CHAPTER 23

Periods of families of elliptic curves

The *periods* of an elliptic curve $E \subset \mathbb{P}^2$ are simply elements of the period lattice $\Lambda_E = \mathbb{Z}\langle \int_\alpha \omega, \int_\beta \omega \rangle$, where α, β are 1-cycles generating $H_1(E, \mathbb{Z})$ and $\omega \in \Omega^1(E)$ is some canonically chosen generator. (For instance, if E is defined over $\mathbb{Q}$, then ω should be the restriction of a rational differential 1-form on $\mathbb{P}^2$ defined over $\mathbb{Q}$.) If E is taken to vary with respect to a parameter $t \in \mathbb{P}^1$, the periods give interesting multivalued transcendental functions (e.g. hypergeometric functions) which are related to modular forms.

In this chapter we explore (via examples) 2 different approaches to computing "period functions" of this sort — the "Euler integral" method and the "Picard-Fuchs" method. The first of these is just a way of computing the integral using Laurent polynomials; the second derives a homogeneous linear ordinary differential equation satisfied by the periods, which yields a recurrence relation for their power-series coefficients. Actually, both methods yield power series at first but one can sometimes recognize what functions they are the power series of. This may sound like complex function theory, but in fact the power series coefficients (esp. when related to modular forms) can have arithmetic meaning, as we shall see in the next chapter; in the context of mirror symmetry (one of several interfaces between algebraic geometry and string theory), power series derived from periods are related to counting curves on threefolds.

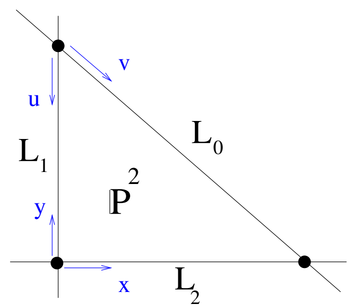
Sections §23.1 and §23.3 will have a bit of overlap with §19.1.

23.1. Holomorphic 1-forms on a smooth cubic $\subset \mathbb{P}^2$

Let $F \in S^3$ define a smooth curve $E = \{F(Z_0, Z_1, Z_2) = 0\}$; by the genus formula $g = \frac{(3-1)(3-2)}{2} = 1$, so that E is elliptic.

23.1.1. EXAMPLE. $F = Z_0Z_1Z_2 - t(Z_0^3 + Z_1^3 + Z_2^3)$, for any $t \in \mathbb{P}^1 \backslash \{0, \frac{1}{3}, \frac{\zeta_3}{3}, \frac{\zeta_3^2}{3}\}$ ($\zeta_3 = e^{\frac{2\pi i}{3}}$).

For the affine forms of the equation we shall use the following notation:



- on $\mathbb{P}^2 \backslash L_0$, the coordinates are $x = \frac{Z_1}{Z_0}$, $y = \frac{Z_2}{Z_0}$, and equation is $f(x,y) := \frac{1}{Z_0^3} F(Z_0, Z_1, Z_2)$;
- on $\mathbb{P}^2 \backslash L_2$, the coordinates are $u = \frac{Z_0}{Z_2}$, $v = \frac{Z_1}{Z_2}$, and equation is $g(u,v) := \frac{1}{Z_2^3} F(Z_0, Z_1, Z_2)$;
- the third neighborhood is left to you;
- on $\mathbb{P}^2 \backslash L_0 \cup L_2$, we have $u = \frac{1}{y}$, $v = \frac{x}{y}$; $y = \frac{1}{u}$, $x = \frac{u}{v}$; and $f(x,y) = y^3 g\left(\frac{1}{y}, \frac{x}{y}\right)$.

Now define a form ω on E by

$$\frac{dx}{f_y}\bigg|_{E\backslash(L_0\cup \mathrm{VT})} = -\frac{dy}{f_x}\bigg|_{E\backslash(L_0\cup \mathrm{HT})} = \frac{du}{g_v}\bigg|_{E\backslash(L_2\cup \mathrm{HT})} = -\frac{dv}{g_u}\bigg|_{E\backslash(L_2\cup \mathrm{VT})} = \cdots$$

where

- the notation $E\backslash(L_0 \cup \mathrm{VT})$ means E minus those points where E intersects L_0 or has a vertical tangent line (similarly, HT means "horizontal tangent");
- equality of any two differentials above is meant in the sense of "where both are defined";
- for example: on E, $f = 0 \implies 0 = df = f_x dx + f_y dy \implies \frac{dx}{f_y} = -\frac{dy}{f_x}$ where ($f = 0$ and) $f_x, f_y \neq 0$;
- the "$\cdots$" means that the third neighborhood stuff is left to you.

