *projective algebraic curve* $C \subset \mathbb{P}^2$ of degree d, is the zero set of a homogeneous polynomial $F \in S_3^d$.

Here, then, is a general procedure for going between affine and projective curves:

$$f(x, y) = 0 \longmapsto Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right) = 0 \tag{2.1.11}$$

corresponds to taking the *projective closure* $\bar{C} \subset \mathbb{P}^2$ of a given affine curve $C \subset \mathbb{C}^2$. Conversely, if the given C is already projective (defined by $F = 0$), then

$$F(Z, X, Y) = 0 \longmapsto F(1, x, y) = 0 \tag{2.1.12}$$

"restricts" C to the affine curve $C \cap \mathbb{C}^2$. Given an affine curve, taking closure then restricting gets you back to where you started.

2.1.13. EXAMPLE. Starting from the homogeneous cubic polynomial $F(Z, X, Y) = ZXY + 3Z^2Y + 4Y^3$, the affinization is $f(x, y) = F(1, x, y) = xy + 3y + 4y^3$. Conversely, if we start from $f(x, y) = x^3y - y^2 + 2x$, the projectivization is $F(Z, X, Y) = Z^4 f(\frac{X}{Z}, \frac{Y}{Z}) = X^3Y - Y^2Z^2 + 2XZ^3$.

Take F, C to be as in Definition 2.1.10. If $F = \prod F_i$ (so that $\deg F = \sum \deg F_i$), then writing C_i for the zero set of F_i, we have $C = \cup C_i$.

2.1.14. DEFINITION. We say that C is *irreducible* if and only if F has no proper ($\deg \geq 1$) homogeneous factors.

Now let's consider our intersection problem (2.1.3) once more, in the complex projective setting. Referring to the discussion up to

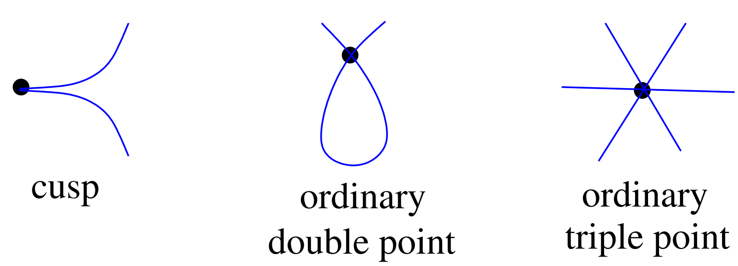
Example 2.1.5, if C and L have an m-fold intersection at infinity, then the degree of the polynomial in (2.1.2) is $d - m$. The Fundamental Theorem of Algebra then says that (2.1.2) has $d - m$ complex roots counted with multiplicity, and we define these to be the intersection multiplicities for C and L in $\mathbb{C}^2$ as indicated in our discussion. We have proved a baby version of Bezout's theorem:

2.1.15. PROPOSITION. *Let $L \subset \mathbb{P}^2$ be a (projective) line in $\mathbb{P}^2$, i.e. an algebraic curve of degree one. A projective algebraic curve of degree d in $\mathbb{P}^2$ not containing L, meets L in d points counted with multiplicity.*

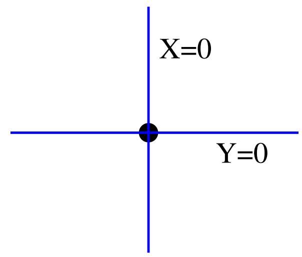
In proving this result we did a tiny bit of complex analysis on L, so were implicitly using its structure as a complex 1-manifold. In general it is quite useful to be able to do analytic computations on curves, but not all irreducible algebraic curves are complex manifolds (at least, without doing something to them called "normalization"). The obstructions are called singularities and will be explored in greater depth later. For now, we will just give a definition and a few examples.

2.1.16. DEFINITION. A *singularity* or *singular point* of an affine algebraic (plane) curve $f(x, y) = 0$ is a point in $\mathbb{C}^2$ where f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are all zero — that is, a point on the curve where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish. A singularity of a projective algebraic curve $F(Z, X, Y) = 0$ is a point where F, $\frac{\partial F}{\partial X}$, $\frac{\partial F}{\partial Y}$, and $\frac{\partial F}{\partial Z}$ are all zero. A curve with one or more singular points is called *singular*; a curve with none is called *smooth*.

