CHAPTER 25

The algebraicity of global analytic objects

To kick off the next part of this course, on curves of higher genus, this chapter will demonstrate two approaches to the following result: *meromorphic (or holomorphic) functions and forms on normalizations of algebraic curves, all arise as pullbacks of functions and forms on projective space constructed from rational functions (quotients of homogeneous polynomials) and their differentials*. This is a special instance of Serre's GAGA principle ("global analytic is global algebraic" in the projective setting), and is proved (in §25.1) using techniques from Chapter 8 together with the primitive element theorem. The related fact that any compact complex-analytic subvariety[1] $X \subset \mathbb{P}^n$ is a projective algebraic variety is checked for curves in $\mathbb{P}^2$ in Exercise (2).

For holomorphic forms, we would like a more precise result (already hinted at in Remark 21.1.1) on how to think of the holomorphic forms on a normalization "rationally". It is important at this point to recall part (B) of the Normalization Theorem 3.2.1, which says that *every Riemann surface can be obtained as the normalization of an algebraic curve in* $\mathbb{P}^2$, even one with only nodal (ordinary double point) singularities. So in the course of analyzing nodal curves in §25.2 we will actually have proved (cf. Prop. 25.2.4(c)) that for any Riemann surface M of genus g,

$$\dim(\Omega^1(M)) = g.$$

Featuring prominently in this section is the space of homogeneous polynomials vanishing at the set of nodes, which will play a key role in the proof of Riemann-Roch in the next chapter.

[1]This means that in a neighborhood (in $\mathbb{P}^n$) of each point $p \in X$, X is the zero-locus of a finite set of holomorphic functions.

25.1. Chow's theorem for algebraic curves

Let $C \subset \mathbb{P}^2$ be an irreducible projective algebraic curve of degree d; applying a projective transformation if necessary we have $[0{:}0{:}1] \notin C$. Start by normalizing C; that is, express it as the image of a morphism $\sigma : \widetilde{C} \to \left(\mathbb{P}^2 \backslash \{[0:0:1]\}\right)$. One may evidently produce meromorphic functions on the Riemann surface $\widetilde{C}$, by pulling back the rational functions $\mathbb{C}(x,y)$ under σ^*. (Recall $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$ on $\mathbb{P}^2$; and the field $\mathbb{C}(x,y)$ consists of all quotients of homogeneous polynomials in X, Y, Z of the same degree.) More precisely,[2] writing $\mathbb{C}(x,y)_C$ for the subring of rational functions whose polar set does not contain C, define the field of rational functions on $\widetilde{C}$ by

$$\mathbb{C}(\widetilde{C}) := \sigma^* \mathbb{C}(x,y)_C.$$

Next we consider the projection

$$\begin{aligned} \pi : \left(\mathbb{P}^2 \backslash \{[0:0:1]\}\right) &\twoheadrightarrow \mathbb{P}^1 \\ [Z:X:Y] &\mapsto [Z:X], \end{aligned}$$

whose composition $\tilde{\pi} := \pi \circ \sigma$ with the normalization presents $\widetilde{C}$ as a d-sheeted[3] branched cover of $\mathbb{P}^1$. Write $\mathfrak{B}\ (\subset \mathbb{P}^1)$ for the branch locus, and Γ for a path containing $\mathfrak{B}$ with $\mathbb{P}^1 \backslash \Gamma$ simply connected (cf. §8.2). We have inclusions

$$\begin{array}{ccccc} \tilde{\pi}^* \mathbb{C}(x) & \subset & \mathbb{C}(\widetilde{C}) & \subset & \mathcal{K}(\tilde{C}) \\ & & \text{rat'l} & & \text{mero.} \\ & & \text{fcns.} & & \text{fcns.} \end{array} , \tag{25.1.1}$$

where the first is obtained by noting $\pi^* \mathbb{C}(x) \subset \mathbb{C}(x,y)_C$ and

$$\tilde{\pi}^* \mathbb{C}(x) = \sigma^* \pi^* \mathbb{C}(x) \subset \sigma^* \mathbb{C}(x,y)_C.$$