Now consider the domains of the first two expressions: since f_x and f_y do not simultaneously vanish (E is smooth!), $\{E\backslash(L_0 \cap \mathrm{VT})\} \cup$

$\{E\backslash(L_0 \cap \text{HT})\}$ is all of $E\backslash L_0$. So the 6 different domains of definition glue to give $(E\backslash L_0) \cup (E\backslash L_1) \cup (E\backslash L_2)$, which is all of E. Moreover, $\frac{dx}{f_y}\Big|_{E\backslash(L_0\cup\text{VT})}$ etc. are all holomorphic where they are defined. We conclude that $\omega \in \Omega^1(E)$.

By Poincaré-Hopf, $\deg((\omega)) = 2g - 2 = 2 - 2 = 0$, and so ω's lack of poles implies it has no zeroes either. Any other $\omega' \in \Omega^1(E)$ has $\frac{\omega'}{\omega} \in \mathcal{O}(E)$, and then by Liouville ω' is a constant multiple of ω. So $\Omega^1(E)$ has dimension 1, and ω spans it.

23.2. Period of a family of cubic curves (Euler integral method)

Now consider the Hesse family E_t of elliptic curves, already given in Example 23.1.1, with affine form

$$f(x,y) = xy - t(x^3 + y^3 + 1) = 0, \qquad t \in \mathbb{C}.$$

(For the four values of t excluded in the example, E_t is singular hence not an elliptic curve. I won't write f_t because the subscript is reserved here for partial derivatives.) An alternate form of the equation, valid on $\mathbb{C}^* \times \mathbb{C}^*$, is

$$1 - \underbrace{t\left(\frac{x^3 + y^3 + 1}{xy}\right)}_{=:\varphi(x,y)} = 0,$$

where φ belongs to the ring of Laurent polynomials $\mathbb{C}[x, x^{-1}, y, y^{-1}]$.

From the last section, we have the family of holomorphic 1-forms

$$\omega_t := \frac{dx}{f_y}\bigg|_{E_t} \in \Omega^1(E_t).$$

To obtain a family of 1-cycles, notice that $\{|x|{=}|y|{=}1\} \cap E_t$ is empty for $|t| < \frac{1}{3}$, since $|\varphi(x,y)| < 3$ for x, y in the unit circle. Indeed,

$$\gamma_t := \{|x| = 1,\ |y| \leq 1\} \cap E_t$$

has this empty set as its boundary $\partial\gamma_t$; and so we would like to compute the period

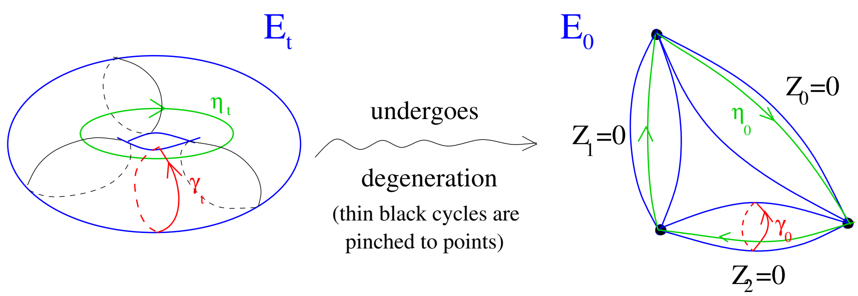
$$P(t) := \int_{\gamma_t} \omega_t$$

as a function of t, on the open disk $|t| < \frac{1}{3}$.

Now since $H_1(E_t, \mathbb{Z})$ has rank 2, there is a complementary 1-cycle η_t on E_t, and corresponding period

$$Q(t) := \int_{\eta_t} \omega_t.$$

Noticing from the homogeneous form of the equation that $E_0 = \{Z_0 Z_1 Z_2 = 0\}$ is a union of 3 lines (each $\cong \mathbb{P}^1$), we can easily visualize what happens to E_t, γ_t, and η_t as t tends to zero:



From the fact that ω_t tends (as $t \to 0$) to $\frac{dx}{x}$ on $Z_2 = 0$, and η_0 passes through the poles of this form while γ_0 traverses the unit circle around them, we infer that $Q(t) \to \infty$ as $t \to 0$ but $P(t) \to 2\pi i$.

We now compute $P(t)$ more precisely, by first noting that the area integral

$$\iint_{|x|=|y|=1} \frac{dx \wedge dy}{f(x,y)} = \iint \frac{dx \wedge df}{f_y \cdot f}$$

(since $df = f_x dx + f_y dy$ and $dx \wedge dx = 0$)

$$= \int_{|x|=1} \left(\int_{|y|=1} \frac{df(x,y)}{f(x,y)} \cdot \frac{1}{f_y(x,y)} \right) dx$$

(where inside the parentheses x is a fixed constant). Now thinking about the equation $f(x,y) = 0$ for $|t|$ small (and x fixed with $|x| = 1$), we have $y^3 + ay + b = 0$ where $a = -\frac{x}{t}$ is big and $b = x^3 + 1$ is not. This means that two of the roots are big and one is small — in particular, there is exactly one solution $y(x)$ with modulus < 1.