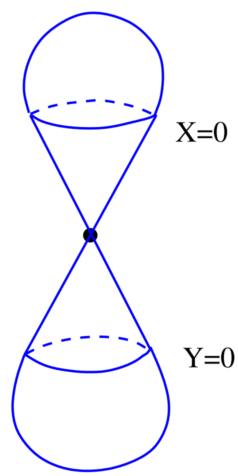
2.1.17. EXAMPLE. Here are some local real (schematic) pictures of plane curve singularities:



An example of a cusp is the point $[1:0:0]$ on $X^3 - Y^2Z = 0$ (or $(0,0)$ on $x^3 = y^2$); the curve $XY = 0$ has an ODP (ordinary double point, or "normal crossing") at $[1:0:0]$:



Now, this is just two $\mathbb{P}^1$'s (namely, $X = 0$ and $Y = 0$) touching at one point. A more (though not completely) "topologcially honest" picture of this is:



which makes it apparent that an ODP is actually a "bi-conical singularity".

Before we pass to the "analytic" side of our story, there are 2 more facts about homogeneous polynomials worth a quick mention. First, the map $S_3^d \to \mathcal{P}_2^d$ given by $F(Z,X,Y) \mapsto F(1,x,y)$ is an isomorphism, so $\dim(S_3^d) = \dim(\mathcal{P}_2^d)$ in particular. Second is the Euler formula

$$\sum_{i=0}^{N} Z_i \frac{\partial F}{\partial Z_i} = d \cdot F \quad \text{for } F \in S_{N+1}^d \tag{2.1.18}$$

which will be used in later chapters (cf. Chapter 6 for a proof).

2.2. Complex 1-manifolds

Recall from basic point-set topology that a *topological space* is a set X together with a collection $\{\mathfrak{U}_{\mathfrak{I}}\}_{\mathfrak{I}\in\Omega}$ of "open sets" containing X, the empty set, and all unions and all finite intersections of its members. (Here Ω is some typically huge index set. A *base* for the topology of X is a sub-collection of the $\{\mathfrak{U}_{\mathfrak{I}}\}_{\mathfrak{I}\in\Omega}$ which generates it under taking unions, and X is said to be *second countable* if it has a countable base.) X is called *Hausdorff* if points can be separated: i.e. given p and q, there exist disjoint open sets U and V containing p and q respectively.

In topology, a *homeomorphism* is a continuous, 1-to-1, open[5] map. Given a point $p \in X$, we like open sets $U \ni p$ that are homeomorphic to $\mathbb{R}^n$ (or equivalently, an open ball in $\mathbb{R}^n$) — these are called open neighborhoods of p. If these always exist, we say X is locally homeomorphic to $\mathbb{R}^n$. A second countable, Hausdorff topological space that is locally homeomorphic to $\mathbb{R}^n$, is called a *real n-manifold.*

In the case $n = 2$, we are going to layer "complex analyticity" onto this construction:

2.2.1. DEFINITION. A *complex* 1-*manifold* consists of

(i) a connected Hausdorff topological space M;[6]

(ii) an open cover $\{U_\alpha\}$ of M (this is a finite set of open sets taken from amongst the $\{\mathfrak{U}_{\mathfrak{I}}\}$, such that $\cup_\alpha U_\alpha = M$); and

(iii) mappings $z_\alpha : U_\alpha \to \mathbb{C}$ that are homeomorphisms onto their image, such that the transition functions[7]