Now, one might initially speculate that the right-hand inclusion of (25.1.1) is proper when C has singularities such as nodes, since: (a)

[2]In more commutative-algebraic terms: if C has affine equation $f = 0$, $f \in R := \mathbb{C}[x,y]$, then the coordinate ring of C is $R/(f)$, with fraction field $\mathbb{C}(\tilde{C})$; while $\mathbb{C}(x,y)$ is the fraction field of R and $\mathbb{C}(x,y)_C$ is the localization $R_{(f)}$.

[3]By Bézout, the mapping degree $\deg(\tilde{\pi}) = d$ $(= \deg(C))$.

a node has 2 preimage points $p, q \in \widetilde{C}$, (b) at first glance the pullback of a function would seem to have the same value at p and q, and (c) meromorphic functions on $\widetilde{C}$ ought to be able to take different values at distinct points, right? The weak link in this chain of reasoning is (b), as you can see from the following

25.1.2. EXAMPLE. $C = \{Y^2Z = X^2(X-Z)\}$ has tangent lies $Y = \pm X$ at its ODP $[1:0:0]$. The pullback of $\frac{x}{y}$ to $\widetilde{C}$ therefore takes values 1 and -1 (resp.) at the 2 points lying over the ODP.

The point is that rational functions are not well-defined at all points of $\mathbb{P}^2$, and this can be used to our advantage to get "more" functions on singular curves. So it becomes plausible that the right-hand inclusion of (25.1.1) is an equality, and that is exactly what we shall prove in the rest of the section.

To that end, let $\varphi \in \mathcal{K}(\widetilde{C})^*$ be a nonzero meromorphic function, and denote by P the set of poles of φ. Writing[4]

$$0 = f(x,y) = y^d + a_1(x)y^{d-1} + \cdots + a_d(x) \tag{25.1.3}$$

for the affine equation of C, we have as in §8.2 distinct solutions $\{y_j(x)\}_{j=1}^d$ to $f(x,\cdot) = 0$ over $\mathbb{P}^1 \backslash \Gamma$, which are interchanged as one passes through $\Gamma \backslash \mathfrak{B}$. Moreover, by irreducibility of C (hence f), (25.1.3) is the minimal polynomial of $\sigma^* y$, proving that

$$[\mathbb{C}(\widetilde{C}) : \tilde{\pi}^*\mathbb{C}(x)] \geq d. \tag{25.1.4}$$

For each $x \in \mathbb{P}^1 \backslash (\Gamma \cup \tilde{\pi}(P))$, one can think of $(x, y_j(x))$ as belonging to $\widetilde{C}$ with $\tilde{\pi}(x, y_j(x)) = x$. Consider the elementary symmetric polynomials ($i = 0, \ldots, d$, with $e_0^\varphi = 1$)

$$e_i^\varphi(x) := e_i\left(\varphi(x, y_1(x)), \ldots, \varphi(x, y_d(x))\right),$$

which are well-defined and holomorphic on $\mathbb{P}^1 \backslash (\mathfrak{B} \cup \tilde{\pi}(P))$. As in §8.2, the fact that they are bounded away from $\tilde{\pi}(P)$ guarantees (by Riemann) their extension to holomorphic functions on $\mathbb{P}^1 \backslash \tilde{\pi}(P)$.

[4] As in §8.2, we may change projective coordinates if necessary to put the equation in this form.