Therefore, by Cauchy's residue theorem, the integral above

$$= \int_{|x|=1} \left(2\pi i \cdot \frac{1}{f_y(x, y(x))}\right) dx$$
$$= 2\pi i \int_{\substack{|x|=1 \\ y=y(x)}} \frac{dx}{f_y} = 2\pi i \int_{\gamma_t} \omega_t,$$

and

$$P(t) = \frac{1}{2\pi i} \iint_{|x|=|y|=1} \frac{dx \wedge dy}{f(x,y)} = \frac{1}{2\pi i} \iint_{|x|=|y|=1} \frac{\frac{dx}{x} \wedge \frac{dy}{y}}{1 - t\varphi(x,y)}$$
$$= \frac{1}{2\pi i} \iint_{|x|=|y|=1} \sum_{n \geq 0} t^n \varphi^n \mathrm{dlog} x \wedge \mathrm{dlog} y,$$

where $\mathrm{dlog}x = \frac{dx}{x}$ and φ^n means simply the n^{th} power of $\varphi(x,y)$. Using Cauchy residue twice, we obtain

$$P(t) = 2\pi i \sum_{n \geq 0} t^n \varphi^n(0,0),$$

in which $\varphi^n(0,0) =: [\varphi^n]_0$ is the constant term of $\varphi^n = (x^2y^{-1} + x^{-1}y^2 + x^{-1}y^{-1})^n$.

We can make this more explicit. Given a product

$$\underbrace{(x^2y^{-1} + x^{-1}y^2 + x^{-1}y^{-1}) \cdot \cdots \cdot (x^2y^{-1} + x^{-1}y^2 + x^{-1}y^{-1})}_{n \text{ times}},$$

each contribution to the constant term comes from exponents summing to $(0,0)$ (so that the monomials multiply to x^0y^0). But the only combinations of $(2,-1)$, $(-1,2)$, $(-1,-1)$ summming to $(0,0)$ are: $m(2,-1) + m(-1,2) + m(-1,-1)$. So φ^n can only have a nonzero constant term if $n = 3m$ (i.e. $3|n$); and this constant term is then given by the number of ways to choose

$$\begin{cases} x^2y^{-1} \text{ from } m \text{ factors} \\ x^{-1}y^2 \text{ from } m \text{ factors} \\ x^{-1}y^{-1} \text{ from } m \text{ factors} \end{cases}$$

which is $\binom{3m}{m,m,m} := \frac{(3m)!}{m!m!m!}$.

We conclude that

$$P(t) \;=\; 2\pi i \sum_{m \geq 0} t^{3m} \cdot \frac{(3m)!}{(m!)^3} \;=\; 2\pi i \, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; (3t)^3\right), \tag{23.2.1}$$

where by definition (writing $(a)_m := a(a+1)\cdots(a+m-1)$ for the "Pochhammer symbol")

$${}_2F_1(a,b;c;u) := 1 + \sum_{m \geq 1} \frac{(a)_m (b)_m}{(c)_m m!} u^m$$

is the *Gauss hypergeometric function*.

Notice that $P(t)$ is "really" a function of $(3t)^3 =: u$. This reflects the symmetry in the family E_t:

$$\begin{aligned} E_t \;&\longrightarrow\; E_{\zeta_3 t} \\ (x,\, y) &\longmapsto (\zeta_3 x,\, y). \end{aligned}$$

The Gauss hypergeometric function satisfies a well-known ODE. In this case, writing $P_0(u) = P(t)$ and $Q_0(u) = Q(t)$,

$$\left\{ u(1-u)\frac{d^2}{du^2} + (1-2u)\frac{d}{du} - \frac{2}{9} \right\} P_0(u) \;=\; 0. \tag{23.2.2}$$

For reasons that will become clear in §23.5, the ODE satisfied by P_0 must be satisfied by the other period Q_0, which turns out to have a term of the form $\frac{\log u}{2\pi i} P_0(u)$ reflecting the fact that "following η_t around $t = 0$" yields $\eta_t + 3\gamma_t$ in $H_1(E_t, \mathbb{Z})$.

23.3. Cohomology of an elliptic curve E

Let V be a finite-dimensional vector space over $\mathbb{C}$, with basis $\{e_k\}_{k=1}^n$. The second tensor power of V, written $V \otimes V$, is the n^2-dimensional vector space consisting of finite sums $\sum_i v_i \otimes w_i$ ($v_i, w_i \in V$) subject to bilinearity (e.g., on the left $(\alpha v + \beta w) \otimes u \;=\; \alpha v \otimes u + \beta w \otimes u$); it has basis $\{e_k \otimes e_\ell\}_{k,\ell=1}^n$. The second exterior power $\bigwedge^2 V$ consists of finite sums $\sum v_i \wedge w_i$ satisfying bilinearity and also $v \wedge w = -w \wedge v$ (so that $v \wedge v = 0$); it may be viewed as a quotientor sub-space of $V \otimes V$, and has basis $\{e_k \wedge e_\ell\}_{1 \leq k < \ell \leq n}$ hence dimension $\binom{n}{2}$. In particular, if $\dim V = 2$, then $\dim(\bigwedge^2 V) = 1$; this is essentially the only case we shall use.

