CHAPTER 26

The Riemann-Roch Theorem

As you know, there are no nonconstant holomorphic functions on a Riemann surface M. What if we allow a simple pole at one point p but no poles anywhere else? Then you still get nothing, *unless* M is $\mathbb{P}^1$ (in which case there is $(z - z(p))^{-1}$). This is because for $g = \text{genus}(M) \geq 1$, there is a nonzero holomorphic form ω which doesn't vanish at p. For any meromorphic function f on M, we know that $\sum_{q \in M} Res_q(f\omega) = 0$; so if f has a simple pole at p, then $Res_p(f\omega) \neq 0$ and f must have another pole to cancel this term.

What if we are prepared to allow a double pole at p (but still no other poles)? Then the answer is more complex; if $g = 0$ or 1 there *are* nonconstant such functions (e.g. the Weierstrass $\wp$-function), while if $g \geq 2$ it can depend on the point p. In general, the vector spaces of meromorphic functions f with (at most) a single pole at p and $\nu_p(f) \geq -k$ has dimension $\geq \max\{1, k - g + 1\}$. You are guaranteed to get something nonconstant as soon as $k - g + 1 \geq 2$.

In the 1850's, Riemann proved a more general inequality which replaces p (and k) by multiple points and orders; a decade later, his student Roch turned this into an exact equality (Theorem 26.2.7 below) incorporating another term related to meromorphic 1-forms. It encompasses the equality $\dim(\Omega^1(M)) = g$ and gives a powerful tool for studying embeddings of Riemann surfaces into higher dimensional projective spaces, among other things. Its statement is in terms of spaces of functions and forms related to divisors, and we will start in §26.1 by defining these spaces precisely.

You may prefer this shorter introduction to the topic from a lecture by Lefschetz: "Well, a Riemann surface is a certain kind of Hausdorff space. You know what a Hausdorff space is, don't you? It's also compact, ok. I guess it is also a manifold. Surely you know what a manifold is. Now let me tell you one nontrivial theorem, the Riemann-Roch Theorem."[1]

26.1. Effective divisors and rational equivalence

Let M be a Riemann surface, and write $D = \sum_{p \in M} m_p[p]$ and $E = \sum_{p \in M} n_p[p]$ for divisors on M. (Of course, only finitely many m_p and n_p are nonzero.) If for all p $m_p \geq n_p$, then we write $D \geq E$.

26.1.1. DEFINITION. $D \in \mathrm{Div}(M)$ is *effective* $\iff$ $D \geq 0$.

26.1.2. EXAMPLE. The divisor (ω) of a holomorphic 1-form ω is effective. (Why?)

We can use this idea to put constraints on meromorphic functions and forms. For instance, suppose $D = 3[q] - 2[r]$, and $f \in \mathcal{K}(M)$ with divisor $(f) = \sum_{p \in M} \nu_p(f)[p]$. Then imposing the inequality $(f) + D \geq 0$ forces $\nu_q(f) + 3 \geq 0$ and $\nu_r(f) - 2 \geq 0$; that is, f is allowed a pole of order no worse than -3 at q, and *must* have a zero of order at least 2 at r. Likewise, if $\omega \in \mathcal{K}^1(M)$ then $(\omega) \geq D$ means ω has a zero of order at least 3 at q, and is allowed a pole of order no worse than -2 at r. The next definition formalizes this and defines the quantities which the Riemann-Roch theorem will relate.

26.1.3. DEFINITION. For any $D \in \mathrm{Div}(M)$, set

$$\mathfrak{L}(D) := \{f \in \mathcal{K}(M)^* \mid (f) + D \geq 0\} \cup \{0\} \text{ and}$$
$$\mathfrak{I}(D) := \{\omega \in \mathcal{K}^1(M)^* \mid (\omega) \geq D\} \cup \{0\}.$$

(The "$\cup\{0\}$" just means that the zero-function is included, so as to produce a vector space.) Write

$$\ell(D) := \dim \mathfrak{L}(D)\,, \quad i(D) := \dim \mathfrak{I}(D).$$

The next step is to define an equivalence relation on divisors which is ubiquitous in algebraic geometry.