Further, if $x_0 \in \tilde{\pi}(P)$ has neighborhood $\Delta_{x_0} \subset \mathbb{P}^1$, then for $k \in \mathbb{N}$ sufficiently large, $\hat{\varphi} := \tilde{\pi}^*((x-x_0)^k) \cdot \varphi$ is holomorphic in $\tilde{\pi}^{-1}(\Delta_{x_0}) \subset \widetilde{C}$. By the same argument (from §8.2), $e_i^{\hat{\varphi}}(x)$ extends holomorphically across x_0; but since $e_i^{\hat{\varphi}}(x) = (x-x_0)^{ik} \cdot e_i^{\varphi}(x)$, $e_i^{\varphi}(x)$ extends *meromorphically* across x_0. Repeating this argument at all points of $\tilde{\pi}(P)$, we find that

$$e_i^{\varphi} \in \mathcal{K}(\mathbb{P}^1);$$

and by Theorem 3.1.7(a) $\mathcal{K}(\mathbb{P}^1) \cong \mathbb{C}(x)$.

Next observe that for any $x \in \mathbb{P}^1 \backslash (\Gamma \cup \tilde{\pi}(P))$ and $j \in \{1, \ldots, d\}$,

$$\begin{aligned} 0 &= \prod_{i=1}^{d} \big(\varphi(x, y_j(x)) - \varphi(x, y_i(x))\big) \\ &= \varphi(x, y_j(x))^d - e_1^{\varphi}(x)\varphi(x, y_j(x))^{d-1} \\ &\qquad + e_2^{\varphi}(x)\varphi(x, y_j(x))^{d-2} - \cdots + (-1)^d e_d^{\varphi}(x)\,; \end{aligned}$$

that is, for a dense subset of points $p \in \widetilde{C}$, $\varphi(p)$ satisfies the equation

$$0 = \sum_{i=0}^{d} (-1)^i e_i^{\varphi}(\tilde{\pi}(p)) \cdot \varphi(p)^{d-i}.$$

Therefore the meromorphic function φ itself satisfies

$$0 = \sum_{i=0}^{d} (-1)^i (\tilde{\pi}^* e_i^{\varphi}) \cdot \varphi^{d-i}, \tag{25.1.5}$$

with coefficients in $\tilde{\pi}^*\mathbb{C}(x)$; and so $\mathcal{K}(\tilde{C})$ is algebraic over $\tilde{\pi}^*\mathbb{C}(x)$.

Finally, in characteristic zero, the *primitive element theorem* says that any finite field extension (of degree n) is generated by a single element (of degree n). (An infinite algebraic field extension will therefore have elements of unbounded degree.) Were $[\mathcal{K}(\widetilde{C}) : \tilde{\pi}^*\mathbb{C}(x)] > d$, finite or not, there would thus be an element of degree $> d$; but as $\varphi \in \mathcal{K}(\widetilde{C})$ was arbitrary, (25.1.5) shows this is not so. Hence

$$[\mathcal{K}(\widetilde{C}) : \tilde{\pi}^*\mathbb{C}(x)] \leq d.$$

Putting this together with (25.1.1) and (25.1.4), we see that

$$\mathcal{K}(\widetilde{C}) = \mathbb{C}(\widetilde{C}),$$

proving the

25.1.6. THEOREM. *Every meromorphic function on the normalization of an irreducible projective algebraic curve is rational, i.e. the pullback of a ratio of homogeneous polynomials.*

25.1.7. COROLLARY. *Every meromorphic 1-form on a normalization is rational (i.e. $f\,dg$ where f, g are rational).*

PROOF. Consider (say) $\sigma^*(dx) =: \omega \in \mathcal{K}^1(\widetilde{C})$, and let $\omega' \in \mathcal{K}^1(\widetilde{C})$ be any other meromorphic 1-form on $\widetilde{C}$. Then $\frac{\omega'}{\omega}$ belongs to $\mathcal{K}(\widetilde{C})$, hence is rational by Theorem 25.1.6. $\square$