Recall the homology groups

$$H_1(E, \mathbb{Z}) = \frac{\mathbb{Z}\,\langle \text{closed paths ("1-cycles") on } E \rangle}{\mathbb{Z}\,\langle \text{boundaries of regions in } E \rangle}$$

from §19.1. The homology class represented by a 1-cycle γ (in the numerator) will be denoted by $[\gamma]$. Dualizing homology, we define *cohomology* groups (with complex coefficients) by

$$H^1(E, \mathbb{C}) := \operatorname{Hom}\left(H_1(E, \mathbb{Z}), \mathbb{C}\right) \ \ (\cong \mathbb{C}^2).$$

We want to argue that cohomology classes (i.e. elements of H^1) can be represented by differential forms.

To that end, write $A^0(E)$ for C^∞ functions, and $A^1(E)$ for the C^∞ 1-forms on E — expressed by $f dx + g dy$ in local coordinates, where $z = x + iy$ and f, g are C^∞ — familiar from Chapter 13. Finally, let $A^2(E)$ denote the C^∞ *2-forms*; these are objects locally of the form

$$G dx \wedge dy \ (= -G dy \wedge dx = -\frac{i}{2} G dz \wedge d\bar{z} = \frac{i}{2} G d\bar{z} \wedge dz)$$

(with G smooth), which you may think of as fields of infinitesimal area elements. In more sophisticated terms, they are C^∞ sections of the bundle $\bigwedge^2 T^*E = \cup_{p \in E} \bigwedge^2 T^*_p E$. (Refer to §13.1 for notation.)

The various degrees of forms are "connected" by exterior differentiation

$$d : A^0(E) \to A^1(E)$$

sending

$$F \mapsto dF := F_x dx + F_y dy = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z},$$

and

$$d : A^1(E) \to A^2(E)$$

sending

$$f dx + g dy \mapsto df \wedge dx + dg \wedge dy = (g_x - f_y) dx \wedge dy.$$

The 1^{st} *de Rham cohomology* group of E is then defined by

$$H^1_{dR}(E,\mathbb{C}) := \frac{\ker(d) \subset A^1(E)}{d(A^0(E))} = \frac{\text{“closed” } C^\infty \text{ forms}}{\text{“exact” } C^\infty \text{ forms}};$$

the class represented by a 1-form ω is written $[\omega]$.

23.3.1. LEMMA. *The map*

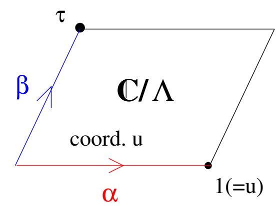
$$\theta : H^1_{dR}(E,\mathbb{C}) \to H^1(E,\mathbb{C})$$
$$[\omega] \mapsto \{[\gamma] \mapsto \textstyle\int_\gamma \omega\}$$

is well-defined, and an isomorphism. (Here, “$[\gamma] \mapsto \int_\gamma \omega$” means the complex-linear functional on homology classes given by integrating ω over a representative 1-cycle.)

PROOF. First we check well-definedness: if γ is closed ($\partial\gamma = 0$) and ω exact ($\omega = d\eta$), then $\theta([\omega]) = 0$ since $\int_\gamma \omega = \int_\gamma d\eta = \int_{\partial\gamma} \eta = 0$. If γ is a boundary ($\gamma = \partial\Gamma$) and ω is closed ($d\omega = 0$) then $\int_\gamma \omega = \int_{\partial\Gamma} \omega = \int_\Gamma d\omega = 0$, so that $\theta([\omega])$ is defined on the level of homology classes. (The middle equality in both cases — swapping ∂ and d — is *Stokes's theorem*, a generalization of the fundamental theorem of calculus for differential forms.)

To see that θ is injective, assume $\theta([\omega]) = 0$, and let p be a point of E. Then $\mathcal{F}(q) = \int_p^q \omega$ defines a C^∞ function $\mathcal{F}$ on E. (The reason $\mathcal{F}$ isn't “multivalued” is that two paths from p to q differ by a cycle γ, and $\int_\gamma \omega = 0$ by the assumption.) Now $\omega = d\mathcal{F}$ by the fundamental theorem of calculus, and so $[\omega] = 0$.

