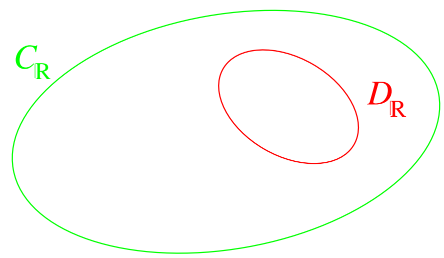
CHAPTER 22

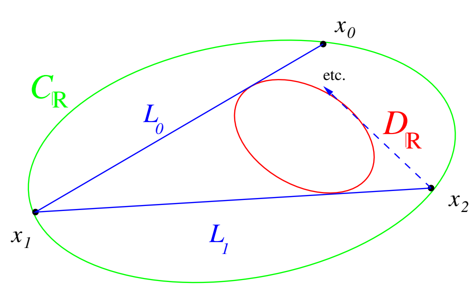
The Poncelet problem

First let's recall the most elementary statement of the "porism" from Chapter 1. One starts with two conics $C_{\mathbb{R}}$, $D_{\mathbb{R}}$ in $\mathbb{R}^2$, which for simplicity we can take to be two ellipses cut out by polynomials $f_C, f_D \in \mathcal{P}^2$ with real coefficients:



We asked in §1.3 whether there exists a closed polygon inscribed in $C_{\mathbb{R}}$ and circumscribed about $D_{\mathbb{R}}$. The result stated there, Theorem 1.3.1, said that if there is *one* then there is an *infinite family*. Our goal in this chapter is not just to flesh out the sketch of proof given there of this "porism", but to actually provide a way of deciding for *which* pairs there does exist a circuminscribed polygon.

A slight reformulation of the theorem is this: starting from some point x_0 on $C_{\mathbb{R}}$, draw a line segment tangent to $D_{\mathbb{R}}$, continue until it hits $C_{\mathbb{R}}$ again. Begin again at this new point, by drawing the other line segment through it and tangent to $D_{\mathbb{R}}$:



Iterating this construction, we may ask whether it ever closes up — i.e. returns to its starting point. (We will not care whether the path crosses itself.) What we will show is that the answer is independent of the choice of starting point x_0.

22.1. Proof of Theorem 1.3.1

Poncelet's Theorem has nothing to do with C and D being ellipses, f_C and f_D being real polynomials, and so forth — it makes sense more generally for pairs of conics in the complex projective plane $\mathbb{P}^2$, and that is the context in which we view it for the proof. Namely, let $C = \{F_C(Z,X,Y) = 0\}$, $D = \{F_D(Z,X,Y) = 0\}$ be the conics cut out by homogeneous degree-2 polynomials F_C, $F_D \in S^2$. *If* the latter have coefficients in $\mathbb{R}$ (not essential for what follows), then the real points $C(\mathbb{R})$, $D(\mathbb{R})$ make sense and then $C_{\mathbb{R}}$, $D_{\mathbb{R}}$ above are just their intersections with affine space. Now, these affine real points need not meet (as in the above picture), but by Bezout C and D *must* meet in four points counted with multiplicity. We will carry out our proof *under the assumption that the multiplicities are all one*, i.e. C and D meet transversely and so $|C \cap D| = 4$.

Consider the incidence correspondence

$$\mathcal{E} := \{(x,L) \mid x \in L\} \subset C \times \check{D}$$

where $\check{D} \subset \check{\mathbb{P}}^2$ is the dual curve consisting of lines tangent to D (at any point). In §1.3 we defined pictorially two involutions $\iota_1 \colon \mathcal{E} \to \mathcal{E}$ and $\iota_2 \colon \mathcal{E} \to \mathcal{E}$. The idea is that each $L \in \check{D}$ meets C in two points (counted with multiplicity), and swapping those points yields ι_1; whereas each $x \in C$ is in two lines tangent to D ("counted with multiplicity"), whose exchange yields ι_2. Composing involutions gives $\jmath := \iota_2 \circ \iota_1$, which is no longer an involution and is the complex geometry analogue of the iteration described just above. If we pick a starting "point" $(x_0, L_0) \in \mathcal{E}$, then we are interested in whether

$$\jmath^n(x_0, L_0) = (x_0, L_0)$$

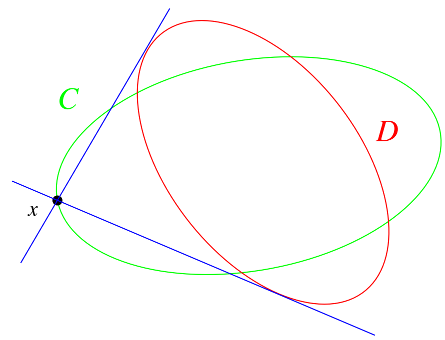
for some $n \in \mathbb{N}$.