25.2. Cohomology of a Riemann surface

Let M be a Riemann surface of genus g. Recall from §21.1 that the 1st homology group $H^1(M, \mathbb{Z}) = \frac{\text{1-cycles}}{\text{boundaries}}$ is an abelian group of rank $2g$, and define M's *1st cohomology group* to be the $2g$-dimensional vector space of complex-linear functionals

$$H^1(M, \mathbb{C}) := Hom(H_1(M, \mathbb{Z}), \mathbb{C}).$$

Exactly as in §23.3 (for elliptic curves) we have the *de Rham cohomology groups*

$$H^1_{dR}(M) := \frac{\ker\{A^1(M) \xrightarrow{d} A^2(M)\}}{\text{image}\{A^0(M) \xrightarrow{d} A^1(M)\}} = \frac{\text{closed } C^\infty \text{ 1-forms}}{\text{exact } C^\infty \text{ 1-forms}}.$$

To any closed 1-form ω we may assign the functional $\gamma \mapsto \int_\gamma \omega$ on loops. By the first 2 paragraphs of the proof of Lemma 23.3.1 (which work for any M), this induces a well-defined injective map

$$H^1_{dR}(M) \hookrightarrow H^1(M, \mathbb{C}). \tag{25.2.1}$$

Surjectivity also holds but will require a little more work than for elliptic curves.

Writing $\overline{\Omega^1(M)}$ for the space of "anti-holomorphic" forms (the complex conjugates of holomorphic ones), we can embed

$$\Omega^1(M) \oplus \overline{\Omega^1(M)} \hookrightarrow H^1_{dR}(M) \tag{25.2.2}$$

via

$$(\omega, \overline{\varphi}) \mapsto [\omega + \overline{\varphi}].$$

The map (25.2.2) is well-defined because $d(\Omega^1(M)) = 0 = d(\overline{\Omega^1(M)})$ (cf. Remark 23.3.2(a)). To prove injectivity, suppose $\omega + \overline{\varphi} = df$, $f \in A^0(M)$. Then

$$d(f\varphi) = f \underbrace{d\varphi}_{=0} + df \wedge \varphi = (\omega + \overline{\varphi}) \wedge \varphi = -\varphi \wedge \overline{\varphi}$$

since $\omega \wedge \varphi$ looks locally like a function times $dz \wedge dz (= 0)$. Now breaking M up into triangular regions Δ_i with local holomorphic coordinates $z_i = x_i + \sqrt{-1}y_i$,

$$\int_M \varphi \wedge \overline{\varphi} = \sum_i \int_{\Delta_i} g_i dz_i \wedge \overline{g_i}\overline{dz_i} = -2\sqrt{-1} \sum \underbrace{\int_{\Delta_i} |g_i|^2 dx_i \wedge dy_i}_{\in \mathbb{R}_{\geq 0}}.$$

Since each integral $= 0 \iff g_i \equiv 0$, we have

$$\int_M \varphi \wedge \overline{\varphi} = 0 \quad \iff \quad \varphi \equiv 0.$$

But using Stokes's theorem and $\partial M = \emptyset$,

$$\int_M \varphi \wedge \overline{\varphi} = -\int_M d(f\varphi) = -\int_{\partial M} f\varphi = 0$$

which implies $\varphi \equiv 0$. So $df = \omega \implies \frac{\partial f}{\partial \bar{z}} = 0 \implies f \in \mathcal{O}(M) \implies f$ constant (by Liouville) $\implies \omega = 0$. So (25.2.2) is injective.

By now you are quite familiar with the fact that $\dim(S_3^{d-3}) = \binom{(d-3)+2}{2} = \frac{(d-1)(d-2)}{2}$. If $\mathcal{S}$ is a set of δ points in $\mathbb{P}^2$, then the homogeneous polynomials of degree $d-3$ vanishing on each of these points are subject to δ (possibly dependent) linear conditions. Denoting the space of such polynomials by $S_3^{d-3}(-\mathcal{S})$, we therefore have

$$\dim(S_3^{d-3}(-\mathcal{S})) \geq \frac{(d-1)(d-2)}{2} - \delta. \tag{25.2.3}$$