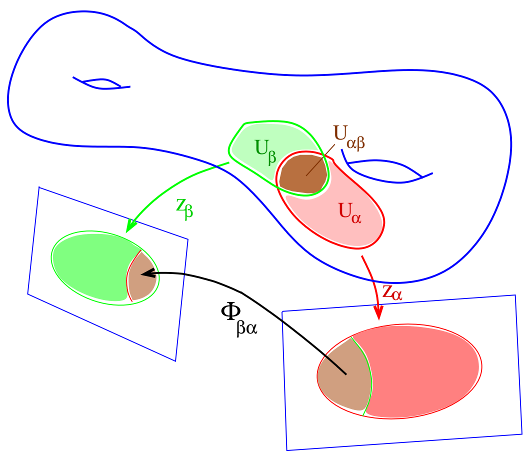
$$\Phi_{\beta\alpha} := z_\beta \circ z_\alpha^{-1} : z_\alpha(U_{\alpha\beta}) \to z_\beta(U_{\alpha\beta})$$

are biholomorphic (i.e., analytic isomorphisms).

[5]A map is open (resp. continuous) if the image (resp. preimage) of any open set under the map is open.

[6]One can include second-countability here, or note that it comes for free by a deep result in complex analysis (Radó's theorem).

[7]If U_α and U_β are distinct open sets in our cover, we write $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Later we will write V_α for $z_\alpha(U_\alpha)$ and V_α^β for $z_\alpha(U_{\alpha\beta})$, so that $\Phi_{\beta\alpha}$ goes from V_α^β to V_β^α.



The z_α are called *local coordinates*, and the $\Phi_{\beta\alpha}$ *transition (or patching) functions*; the entire collection $\{z_\alpha\}, \{\Phi_{\alpha\beta}\}$ is called an *analytic atlas*.

The functions $\Phi_{\beta\alpha}$ are key: M is an complex analytic manifold because they are complex analytic. If in (iii) we replace $\mathbb{C}$ by $\mathbb{R}^n$ and require the transition functions to be smooth (i.e., have continuous partial derivatives of all orders), then M would have been a *smooth* (or "differentiable") real n-manifold instead.

If we think of the (complex analytic) transition functions in Definition 2.2.1 as maps from $\mathbb{R}^2$ to $\mathbb{R}^2$, then

$$\Phi(\overbrace{x}^{\substack{\text{real}\\\text{part}}}, \overbrace{y}^{\substack{\text{imag.}\\\text{part}}}) = (\overbrace{u(x,y)}^{\substack{\text{real}\\\text{part}}}, \overbrace{v(x,y)}^{\substack{\text{imag.}\\\text{part}}})$$

is smooth *and* u, v satisfy the *Cauchy-Riemann equations*. These may be expressed in terms of the Jacobian matrix

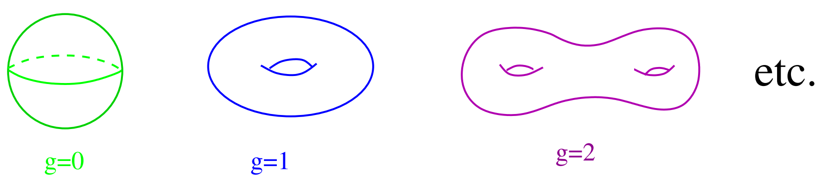
$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix},$$

which consequently has positive determinant: $u_x^2 + v_x^2$ (obviously ≥ 0) cannot equal zero since Φ is *bi*holomorphic. Therefore Φ preserves orientation, so M is orientable as a real 2-manifold.[8]

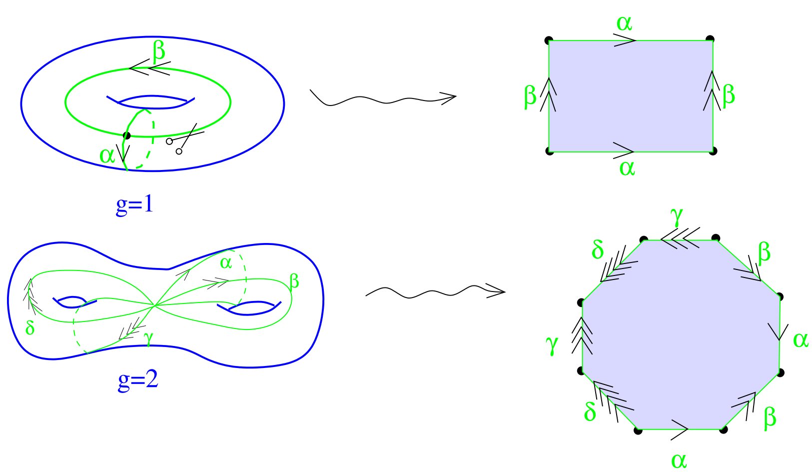
[8]in fact, a matrix of the form $\left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$ is a rotation times a dilation, hence *preserves angles* – that is to say, under the assumption of the CR-equations, Φ is *conformal*. So