Now the projection

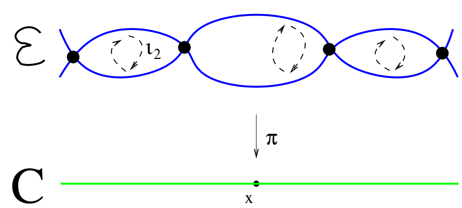
$$\pi\colon \mathcal{E} \to C(\cong \mathbb{P}^1)$$
$$(x, L) \mapsto x$$

has

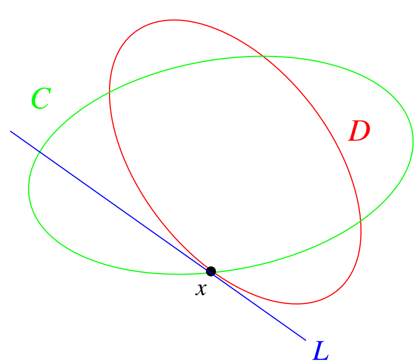
- mapping degree 2: there exist two lines L, L' tangent to D through a general point $x \in C$



- 4 ramification points (each of order two): namely, the points of $\mathcal{E}$ fixed by the involution ι_2



In particular, the ramification points of π identify with the points of $C \cap D$, since through each of these there is a *unique* tangent to D (rather than two):

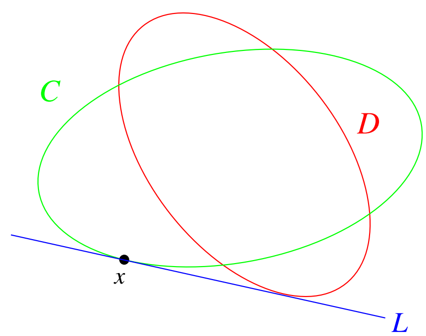


By the Riemann-Hurwitz formula (for π),

$$\chi_{\mathcal{E}} = d \cdot \chi_C - r = 2 \cdot 2 - 4 = 0.$$

This implies $\mathcal{E}$ is elliptic, and so has an Abel map u mapping it isomorphically to a 1-torus $\mathbb{C}/\Lambda$ (where Λ depends on[1] $\mathcal{E}$ hence ultimately on C and D).

Alternatively, we could have carried out this same computation using $\check{\pi}\colon \mathcal{E} \to \check{D}$ (sending $(x, L) \mapsto L$), whose ramification points (in $\mathcal{E}$) are the fixed points of ι_1 and hence identify with bitangents:



There are four of these since $\check{C}$ and $\check{D}$ are conics in $\check{\mathbb{P}}^2$ hence have $|\check{C} \cap \check{D}| = 4$.

Now consider an arbitrary involution I of $\mathbb{C}/\Lambda$, where the coordinate on $\mathbb{C}$ is denoted u. Any automorphism of $\mathbb{C}/\Lambda$ (in particular I) takes the form $u \mapsto au + b$ by Exercise (5) of Chapter 14, and squaring this gives

$$u \longmapsto au + b \longmapsto a(au + b) + b = a^2 u + b(a + 1).$$

If this is to be the identity, we must either have (i) $a = 1$ and $b \in \Lambda/2$, or (ii) $a = -1$ and $b \in \mathbb{C}$ arbitrary. Case (i) has no fixed points as it is a translation by a 2-torsion point.

By abuse of notation[2] we will think of $\iota_1, \iota_2, \jmath$ as automorphisms of $\mathbb{C}/\Lambda$. Since ι_1 and ι_2 are involutions of $\mathbb{C}/\Lambda$ with fixed points,

[1]To define the Abel map you also have to choose a holomorphic 1-form on $\mathcal{E}$; this affects the scaling of the lattice but not its isomorphism class.

[2]Strictly speaking, one should write $u \circ \iota_1 \circ u^{-1}$ for the involution of $\mathbb{C}/\Lambda$ corresponding to ι_1 on $\mathcal{E}$.

they belong to case (ii):

$$\iota_1(u) \equiv b_1 - u\,, \quad \iota_2(u) \equiv b_2 - u \quad \pmod{\Lambda}.$$

Therefore

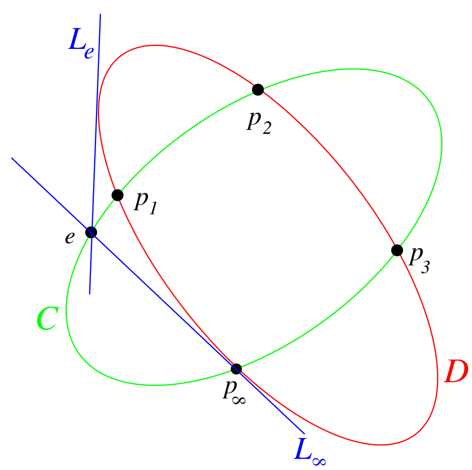
$$\jmath(u) = \iota_2(\iota_1(u)) \equiv b_2 - (b_1 - u) = u + \underbrace{(b_2 - b_1)}_{=:\beta}, \tag{22.1.1}$$

i.e. $\jmath$ is a *translation* on $\mathbb{C}/\Lambda$.