Now assume $\sigma : M \to \mathbb{P}^2$ is injective off a finite point set, with image an irreducible algebraic curve $C = \{F(Z, X, Y) = 0\}$ of degree d ($F \in S_3^d$), having only nodal singularities (as in part (B) of the

Normalization Theorem). Write $\mathcal{S}$ for the collection of these nodes, and note that $M = \widetilde{C}$. By (25.2.3) and the genus formula,

$$\dim(S_3^{d-3}(-\mathcal{S})) \geq g.$$

By Exercises (3)-(4), we have a map

$$S_3^{d-3}(-\mathcal{S}) \to \Omega^1(M)$$

given by

$$G \longmapsto \sigma^* \left(\frac{g\,dx}{f_y} \right) =: \omega_G$$

where $g(x,y) = G(1,x,y)$ etc. This is necessarily injective: were $\frac{g dx}{f_y}$ to vanish on C, we would have $G \equiv 0$ on C; since F is irreducible then $F|G$ by Study, which is impossible (unless G is trivial) as $\deg(G) = d-3 < d = \deg(F)$.

Likewise, sending $G \mapsto \overline{\omega_G}$ gives an injective map from $S^{d-3}(-\mathcal{S})$ to $\overline{\Omega^1(M)}$. All told, we have a sequence of injective maps of complex vector spaces

$$S_3^{d-3}(-\mathcal{S}) \oplus S_3^{d-3}(-\mathcal{S}) \hookrightarrow \Omega^1(M) \oplus \overline{\Omega^1(M)} \hookrightarrow H^1_{dR}(M) \hookrightarrow H^1(M,\mathbb{C}).$$

Notice that the left-hand side has dimension $\geq 2g$ and the right-hand side has dimension exactly $2g$. All the injections are therefore isomorphisms and we conclude:

25.2.4. PROPOSITION. *For a Riemann surface M (of genus g) normalizing an algebraic curve of degree d with nodes $\mathcal{S} = \{p_1, \dots, p_\delta\}$, the holomorphic 1-forms are all pullbacks of rational forms $\frac{g dx}{f_y}$ (as described above). Moreover, we have:*

(a) [DE RHAM THEOREM] $H^1(M,\mathbb{C}) \cong H^1_{dR}(M)$;
(b) [HODGE DECOMPOSITION] $H^1_{dR}(M) \cong \Omega^1(M) \oplus \overline{\Omega^1(M)}$; *and*
(c) $\dim \Omega^1(M) = g = \frac{(d-1)(d-2)}{2} - \delta = \dim(S_3^{d-3}(-\mathcal{S}))$.

We also get an application to the period matrices Π described in §21.1. Recall that if $\gamma_1, \dots, \gamma_{2g}$ is a basis for $H_1(M,\mathbb{Z})$ and $\omega_1, \dots, \omega_g$

a basis for $\Omega^1(M)$, then

$$\Pi = \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \cdots & \int_{\gamma_{2g}} \omega_1 \\ \vdots & \ddots & \ddots & \vdots \\ \int_{\gamma_1} \omega_g & \cdots & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}.$$

25.2.5. PROPOSITION. *Viewed as vectors in $\mathbb{R}^{2g} (\cong \mathbb{C}^g)$, the columns π_j of Π are $\mathbb{R}$-linearly independent.*

PROOF. Suppose otherwise, i.e. that there exists a nonzero vector $\underline{a} \in \mathbb{R}^{2g}$ satisfying

$$\underline{0} = \Pi \underline{a}\,;$$

then we have also (by complex conjugating)

$$\underline{0} = \overline{\Pi}\underline{a}.$$

That is,

$$\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} \underline{a} = \underline{0}$$

and so the rank of $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ is less than $2g$. But then there is a nonzero $\underline{b} \in \mathbb{C}^{2g}$ such that

$${}^t\underline{b} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = {}^t\underline{0}\,,$$

which means explicitly for each j that

$$\int_{\gamma_j} (\underbrace{\sum_{i=1}^{g} b_i \omega_i}_{=:\omega} + \underbrace{\sum_{i=1}^{g} b_{g+1} \overline{\omega_i}}_{=:\overline{\varphi}}) \;=\; 0.$$