2.2.2. EXAMPLE. $\mathbb{C}$, $\mathbb{C}^*$, $\mathfrak{H}$, $\mathbb{P}^1$, and $\mathbb{C}/\Lambda$ are (the simplest) examples of complex 1-manifolds. For the first three, producing an analytic atlas is trivial (since you only need one U_α), and we will do this below for the latter two.

Now assume M is *compact*, that is, every open cover has a finite subcover. (In fact, since a complex 1-manifold always admits a metric, compactness is equivalent to every sequence of points having a convergent subsequence. Clearly $a_n = n$ has no limit in $\mathbb{C}$ but $a_n = [1 : n] = [\frac{1}{n} : 1]$ does limit to $[0 : 1]$ in $\mathbb{P}^1$, which is compact.) Then viewed over $\mathbb{R}$, M is an orientable, compact, connected, smooth 2-manifold. By a theorem in topology, this means that M is homeomorphic to a sphere with g handles, and we say M has *genus* g:



It is a fact that all $g \geq 0$ occur for complex manifolds; we'll show this in a moment for $g = 0, 1$. To do complex analysis on M, you can use the local coordinates, but for some purposes it is also convenient to cut M into a simply connected region, e.g.

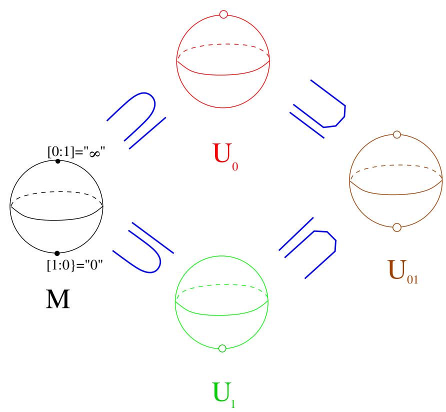


a complex 1-manifold is essentially a differentiable real 2-manifold with conformal transition functions.

Extrapolating from this, one sees that if you begin with a sphere with g handles, the cut-open version is a polygon with $4g$ edges identified in pairs. (The black points on the right-hand side are *all* identified.) Using this[9] we can do a quick computation of the Euler characteristic of M:

$$\chi_M := \text{faces} - \text{edges} + \text{vertices} = 1 - 2g + 1 = 2 - 2g. \tag{2.2.3}$$

2.2.4. EXAMPLE. ($g = 0$) Let $M := \mathbb{P}^1$ with homogeneous coordinates $[X : Y]$. Consider the open cover $\{U_0, U_1\}$ of M given by:



That is, $U_0 = M\backslash\{[0:1]\}$ and $U_1 = M\backslash\{[1:0]\}$. For local coordinates, we take

$$\begin{array}{rccc} z_0 : & U_0 & \to & \mathbb{C} \\ & [X:Y] & \mapsto & \frac{Y}{X} \end{array}$$

and

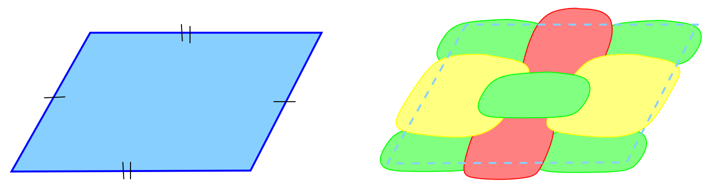
$$\begin{array}{rccc} z_1 : & U_1 & \to & \mathbb{C} \\ & [X:Y] & \mapsto & \frac{X}{Y} \end{array}.$$