Write u_0 for the image of (x_0, L_0) under the Abel map. Clearly $\jmath^n(x_0, L_0) = (x_0, L_0)$ iff $\jmath^n(u_0) \equiv u_0 \pmod{\Lambda}$. But $\jmath^n(u_0) = u_0 + n\beta$, which $\equiv u_0$ iff $n\beta \equiv 0$, i.e. $n\beta \in \Lambda$. We conclude that *the Poncelet construction* (starting from (x_0, L_0)) *closes up at the n^{th} iteration if and only if β is n-torsion relative to the lattice*. Since β depends only on $\jmath$, this has nothing to do with the choice of (x_0, L_0). Q.E.D.

22.2. Explicit solution of the Poncelet problem

The flexes are the preferred choices of origin for the group law on a cubic plane curve. On the incidence-correspondence elliptic curve $\mathcal{E}$, it turns out that the best choice for $\mathcal{O}$ is one of the fixed points of ι_2 (the four (x, L) with $x \in C \cap D$). Writing $C \cap D = \{p_1, p_2, p_3, p_\infty\}$, we set $\mathcal{O}_{\mathcal{E}} := (p_\infty, L_\infty) \in \mathcal{E}$.



Here (e, L_e) is the first point in the "Poncelet iteration" applied to this "origin", i.e. $\jmath(\mathcal{O})$.

Clearly $\beta = u((e, L_e))$ in (22.1.1), with $u : \mathcal{E} \to \mathbb{C}/\Lambda$ the usual Abel isomorphism. The question of whether $\jmath^n$ is the identity can be restated in terms of the (unique) group law on $\mathcal{E}$ with origin $\mathcal{O}$:

(22.2.1) Is (e, L_e) an N-torsion point?

The approach we take to its solution in this section is work of CAYLEY as presented in the nice expository article [P. Griffiths and J. Harris, *On Cayley's explicit solution to Poncelet's porism*, L'Enseignement Math. 24 (1978), 31-40.].

A family of conics. Consider the collection of conic curves

$$D_t := \{p \in \mathbb{P}^2 \mid tF_C(p) + F_D(p) = 0\}$$

depending on $t \in \mathbb{P}^1$, with $D_\infty = C$ and $D_0 = D$. Each D_t contains p_1, p_2, p_3, p_∞ since F_C and F_D both vanish at these points. For each $t \in \mathbb{P}^1$, let $\ell_t := T_{p_\infty} D_t$ and define q_t by $\ell_t \cap C =: p_\infty + q_t$. Note that $q_\infty = p_\infty$ (double intersection) and $q_0 = e$ (by the last picture).

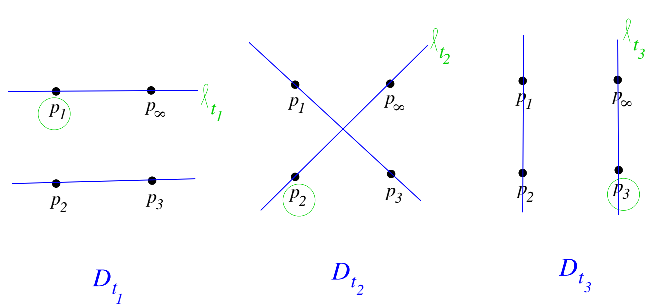
Recall that the equation of a conic may always be written

$$^t\underline{p}.M.\underline{p} = 0\,, \qquad M \text{ a symmetric 3x3 matrix;}$$

the conic is singular if and only if $\det M = 0$. Write M_C, M_D for the matrices corresponding to C, D, so that $tM_C + M_D$ corresponds to D_t. Those t for which D_t is singular, are then just the t_i in

(22.2.2) $$\det\,(tM_C + M_D) = \kappa(t - t_1)(t - t_2)(t - t_3).$$

There are three singular conics through the $\{p_i\}_{i=1,2,3,\infty}$:



and we may order the t_i (as shown) to have $q_{t_i} = p_i$ for $i = 1, 2, 3$.