Finally, write α, β for a basis of $H_1(E,\mathbb{Z})$. Using the identification $H^1(E,\mathbb{C}) \xrightarrow{\cong} \mathbb{C}^2$ which evaluates a functional against this basis, a nice way to think about the map θ is as sending $[\omega] \mapsto \begin{pmatrix} \int_\alpha \omega \\ \int_\beta \omega \end{pmatrix} \in \mathbb{C}^2$. Moreover, the Abel map $E \xrightarrow{\cong} \mathbb{C}/\Lambda_E$ identifies ω with du. Rescaling ω (by a complex constant) so that $\int_\alpha \omega = 1$, the 2-vector becomes $\theta([du]) = \begin{pmatrix} \int_\alpha du \\ \int_\beta du \end{pmatrix} = \begin{pmatrix} 1 \\ \tau \end{pmatrix}$ (where we may assume $\tau \in \mathfrak{H}$), and we have the standard picture



Noting that $\theta([d\bar{u}]) = \begin{pmatrix} \int_\alpha d\bar{u} \\ \int_\beta d\bar{u} \end{pmatrix} = \begin{pmatrix} \overline{\int_\alpha du} \\ \overline{\int_\beta du} \end{pmatrix} = \begin{pmatrix} 1 \\ \bar{\tau} \end{pmatrix}$, we conclude that θ is surjective since $\binom{1}{\tau}, \binom{1}{\bar{\tau}}$ span $\mathbb{C}^2$. $\square$

23.3.2. REMARK. (a) Meromorphic 1-forms on a Riemann surface are always closed on the complement of their poles (which is where "d" makes sense), since locally [f mero.] $d\{fdz\} = df \wedge dz = \frac{\partial f}{\partial z} dz \wedge dz$ and $dz \wedge dz = -dz \wedge dz = 0$. (Here we have used $\frac{\partial f}{\partial \bar{z}} = 0$, which expresses the holomorphicity of f away from said poles.) Using the same formula as above (i.e. $\omega \mapsto \{\gamma \mapsto \int_\gamma \omega\}$), we can define a map

$$\frac{\ker(Res) \subset \mathcal{K}^1(E)}{d(\mathcal{K}(E))} \xrightarrow{\tilde{\theta}} H^1(E, \mathbb{C})$$

which also turns out to be an isomorphism. (Here $\ker(Res)$ consists of forms with no residues – in particular, with no simple poles. This doesn't mean they're holomorphic though!)

(b) A nonzero holomorphic 1-form ω cannot be d of a smooth function G or meromorphic function f. (Locally the integral of ω is a holomorphic function, so in either case G or f would actually have to be holomorphic, hence by Liouville constant, making ω zero.) So we have a commuting diagram of injective homomorphisms

(23.3.3)
$$\begin{array}{ccc} \Omega^1(E) & \hookrightarrow & \frac{\ker(d) \subset A^1(E)}{\text{exact}} \\ \big\downarrow & \overset{\hat{\theta}}{\searrow} & \cong \big\downarrow \theta \\ \frac{ker(Res) \subset \mathcal{K}^1(E)}{\text{exact}} & \underset{\cong}{\xrightarrow{\tilde{\theta}}} & H^1(E, \mathbb{C}). \end{array}$$

In what follows $\hat{\theta}, \tilde{\theta}, \theta$ will all just be denoted θ, which you should read "take the period vector associated to this 1-form".

23.4. Differentiating cohomology classes

Given a family $\{E_t\}_{t\in\mathbb{P}^1}$ of elliptic curves (smooth but for finitely many t) with holomorphic forms $\omega_t \in \Omega^1(E_t)$, write

$$\theta(\omega_t) =: \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix}.$$

This assumes a choice (this unfortunately only works locally in t) of basis α_t, β_t for $H_1(E_t, \mathbb{Z})$, so that $P(t) = \int_{\alpha_t} \omega_t$, $Q(t) = \int_{\beta_t} \omega_t$. We can differentiate this period vector to obtain

$$\begin{pmatrix} P'(t) \\ Q'(t) \end{pmatrix},$$

which for each t (considering the isomorphisms in (23.3.3)) is θ of something in $\ker(d) \subset A^1(E_t)$ or $\ker(Res) \subset \mathcal{K}^1(E)$ (but not $\Omega^1(E_t)$). We will use the latter, and we write ω_t' for a family of residue-free meromorphic 1-forms satisfying

$$\theta(\omega_t') = \begin{pmatrix} P'(t) \\ Q'(t) \end{pmatrix}.$$

The point is that by differentiating families of cohomology classes you get a new family of cohomology classes.