Writing $U_{01} := U_0 \cap U_1$, we have $z_0(U_{01}) = \mathbb{C}^* \subset \mathbb{C}$ and $z_1(U_{01}) = \mathbb{C}^* \subset \mathbb{C}$. The transition function (which goes from $z_0(U_{01})$ to $z_1(U_{01})$ by definition) is then

$$\begin{array}{rccc} \Phi_{10} : & \mathbb{C}^* & \to & \mathbb{C}^* \\ & u & \mapsto & \frac{1}{u} \end{array}.$$

[9]traditionally one would use the numbers of faces, edges, and vertices in a *triangulation* of M, but using a polygonal decomposition like this is also OK

2.2.5. EXAMPLE. ($g = 1$) Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Then $\Lambda := \mathbb{Z}\langle \lambda_1, \lambda_2 \rangle = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ is a lattice, and we set $M := \mathbb{C}/\Lambda$. (This means that $z, z' \in \mathbb{C}$ give the same point in M if and only if $z - z' \in \Lambda$.) We endow M with local coordinates on the neighborhoods shown



M

covering by coordinate neighborhoods

basically by using the coordinate on $\mathbb{C}$ (before quotienting by Λ), and find that the transition functions Φ_{ij} are all either the idenitity or translation by some $\lambda \in \Lambda$. Topologically, M is a torus (cf. the $g = 1$ pictures above), which is evident from performing the identifications on the sides of a fundamental region for $\mathbb{C}/\Lambda$ as shown.

2.3. Riemann surfaces

Traditionally, a *Riemann surface* M is defined to be a compact complex 1-manifold obtained as the "existence domain" of an algebraic (typically multivalued) function over $\mathbb{P}^1$. That isn't the definition I'll use here, but I do want to explain the concept.

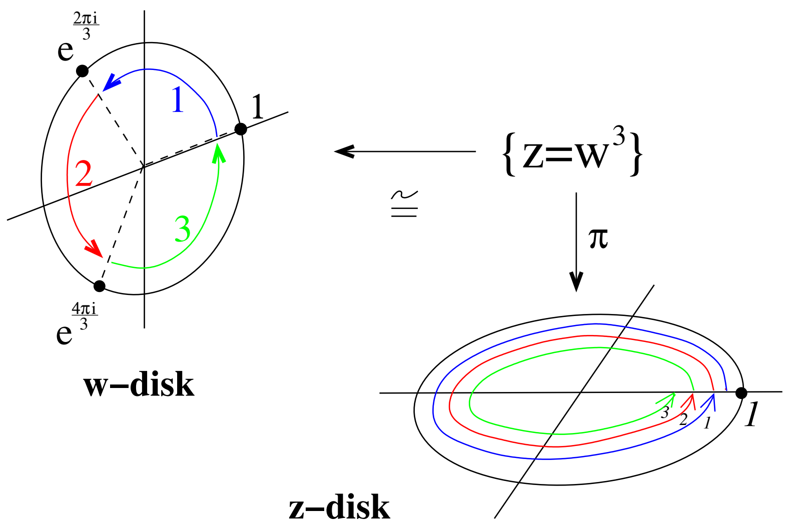
For example, given distinct complex numbers α_i, the algebraic function

$$\mathfrak{F}(z) := \sqrt{\prod_{i=1}^{2g+2}(z - \alpha_i)}$$

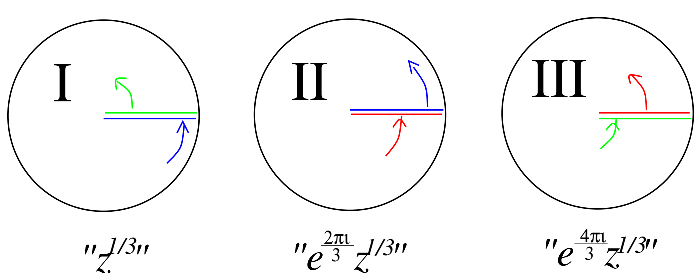
on $\mathbb{P}^1$ can be made single-valued on a complex manifold M (of genus g, it turns out) constructed as follows: On two copies of $\mathbb{P}^1$, cut identical nonintersecting slits from α_{2j-1} to α_{2j} for $j = 1, \ldots, g+1$. Then glue the two copies of $\mathbb{P}^1$ together on these slits, forming a set on which $\mathfrak{F}$ becomes single-valued; finally, endow this set with an analytic atlas to get a complex manifold M. This manifold has a distinguished morphism $M \xrightarrow{\pi} \mathbb{P}^1$ presenting it as a finite branched cover of the projective line. We won't do this explicitly here — especially endowing it with an analytic atlas, since that is really a special case of normalizing an algebraic curve (cf. §3.1).[10]