Thus we have constructed a morphism

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow C \\ t &\longmapsto q_t \end{aligned} \tag{22.2.3}$$

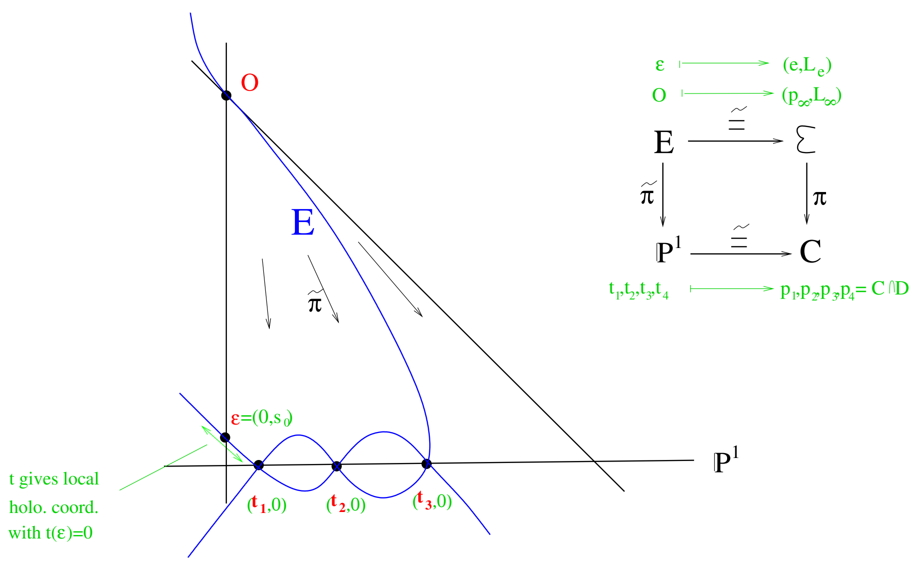
which sends $0 \mapsto e$, $\infty \mapsto p_\infty$, and $t_i \mapsto p_i$ ($i = 1, 2, 3$). That it is an isomorphism, hence may be viewed as the usual "normalization by stereographic projection", is checked in Exercise (3).

Since $\mathcal{E}$ was already a double cover of C branched at p_1, p_2, p_3, p_∞, (22.2.3) exhibits $\mathcal{E}$ as a double-cover of $\mathbb{P}^1$ branched at t_1, t_2, t_3, ∞ — i.e. the "existence domain" (cf. §2.3) of $\sqrt{(22.2.2)}$, which is to say the Riemann surface

$$\left\{ s^2 = \det(tM_C + M_D) \right\} =: E. \tag{22.2.4}$$

The point (e, L_e) on $\mathcal{E}$ corresponds to a point over $t = 0$ on E; call this ε. (Moreover, $\mathcal{O}_{\mathcal{E}} \in \mathcal{E}$ corresponds to $[0 : 0 : 1] =: \mathcal{O} \in E$, as it should.)

Summarizing everything in a picture:



Our main question (22.2.1) becomes:

Is ε N-torsion on E?

Now $t_1 + t_2 + t_3$ may not be zero and we are lacking a factor of 4, so E is not quite in Weierstrass form. But it is easy to see that we have a normalization

$$\mathcal{P} : \mathbb{C}/\Lambda \xrightarrow{\cong} E$$

given by

$$u \longmapsto \left(\wp(u) + \frac{\sum t_i}{3}, \, \frac{\wp'(u)}{2\kappa^{-\frac{1}{2}}} \right).$$

Clearly this sends $0 \mapsto \mathcal{O}$; define $u_0 \in \mathbb{C}/\Lambda$ to be the point sent to ε. The question is now:

$$\text{Is } u_0 \text{ } N\text{-torsion on } \mathbb{C}/\Lambda?$$

"Normal" elliptic curves and a "multiple addition" theorem. Put $u_j := u_0 + \Delta_j$, where $\Delta_j \in \mathbb{C}$. Abel's theorem implies

22.2.5. PROPOSITION. *There exists a Λ-periodic meromorphic function F with order-N pole at* 0 *and simple zeroes at $u_1, \ldots, u_N$, if and only if $u_1 + \cdots + u_N \equiv 0 \pmod{\Lambda}$.*

What we are really after here is the vector space V of meromorphic functions on E with at worst an order-N pole at $\mathcal{O}$ (and no other poles). There are $N - 1$ degrees of freedom coming from pushing around the $\{u_j\}$ (while keeping $\sum u_j \equiv 0$) and one degree of freedom from multiplying the function by a constant. So $\dim V = N$; let $\{f_1, f_2, \ldots, f_N\}$ (with f_1 constant) be a basis, and define

$$\varphi_N : E \longrightarrow \underset{\substack{\text{(coords.}\\ w_1,\ldots,w_N)}}{\mathbb{P}^{N-1}}$$

by

$$([1:t:s] =:) \; \underline{z} \;\longmapsto\; [f_1(\underline{z}) : \cdots : f_N(\underline{z})].$$

22.2.6. DEFINITION. The image of φ_N, denoted E_N, is called a *normal* elliptic curve of degree N. (Note that E_3 is essentially E — take f_1, f_2, f_3 to be $1, t, s$.)