23.4.1. EXAMPLE. Consider the Legendre family $E_t \subset \mathbb{P}^2$ given by the projective closure of

$$y^2 = x(x-1)(x-t),$$

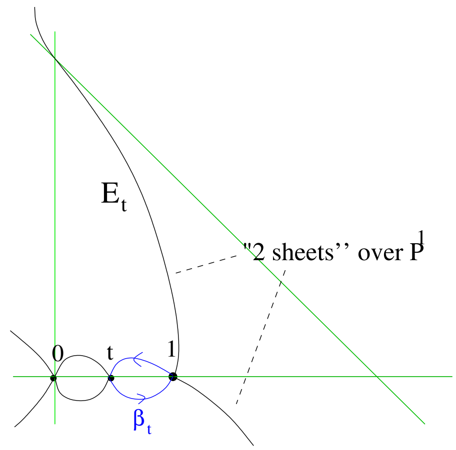
with holomorphic 1-form family

$$\omega_t = \left.\frac{dx}{y}\right|_{E_t} \in \Omega^1(E_t).$$

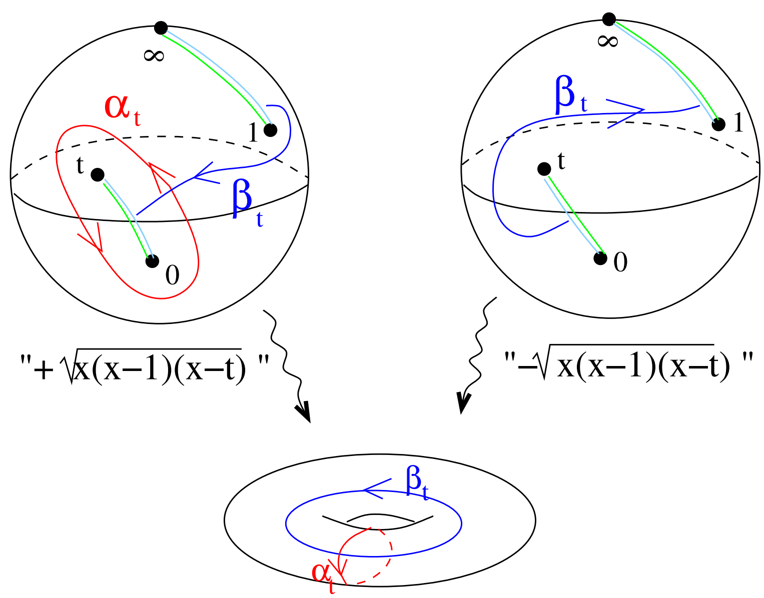
We have

$$\theta(\omega_t) = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} = \begin{pmatrix} \int_{\alpha_t} \omega_t \\ \int_{\beta_t} \omega_t \end{pmatrix} = \begin{pmatrix} \int_{\alpha_t} \frac{dx}{\pm\sqrt{x(x-1)(x-t)}} \\ \int_{\beta_t} \frac{dx}{\pm\sqrt{x(x-1)(x-t)}} \end{pmatrix},$$

where α_t, β_t are the 1-cycles exhibited in the schmatic picture



or the topological picture



— which shows the two sheets (each is a $\mathbb{P}^1$ with slits from 0 to t and 1 to ∞) being glued together to give E_t (cf. §2.3).

From the latter picture, it is clear that for t small we may take α_t to be stationary on its sheet as t moves, and the two "pieces" of β_t on the different sheets not to change either. Therefore we may differentiate the above integrals under the integral sign (by $\frac{d}{dt}$) to obtain

$$\begin{pmatrix} P'(t) \\ Q'(t) \end{pmatrix} = \begin{pmatrix} \int_{\alpha_t} \frac{\frac{1}{2}dx}{\pm(x-t)\sqrt{x(x-1)(x-t)}} \\ \int_{\beta_t} \frac{\frac{1}{2}dx}{\pm(x-t)\sqrt{x(x-1)(x-t)}} \end{pmatrix}$$

and

$$\begin{pmatrix} P''(t) \\ Q''(t) \end{pmatrix} = \begin{pmatrix} \int_{\alpha_t} \frac{\frac{3}{4}dx}{\pm(x-t)^2\sqrt{x(x-1)(x-t)}} \\ \int_{\beta_t} \frac{\frac{3}{4}dx}{\pm(x-t)^2\sqrt{x(x-1)(x-t)}} \end{pmatrix};$$

obviously the first is $\theta\left(\frac{1/2}{x-t}\frac{dx}{y}|_{E_t}\right)$ and the second $\theta\left(\frac{3/4}{(x-t)^2}\frac{dx}{y}|_{E_t}\right)$, and so we have

$$\omega_t' = \frac{1/2}{(x-t)}\left.\frac{dx}{y}\right|_{E_t}, \qquad \omega_t'' = \frac{3/4}{(x-t)^2}\left.\frac{dx}{y}\right|_{E_t}.$$

These both belong to $\ker(Res) \subset \mathcal{K}^1(E_t)$, since their only poles are at $(t,0)$ (orders 2 and 4 resp.) and the sum of the residues of a meromorphic form must always be zero.