Instead, let's visualize what a couple of "existence domains" for algebraic functions look like, starting with

2.3.1. EXAMPLE. How should we think about the "Riemann surface of $(w =)z^{\frac{1}{3}}$ over the unit disk"? This is some object fitting (as "$\{z = w^3\}$") into the following picture:

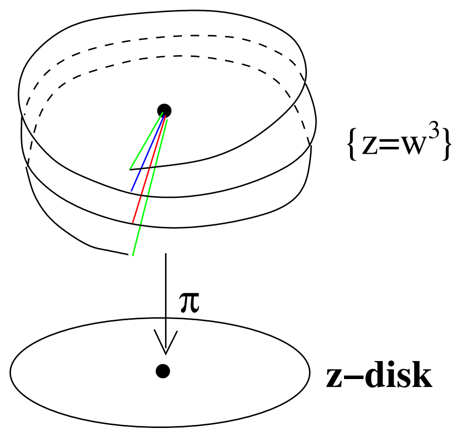


To construct it, think about following $z^{\frac{1}{3}}$ around the disk once counterclockwise: when you reach your starting point the function has become $e^{\frac{2\pi i}{3}}$ times the branch of $z^{\frac{1}{3}}$ you started with; going around once more, you get $e^{\frac{4\pi i}{3}} z^{\frac{1}{3}}$; and one more time gets you back to your original branch. So taking three unit disks, slitting them along the positive reals, and gluing them as indicated



[10]It makes a very instructive exercise though, and the next example gives a hint on how to do it.

we get the "parking lot"

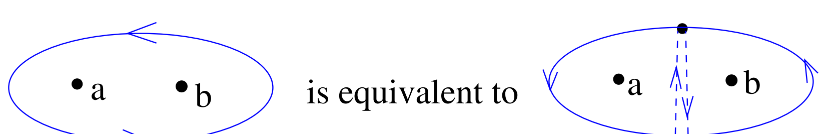


(The green segments are glued but I can't draw in 4 dimensions.) An easier way to visualize this "Riemann surface" is this: it's just the w-disk (and w is the local coordinate). The difficulty is in seeing the w-disk "over" the z-disk.

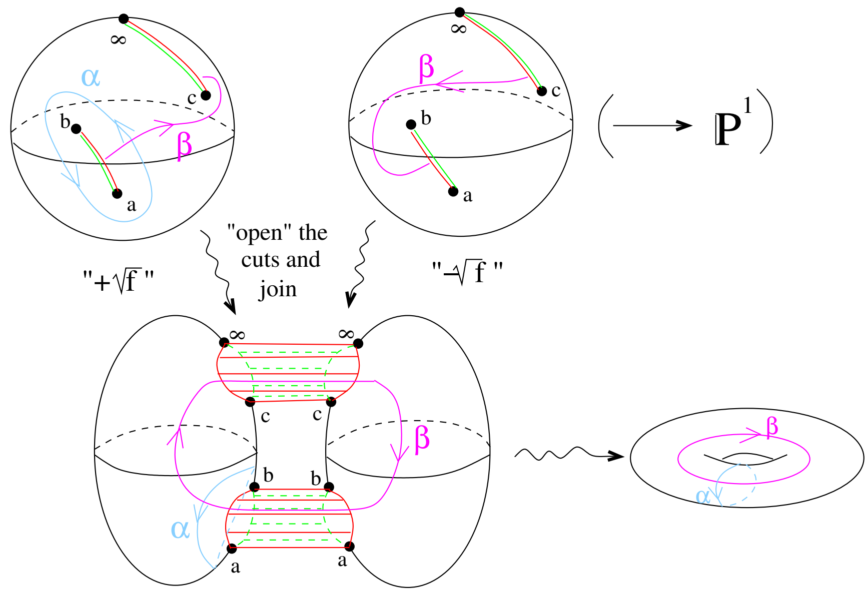
2.3.2. EXAMPLE. Next, let's construct an existence domain for