For $\sum u_j \equiv 0$, there exists a function $\mathfrak{F}$ on E_N with zeroes at $\varphi_N(\mathcal{P}(u_j))$, and order N pole at $\varphi_N(\mathcal{O})$. Now the "hyperplane at infinity" $\{w_1 = 0\} \subset \mathbb{P}^{N-1}$ intersects E_N only at $\varphi_N(\mathcal{O})$ (with multiplicity N, cf. Exercise (4)). If written as the pullback to E_N of a rational function, it follows that $\mathfrak{F}$ has "denominator" w_1; the numerator must then also be a homogeneous linear form $H \in S^1_N$, i.e. $\mathfrak{F} = \left.\frac{H(\underline{w})}{w_1}\right|_{E_N}$. It follows that the $\varphi_N(\mathcal{P}(u_j))$ all lie on $\{H = 0\} \subset \mathbb{P}^{N-1}$, and so

$$0 \;=\; \det[\overbrace{\underbrace{f_i}_{=:F_i}(\mathcal{P}(u_j))}^{\substack{\text{coords. of}\\ \varphi_N(\mathcal{P}(u_j))}}]. \tag{22.2.7}$$

Conversely if this is satisfied then the $\varphi_N(\mathcal{P}(u_j))$ lie on a hyperplane $\{H = 0\}$; one writes down the function $\left.\frac{H(\underline{w})}{w_1}\right|_{E_N}$ and computes its divisor $\sum_{j=1}^N [u_j] - N[0]$, and concludes (by Abel) that $\sum u_j \equiv 0$.

We can push this computation further. Expand $F_i(u_0 + \Delta_j) =$

$$F_i(u_0) + F_i'(u_0)\Delta_j + \cdots + \frac{F_i^{(N-1)}(u_0)}{(N-1)!}\Delta_j^{N-1} + \Delta_j^N(\,\cdots\,)\,,$$

then apply multilinearity of the determinant to expand the RHS of (22.2.7):[3]

$$0 \;=\; \text{const.} \times \prod_{k>\ell}(\Delta_k - \Delta_\ell) \times \det\left[F_i^{(j-1)}(u_0)\right] \;+\; \begin{pmatrix} \text{terms of higher} \\ \text{homog. degree} \\ \text{in the } \{\Delta_j\} \end{pmatrix}.$$

Dividing by $\prod_{k>\ell}(\Delta_k - \Delta_\ell)$ and taking the limit as all $\Delta_j \to 0$ (i.e. all $u_j \to u_0$), this becomes

$$0 \;=\; \det\left[F_i^{(j-1)}(u_0)\right]_{\substack{i=1,\ldots,N\\ j=1,\ldots,N}}. \tag{22.2.8}$$

The determinant on the RHS of (22.2.8) is called the *Wronskian* of $\varphi_N \circ \mathcal{P}$. Notice that in the limit $\sum_{j=1}^N u_j \equiv 0$ becomes $Nu_0 \equiv 0$; so this last condition is equivalent to (22.2.8)!

22.2.9. EXAMPLE. Here is what the above calculation (using multilinearity of the determinant) looks like for $N = 2$, ignoring terms

[3]Note: "higher homog. degree in the $\{\Delta_j\}$" means higher than $\prod_{k>\ell}(\Delta_k - \Delta_\ell)$.

of degree higher than 1 in the $\{\Delta_j\}$:

$$\begin{vmatrix} F_1+\Delta_1 F_1' & F_1+\Delta_2 F_1' \\ F_2+\Delta_1 F_2' & F_2+\Delta_2 F_2' \end{vmatrix} = \begin{vmatrix} F_1+\Delta_1 F_1' & (\Delta_2-\Delta_1)F_1' \\ F_2+\Delta_1 F_2' & (\Delta_2-\Delta_1)F_2' \end{vmatrix}$$

$$= \begin{vmatrix} F_1 & (\Delta_2-\Delta_1)F_1' \\ F_2 & (\Delta_2-\Delta_1)F_2' \end{vmatrix} = (\Delta_2-\Delta_1)\begin{vmatrix} F_1 & F_1' \\ F_2 & F_2' \end{vmatrix}.$$

Using the chain rule and again multilinearity of "det", one finds that the vanishing of the Wronskian is independent of the choice of local coordinate on E. So we can replace u by t (and hence F by f), which yields our "multiple addition theorem":

22.2.10. THEOREM. *u_0 is N-torsion in $\mathbb{C}/\Lambda$ (and the Poncelet iteration closes up at the N^{th} step) if and only if*

$$\det\left[f_i^{(j-1)}(0)\right]_{\substack{i=1,\dots,N\\ j=1,\dots,N}} = 0. \tag{22.2.11}$$

22.2.12. REMARK. The meaning of $f_i^{(j-1)}(0)$ probably requires explanation: first, we are viewing f locally as a function of t (rather than of $[1:t:s] =: \underline{z}$ on E), and the $(j-1)^{\text{st}}$ derivative is (total derivative) with respect to t. The "0" just means t is set to 0 at the end; this is because we are evaluating at ε (i.e. u_0), which has coordinates $[(t,s)=]\,(0,s_0)$.