Of course, $\theta(\omega_t)$, $\theta(\omega_t')$, and $\theta(\omega_t'')$ must be linearly dependent in $\mathbb{C}^2$! Therefore, $[\omega_t]$, $[\omega_t']$, and $[\omega_t'']$ are linearly dependent in $\mathcal{K}^1(E_t)$ modulo $d(\mathcal{K}(E_t))$, i.e. as cohomology classes.

23.5. The Picard-Fuchs equation

Since what is being differentiated in the last section is really cohomology *classes* (via the identification with $\mathbb{C}^2$), it makes sense to write

$$D_t[\omega_t] = [\omega_t'], \qquad D_t^2[\omega_t] = [\omega_t''].$$

With this notation, the linear dependence observation above implies an ODE of the form

$$\underbrace{\left(A(t)D_t^2 + B(t)D_t + C(t)\right)}_{=:D_{\text{PF}}}(\cdot) = 0 \tag{23.5.1}$$

satisfied by $[\omega_t]$ (as a varying cohomology class) *hence by $P(t)$ and $Q(t)$!*

However, to find A, B, and C, we have to compute. We start by differentiating a meromorphic function

$$\underbrace{d\left(\frac{2y}{(x-t)^2}\bigg|_{E_t}\right)}_{\in\mathcal{K}(E)} = \frac{-4y\,dx}{(x-t)^3}\bigg|_{E_t} + \frac{2dy}{(x-t)^2}\bigg|_{E_t},$$

which using $y^2 = x(x-1)(x-t) \implies dy = \frac{3x^2-2(1+t)x+t}{2y}dx$ becomes

$$= \left(\frac{-4y^2}{(x-t)^3y} + \frac{3x^2-2(1+t)x+t}{(x-t)^2y}\right)dx\bigg|_{E_t}$$

and using $y^2 = x(x-1)(x-t)$ again

$$= \frac{-x^2+(2-2t)x+t}{(x-t)^2y}dx\bigg|_{E_t}$$

$$= \frac{-(x-t)^2-2tx+t^2+(2-2t)x+t}{(x-t)^2}\frac{dx}{y}\bigg|_{E_t}$$

$$= -\omega_t + \frac{(2-4t)x+t^2+t}{(x-t)^2}\frac{dx}{y}\bigg|_{E_t}$$

$$= \cdots = -\omega_t + 4(1-2t)\omega_t' + 4t(1-t)\omega_t''.$$

So this last expression is d of a meromorphic function, hence (has both its periods 0 and) is trivial in $H^1(E_t, \mathbb{C})$. We conclude that (dividing through by 4 to simplify)

$$D_{\text{PF}} = t(t-1)D_t^2 + (2t-1)D_t + \frac{1}{4} \tag{23.5.2}$$

kills $[\omega_t]$. From ODE theory, the associated indicial equation is

$$r(r-1) + \left(\lim_{t\to 0}\frac{B(t)}{A(t)}t\right)r + \left(\lim_{t\to 0}\frac{C(t)}{A(t)}\right) = r^2$$

which has a double root, implying one holomorphic solution (unique up to scale) and one logarithmic solution near $t = 0$.

23.6. Computation of a period (Picard-Fuchs method)

First let's compute its limit

$$\lim_{t\to 0} \underbrace{\int_{\alpha_t} \omega_t}_{P(t)} \;=\; \lim_{t\to 0} \int_{\alpha_t} \frac{dx}{\pm\sqrt{x(x-1)(x-t)}}.$$

Referring to the picture of α_t (on the slit $\mathbb{P}^1$) above, this

$$= \oint \frac{dx}{x\sqrt{x-1}} \;=\; 2\pi i \cdot \mathrm{Res}_0 \left(\frac{dx}{x\sqrt{x-1}} \right)$$

$$= \; 2\pi i \cdot \frac{1}{\sqrt{-1}} \;=\; 2\pi,$$

and so $P(t)$ must be "the" holomorphic solution.

Next write $P(t) = 2\pi \sum a_n t^n$, $a_0 = 1$, and apply D_{PF}:

$$0 \;=\; D_{\mathrm{PF}} \sum a_n t^n$$

$$= \sum_{n\geq 0} \left[t(t-1)(n+2)(n+1)a_{n+2} + (2t-1)(n+1)a_{n+1} + \frac{1}{4}a_n \right] t^n$$

where we have shifted indices after differentiating. Collecting terms with like powers of t, this

$$= \sum_{n\geq 0} \left[\frac{1}{4}a_n - (n+1)a_{n+1} \right] t^n \;+\; \sum_{n\geq 0} \begin{bmatrix} 2(n+1)a_{n+1} \\ -(n+1)(n+2)a_{n+2} \end{bmatrix} t^{n+1}$$

$$+ \sum_{n\geq 0} (n+1)(n+2)a_{n+2}t^{n+2}.$$

Shifting indices once more, we have

$$= \sum_{n\geq 0} \left[\frac{1}{4}a_n - (n+1)a_{n+1} + 2na_n - n(n+1)a_{n+1} + n(n-1)a_n \right] t^n$$

$$= \sum_{n\geq 0} \left[\left(n+\frac{1}{2}\right)^2 a_n - (n+1)^2 a_{n+1} \right] t^n.$$