$$\mathfrak{F}(z) = \sqrt{(z-a)(z-b)(z-c)}$$

over $\mathbb{P}^1$. In a neighborhood of $z_0 = a, b, c$ this looks like the "Riemann surface of $(z-z_0)^{\frac{1}{2}}$ over a disk", which is the same as the construction we just did except with 2 unit disks instead of 3. Indeed, going once around a, b, or c takes $\mathfrak{F} \mapsto -\mathfrak{F}$; and furthermore, because the degree of the polynomial under the square root is odd, going once around ∞ does the same thing. Since



going around two points at once gives no change. So taking two $\mathbb{P}^1$'s and cutting and pasting them as indicated, we end up with a manifold of genus 1 on which $\mathfrak{F}$ becomes well-defined:



(In the picture, α and β are called 1-cycles; there just there to make the topology clear.) The same construction works if we replace $\mathfrak{F}(z)$ by $\sqrt{(z-a)(z-b)(z-c)(z-d)}$, with d replacing ∞.

In fact, by a deep result (on existence of nonconstant meromorphic functions on complex 1-manifolds) any compact complex 1-manifold is an "existence domain" of the sort we have just discussed: they are equivalent objects in the end. The following is motivated by this, and the desire to keep things simple:

2.3.3. DEFINITION. A *Riemann surface* is a compact complex 1-manifold.

Exercises

(1) Take projective closures of $\{y^2 = (x-1)(x-2)(x-3)(x-4)\} =: C$ and $\{x = 0\} =: L$ in $\mathbb{P}^2$ (by finding the associated homogeneous equations), and determine all intersections and their multiplicities (give the projective coordinates of the points). What is the sum of multiplicities?

(2) Find the affine equation associated to $Z_0^3 + Z_1^3 + Z_2^3 = \lambda Z_0 Z_1 Z_2$. (This equation is homogeneous of degree 3 — λ is a scalar, not a coordinate). Is the curve it defines is smooth for $\lambda = 0$? $\lambda = 3$?

(3) Let F be a polynomial in 3 variables. Prove that $F(z_0, z_1, z_2) = 0 \implies F(\alpha z_0, \alpha z_1, \alpha z_2) = 0$ $(\forall\, \alpha \in \mathbb{C}^*)$ forces F to be homogeneous (of some degree). [You will have to assume the following result: given two polynomials f and g (in (z_0, z_1, z_2)), with vanishing locus $\{f = 0\} \subseteq \{g = 0\}$, then f divides a power of g. This is called *Study's lemma* and will be proved later.] Also write out the details for the (easier) converse.

(4) Show that the existence domain M of $\sqrt{\prod_{i=1}^{2g+2}(z - \alpha_i)}$ is a complex 1-manifold of genus g: (a) construct an atlas (what is the local coordinate about a branch point $z = \alpha_i$, in view of Example 2.3.1?); then (b) working in analogy to Example 2.3.2, use the slits as part of a triangulation of M and apply (2.2.3).

(5) As we'll see in Chapter 8, irreducible complex algebraic curves are connected. Suppose $C \subset \mathbb{C}^2$ is defined by a real polynomial $f \in \mathbb{R}[x, y]$, and consider the real solution set $C(\mathbb{R}) = C \cap \mathbb{R}^2$. The potential (in fact, usual) failure of connectedness for $C(\mathbb{R})$ is yet another weirdness that afflicts real algebraic curves: for instance, we can have an *isolated point* $p \in \mathbb{R}^2$ such that an open disk $B \subset \mathbb{R}^2$ about p has $B \cap C(\mathbb{R}) = \{p\}$. (a) Give an example. (b) Show that at any such point, C is singular.