Application in the case N odd. Obviously we can't compute the Wronskian (22.2.11) unless we know the f_i.

Take $N = 2m+1$. Then for $f_1,\dots,f_{m+1}; f_{m+2},\dots,f_{2m}$ we may choose

$$1, t, \dots, t^m;\ s, st, \dots, st^{m-1}.$$

These have order of pole at 0

$$0, 2, \dots, 2m;\ 3, 5, \dots, 2m+1.$$

The determinant in (22.2.11) is then (using that $\left.\frac{d^{j-1}t^{i-1}}{dt^{j-1}}\right|_0 = 0$ unless $j = i$)

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ddots & & \vdots & \ddots & \vdots \\ 0 & m! & 0 & \cdots & 0 \\ \cdots & * & \left.\frac{d^{m+1}s}{dt^{m+1}}\right|_\varepsilon & \cdots & \left.\frac{d^{2m}s}{dt^{2m}}\right|_\varepsilon \\ \vdots \;\; \ddots & \vdots & \vdots & \ddots & \vdots \\ \cdots & * & \left.\frac{d^{m+1}(st^{m-1})}{dt^{m+1}}\right|_{(0,\varepsilon)} & \cdots & \left.\frac{d^{2m}(st^{m-1})}{dt^{2m}}\right|_{(0,\varepsilon)} \end{vmatrix}.$$

Writing $s = s(t) = \sqrt{\det(tM_C + M_D)} = A_0 + A_1 t + A_2 t^2 + \cdots$ (here $A_0 = s_0$), this becomes a nonzero constant times

$$\begin{vmatrix} A_{m+1} & \cdots & A_{2m} \\ \vdots & \ddots & \vdots \\ A_2 & \cdots & A_{m+1} \end{vmatrix}. \tag{22.2.13}$$

We conclude that there is a circuminscribed $(2m+1)$-gon (and hence a family of such) for the pair C, D iff (22.2.13) vanishes.

22.2.14. EXAMPLE. We work out the case $N = 3$, i.e $m = 1$. The determinant (22.2.13) is just A_2, so we can get a "Poncelet triangle" $\iff A_2 = 0$. Writing $T_i = \frac{1}{t_i}$, calculate

$$\begin{aligned} s &= \sqrt{\det(tM_C + M_D)} = \sqrt{\kappa \prod_{i=1}^3 (t - t_i)} \\ &= C \prod_{i=1}^{3} \sqrt{1 - \frac{t}{t_i}} = C \prod_{i=1}^{3} \left(1 - \frac{T_i}{2} t - \frac{T_i^2}{8} t^2 - \cdots \right) \\ \implies \quad & \frac{A_2}{C} = -\frac{1}{8} \sum_{i=1}^{3} T_i^2 + \frac{1}{4} \left(T_1 T_2 + T_2 T_3 + T_1 T_3\right). \end{aligned}$$

If $T_1 = 1$, solving a quadratic equation we find

$$A_2 = 0 \iff T_2 = \frac{(1+T)^2}{4}, \; T_3 = \frac{(1-T)^2}{4} \text{ for some } T \iff$$

$$\text{equation of } E \text{ reads } s^2 = \kappa(t-1)\left(t - \frac{4}{(1+T)^2}\right)\left(t - \frac{4}{(1-T)^2}\right).$$

If we take

$$M_D = \begin{pmatrix} \frac{-4}{(1+T)^2} & 0 & 0 \\ 0 & \frac{-4}{(1-T)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

corresponding to

$$C = \left\{ \frac{4x^2}{(1+T)^2} + \frac{4y^2}{(1-T)^2} = 1 \right\}, \quad D = \left\{ x^2 + y^2 = 1 \right\},$$

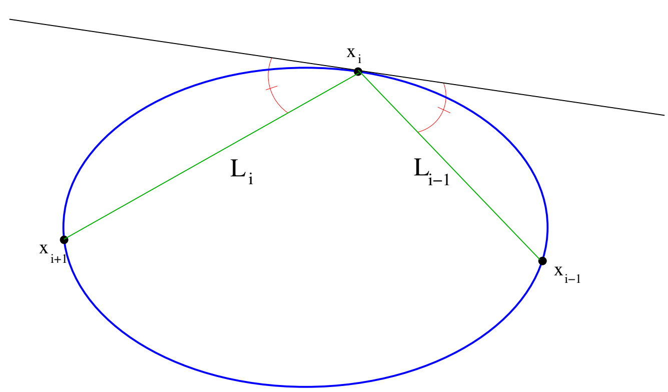
then $\kappa = -1$ and indeed

$$\det (tM_C + M_D) = \left(t - \frac{4}{(1+T)^2} \right) \left(t - \frac{4}{(1-T)^2} \right) (1-t) .$$

This recovers Example 1.3.4(b) from the beginning of the course! It's easy to draw one triangle, but seems quite nontrivial that you get one independent of the starting point.