Since this power series is zero, we get a recurrence relation for the coefficients of $P(t)$:

$$a_{n+1} = \left(\frac{n+\frac{1}{2}}{n+1}\right)^2 a_n,$$

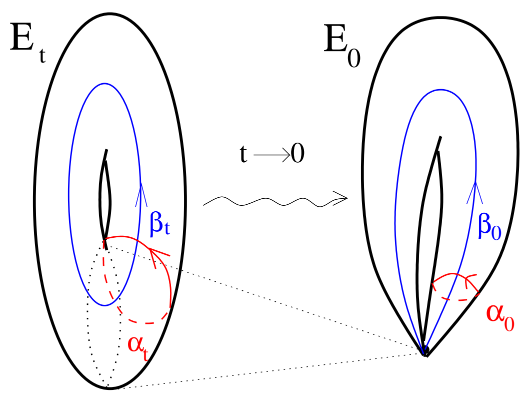
so that

$$\begin{aligned} a_n &= \underbrace{a_0}_{=1} \cdot \left(\frac{1/2}{1}\frac{3/2}{2}\cdots\frac{1/2+n-1}{n}\right)^2 \\ &= \left(\frac{-1/2\cdot -3/2\cdot\cdots\cdot(-1/2-n+1)}{n!}\right)^2 = \binom{-1/2}{n}^2, \end{aligned}$$

and

$$P(t) = 2\pi \sum_{n\geq 0} \binom{-1/2}{n}^2 t^n = 2\pi\, {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;t). \tag{23.6.1}$$

Again, the situation as $t \to 0$ looks like



In the next chapter the formula for $P(t)$ will be connected to counting rational points on cubics over $\mathbb{F}_p$.

Exercises

(1) Check that ${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;(3t)^3\right) = \sum_{m\geq 0} t^{3m}\cdot\frac{(3m)!}{(m!)^3}$ by writing out the Pochhammer symbols.

(2) Show that the curves

$$E_t := \{Z_0Z_1W_0W_1 - t(Z_1-Z_0)^2(W_1-W_0)^2 = 0\} \subset \mathbb{P}^1_{Z_0:Z_1}\times\mathbb{P}^1_{W_0:W_1}$$

are in fact elliptic (except at those finitely many t — which ones? — for which they are singular). You could do this by projecting to the first $\mathbb{P}^1$ and using Riemann-Hurwitz to compute the genus. (To use R-H in this way, first find all "vertical tangents", i.e. places on the curve where the partials with respect to W_0 and W_1 vanish.)

(3) Writing the family of curves from the last exercise in affine form, $xy - t(x-1)^2(y-1)^2 = 0$ (or $1 - t\varphi(x,y) = 0$), define a family of loops $\gamma_t \in H_1(E_t, \mathbb{Z})$ for small t, and a family of holomorphic forms $\omega_t \in \Omega^1(E_t)$, exactly as in the text. Compute the period $P(t) := \int_{\gamma_t} \omega_t$ as a power series, using the computation done above as a model.

(4) Check that $\tilde{\theta}$ in Remark 23.3.2(a) is well-defined and an isomorphism. [Hint: the difficult part here is surjectivity. Working with $\mathbb{C}/\Lambda$ ($\Lambda = \mathbb{Z}\langle \lambda_1, \lambda_2 \rangle$) in lieu of $E \subset \mathbb{P}^2$, let $\zeta(u)$ be a primitive of $\wp(u)$ on $\mathbb{C} \setminus \Lambda$, and note that $\zeta(u + m\lambda_1 + n\lambda_2) = m\eta_1 + n\eta_2$ for some $\eta_i \in \mathbb{C}$ (why?). Let $\mathfrak{F}$ be a fundamental domain with 0 in its interior, and consider $\int_{\partial\mathfrak{F}} \zeta(u)du$. You will find that a certain determinant doesn't vanish, which shows that a certain map has rank 2.]

(5) Find $Q(t)$ in the Legendre example by plugging the ansatz

$$\frac{\log t}{2\pi} P(t) + \sum_{n \geq 1} b_n t^n$$

into the Picard-Fuchs equation.

(6) Let E_t denote the family of elliptic curves in $\mathbb{P}^2$ with affine equation $(F(x,y) =)\, xy - (3x^3 + 2y^3 + 1)t = 0$. This comes with a family of holomorphic 1-forms given (in affine form) by $\omega_t = \frac{dx}{(2\pi i)F_y}|_{E_t}$. (a) Compute (as a power series in t) a period of ω_t. (b) Deduce from this the smallest nonzero value of $|t|$ for which E_t is singular.