Thus $(\omega, \overline{\varphi}) \in \Omega^1(M) \oplus \overline{\Omega^1(M)}$ goes to zero in $Hom(H_1(M, \mathbb{Z}), \mathbb{C}) = H^1(M, \mathbb{C})$. By our sequence of injections above, $\omega = \varphi = 0$. But since $\underline{b} \neq \underline{0}$, this contradicts linear independence of $\omega_1, \ldots, \omega_g$ in $\Omega^1(M)$. $\square$

Exercises

(1) Write a basis for the holomorphic 1-forms on the (smooth) curve $C \subset \mathbb{P}^2$ with affine equation $1 + x^6 + y^6 - xy^5 = 0$. What is $\dim(\Omega^1(C))$?

(2) Adapt the proof of Proposition 8.2.7 to show that *any* (closed) *complex analytic curve* $\mathcal{C} \subset \mathbb{P}^2$ (i.e., a subset which in a neighborhood of any point is cut out by the vanishing of a nonconstant holomorphic function) *is in fact algebraic* (cut out globally by a homogeneous polynomial). [Suggestion: by applying a projectivity, you may assume that [0:1:0] is not in $\mathcal{C}$. Note that the intersection of $\mathcal{C}$ with any vertical line $x = x_0$ is finite; in a neighborhood of each intersection point (x_0, y_0), $\mathcal{C}$ can be described by a Weierstrass polynomial in $y - y_0$ as in Chapter 10. Multiply these together to get an element of $\mathbb{C}\{x - x_0\}[y]$, monic in y, cutting out $\mathcal{C}$ for $|x - x_0| < \rho$ (and all y). Argue that these local elements patch together to give an element "$\prod_{\lambda=1}^{m}(y - y_\lambda(x))$" in $\mathcal{O}(\mathbb{C})[y]$, and then show that $\mathcal{O}(\mathbb{C})$ can be replaced by $\mathbb{C}[x]$.]

(3) For this and the next problem,[5] let $C \subset \mathbb{P}^2$ be cut out by an irreducible polynomial $F \in S_3^d$ of degree d, with affinization $f(x, y) := F(1, x, y)$. In this exercise, C is also assumed to be smooth. Show that, for any $G \in S_3^{d-3}$ with affinization g, the form $\omega_G := \frac{g\,dx}{f_y}|_C$ is holomorphic. [See the hint for Exercise (4) of Chapter 19, which this problem continues, and be sure to treat points at infinity and points with vertical tangent.]

(4) Now suppose that C has a finite set $\mathcal{S}$ of singular points, which are nodes. Write $S_3^{d-3}(-\mathcal{S}) \subseteq S_3^{d-3}$ for the polynomials vanishing on this set. Show that $G \mapsto \omega_G := \sigma^* \frac{g\,dx}{f_y}$ defines a map from $S_3^{d-3}(-\mathcal{S})$ to $\Omega^1(\tilde{C})$, where $\tilde{C} \xrightarrow{\sigma} C$ is the normalization. [Hint: apply a projectivity to move a node to [1:0:0], with tangent lines $X = 0$ and $Y = 0$; write an expression for the resulting $f(x, y)$. It suffices to show that if $g(0, 0) = 0$, then $\sigma^*\omega_G$ is holomorphic

[5]The result of Exercises (3)-(4) is also proved in Chapter 3 of Griffiths's Introduction to Algebraic Curves.

at the two preimage points. You don't need much detail about the local normalizations; e.g., that the branch with tangent $y = 0$ has a parametrization of the form $t \mapsto (t, t^n h(t))$ (with $n \geq 2$ and $h(0) \neq 0$) is more than enough.]