22.3. Elliptic billards

Returning to the "real" world, let $C_{\mathbb{R}} \subset \mathbb{R}^2$ be an ellipse with foci F_1 and F_2. ($C_{\mathbb{R}}$ consists of all points in $\mathbb{R}^2$, the sum of whose distances from F_1 and F_2 is a fixed constant.) We imagine that $C_{\mathbb{R}}$ is the boundary of a pool table (frictionless, of course!). A *billiard trajectory* for $C_{\mathbb{R}}$ is a sequence of pairs $(x_i, L_i)_{i \geq 0}$ with x_i, $x_{i+1} \in C \cap L_i$ and where L_{i-1}, L_i make equal angles with $T_{x_i}C_{\mathbb{R}}$



— i.e. one has "equality of angles of incidence and reflection".

If $D_{\mathbb{R}}$ is another conic (ellipse or hyperbola) then a (real) *Poncelet trajectory* for $(C_{\mathbb{R}}, D_{\mathbb{R}})$ is a sequence of pairs $(x_i, L_i)_{i\geq 0}$ with $x_i,\ x_{i+1} \in C \cap L_i$ and L_i tangent to $D_{\mathbb{R}}$.

22.3.1. THEOREM. [L. FLATTO, 2003] (a) *Assume $D_{\mathbb{R}}$ is confocal with*[4] *$C_{\mathbb{R}}$. Then the* (*real*) *Poncelet trajectories are billiard trajectories with respect to $C_{\mathbb{R}}$.*

(b) *Conversely, any billiard trajectory for $C_{\mathbb{R}}$ not passing through F_1 or F_2 and not along the minor axis, is a Poncelet trajectory for $C_{\mathbb{R}}$ and some $D_{\mathbb{R}}$ confocal with $C_{\mathbb{R}}$.*

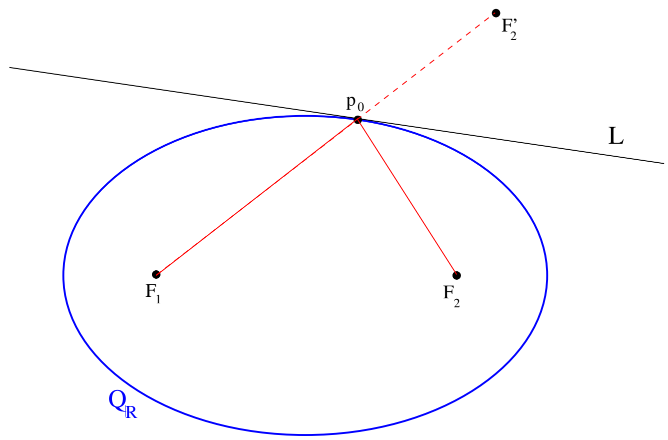
We will prove only (a); Flatto does (b) in Appendix E of his book [L. Flatto, "Poncelet's Theorem," AMS, 2009]. At any rate, the two proofs are very similar.

22.3.2. REMARK. It's worth pointing out right away that given (x_0, L_0) (L_0 not containing F_1 or F_2, and not the minor axis), there is a unique conic $D_{\mathbb{R}}$ confocal with $C_{\mathbb{R}}$ and tangent to L_0. If L_0 passes between F_1 and F_2, $D_{\mathbb{R}}$ is a hyperbola; otherwise, it's an ellipse. One determines this $D_{\mathbb{R}}$, and then from $\sqrt{\det(tM_C + M_D)}$ obtains information (as in §22.2) on whether the Poncelet trajectory closes up. By the Theorem, this is also the billiard trajectory! You'll use this to do a computation in Exercise (2) below. But I should emphasize that if you change (x_0, L_0) (i.e. the choice of billiard trajectory), you have to change the choice of $D_{\mathbb{R}}$ accordingly.

PROOF OF 22.3.1(a). In what follows we will write, given p, q distinct points in $\mathbb{R}^2$, pq for the segment and $|pq|$ for its length.

We begin with a general principle, for a conic $Q_{\mathbb{R}}$ with foci F_1, F_2. Given $p_0 \in Q_{\mathbb{R}}$, let $L := T_{p_0}Q_{\mathbb{R}}$ and denote by F_2' the reflection of F_2 in L as in the picture:

[4]That is, F_1 and F_2 are the foci of $D_{\mathbb{R}}$. If $D_{\mathbb{R}}$ is a hyperbola, this just means that the difference of distances from its points to F_1 and F_2 must remain constant.



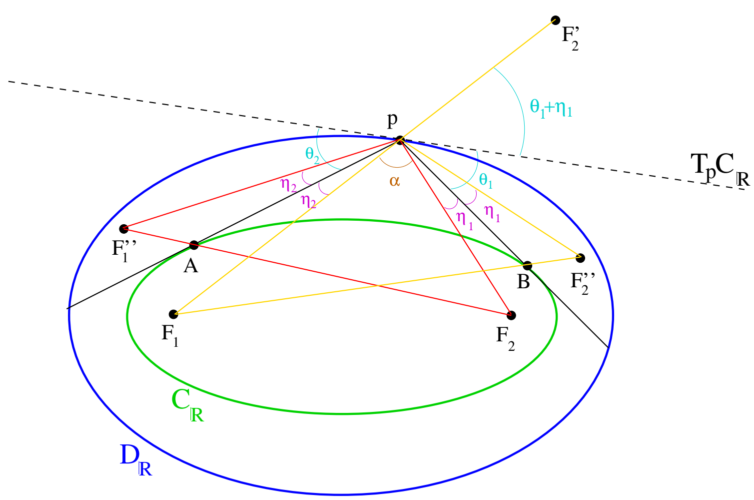
Set $\beta := |F_1 p_0| + |p_0 F_2|$ and note that by the definition of ellipse,

$$|F_1 q| + |q F_2| = \beta \qquad (\forall q \in Q_{\mathbb{R}}).$$

If $p \in L \backslash \{p_0\}$, $|F_1 p| + |pF_2'| = |F_1 p| + |pF_2|$ exceeds β (cf. Exercise (5)), meaning that taking $p = p_0$ minimizes $|F_1 p| + |pF_2'|$. It follows that

$$F_1 p_0 \cup p_0 F_2' = F_1 F_2'. \tag{22.3.3}$$

Now let $C_{\mathbb{R}}$, $D_{\mathbb{R}}$ be confocal — assume that $D_{\mathbb{R}}$ is an ellipse. Applying the principle that (22.3.3) holds for the above construction, leads to a picture



in which the solid black lines are part of a Poncelet iteration and we must show $\theta_1 = \theta_2$ (so that it is a billiard trajectory). Reflection in $T_pC_{\mathbb{R}}$ (dotted black) is denoted by one prime, reflection in solid black lines by two primes.

By definition of ellipse, $F_1A + AF_2 = F_1B + BF_2$, which implies

$$|F_1''F_2| = |F_1F_2''|.$$

From there it is clear that the triangles $F_1''pF_2$ and F_1pF_2'' are rotations of each other (through p), so that $\alpha + 2\eta_1 = \alpha + 2\eta_2$ ($\implies \quad \eta_1 = \eta_2$). It is obvious from the picture that $\theta_1 + \eta_1 = \theta_2 + \eta_2$, and so we indeed conclude that $\theta_1 = \theta_2$. $\square$

Exercises

(1) Consider the pair of conics C, D from Exercise (2) of Chapter 1 once more — but in the following form: write

$$M_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad M_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{pmatrix}$$

and use these to define quadratic forms by e.g.

$$Q_C(X,Y,Z) = \begin{pmatrix} X & Y & Z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = X^2 + Y^2 - Z^2$$

So $Q_C = 0$ defines C and $Q_D = 0$ defines D as conics in $\mathbb{P}^2$. Working in homogeneous coordinates $[V : T : U]$, define an elliptic curve by

$$U^2V = \det\left(T \cdot M_C + V \cdot M_D\right).$$

In affine coordinates, this is $u^2 = \det(t \cdot M_C + M_D)$, where $t = \frac{T}{V}$, $u = \frac{U}{V}$. This is the general prescription for the elliptic curve $\mathcal{E}$ arising in the Poncelet construction, exactly as above. All you have to show is that in the present situation, with M_C and M_D as given, the elliptic curve is singular. It's practically a one-line problem, and you may use the affine setup. But now you are in a position to see "why" the curve being singular should make the Poncelet problem easier, and even to see "why" (from the abstract perspective) the solution involved trigonometric functions.

(2) Let $a > b > 0$ and

$$C_{\mathbb{R}} = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\};$$

it has foci $(\pm\sqrt{a^2 - b^2}\,,\ 0)$. We plan to shoot our pool ball vertically along the line $(L_0 =) \{x = c\}$, where $0 < c < a$ (and $c \neq \sqrt{a^2 - b^2}$). For what value of c does the resulting billiard trajectory yield a triangle? [Hint: the conics confocal with $C_{\mathbb{R}}$ are all of the form $\left\{ \frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} = 1 \right\}$. Also: while straightforward, this is not a 1-line computation!]

(3) Prove that the map (22.2.3) from $\mathbb{P}^1 \to C$ has degree 1.

(4) Show that the normal elliptic curve E_N meets $\{w_1 = 0\}$ with multiplicity N.

(5) Verify the claim just above (22.3.3) that $|F_1 p| + |pF_2| > |F_1 p_0| + |p_0 F_2|$, and explain why (22.3.3) implies $\theta_1 + \eta_1 = \theta_2 + \eta_2$.