[1]from *A Beautiful Mind* by S. Nasar

26.1.4. DEFINITION. Divisors $D, E \in \mathrm{Div}(M)$ are *rationally equivalent* iff there exists[2] $f \in \mathcal{K}(M)^*$ with $(f) = D - E$; we write $D \overset{\text{rat}}{\equiv} E$.

26.1.5. PROPOSITION. *If $D \overset{rat}{\equiv} E$, then*

(i) $\deg(D) = \deg(E)$;
(ii) $\mathfrak{L}(D) \cong \mathfrak{L}(E)$;
(iii) $\mathfrak{I}(D) \cong \mathfrak{I}(E)$; *and*
(iv) $\ell(D) = \ell(E)$ *and* $i(D) = i(E)$.

Furthermore, $\overset{rat}{\equiv}$ *respects the abelian group structure of* $\mathrm{Div}(M)$.

PROOF. By assumption $D - E = (f)$. Now Exercise (2) of Ch. 3 says that $\deg((f)) = 0$, which yields (i). Given $g \in \mathfrak{L}(D)$,

$$(fg) + E = (f) + (g) + E = (g) + D \geq 0\,;$$

so $g \mapsto fg$ defines a map $\mathfrak{L}(D) \to \mathfrak{L}(E)$, and $h \mapsto \frac{h}{f}$ defines an inverse map. This gives (ii), and (iii) is done in the same way. (iv) obviously follows from (ii)-(iii). The last statement about $\overset{\text{rat}}{\equiv}$ is essentially just that $(D + (f)) + (E + (g)) = (D + E + (fg))$. $\square$

26.1.6. REMARK. The *Picard group* $\mathrm{Pic}(M)$ of §21.1 is the group of equivalence classes

$$\frac{\mathrm{Div}(M)}{\overset{\text{rat}}{\equiv}},$$

and Proposition 26.1.5 says (in part) that deg, ℓ, and i give well-defined functions from $\mathrm{Pic}(M)$ to $\mathbb{Z}$. In particular, deg is a homomorphism, and writing $\mathrm{Pic}^0(M) := \ker(\deg) \subset \mathrm{Pic}(M)$ recovers the "degree-zero part" seen in Ch. 21.

26.1.7. DEFINITION. A *canonical divisor* $K \in \mathrm{Div}(M)$ is just the divisor of any meromorphic 1-form $\omega \in \mathcal{K}^1(M)$. Since any two such are rationally equivalent (easy exercise), there is a single canonical divisor *class* $[K] \in \mathrm{Pic}(M)$.

The next (basic) result is sometimes called "Brill-Noether reciprocity":

[2] By Chow's theorem (§25.1), all meromorphic functions are *rational*, hence the terminology "rational equivalence" (sometimes also called "linear equivalence").

26.1.8. PROPOSITION. *Let $D \in Div(M)$ be arbitrary, and K a canonical divisor. Then*

$$\mathfrak{I}(D) \cong \mathfrak{L}(K-D),$$

and so $i(D) = \ell(K-D)$.

PROOF. Let $K = (\omega)$; if $(f) + K - D \geq 0$, then $(f\omega) = (f) + K \geq D - K + K = D$. So $f \mapsto f\omega$ maps $\mathfrak{L}(K-D) \to \mathfrak{I}(D)$, and $\eta \mapsto \frac{\eta}{\omega}$ gives an inverse. □

26.2. Proof and statement

Throughout this section we take C to be an irreducible degree d projective algebraic curve with nodal singularities $\mathcal{S} = \{p_1, \ldots, p_\delta\}$. Let $M := \widetilde{C} \xrightarrow{\sigma} \mathbb{P}^2$ ($\sigma(M) = C$) be its normalization. According to part (B) of the Normalization Theorem 3.2.1, every Riemann surface M arises in this way.

Writing $\sigma^{-1}(p_i) =: \{q_i, r_i\}$, we define a divisor

$$\mathcal{E} := \sigma^{-1}(\mathcal{S}) = \sum_{i=1}^{\delta} [q_i] + [r_i] \ \in \ \mathrm{Div}(M)$$

of degree 2δ. Given any line $H \subset \mathbb{P}^2$ ("H" for "hyperplane"), put

$$\mathcal{H} := \sigma^{-1}(H \cdot C) \ \in \ Div(M)$$

for the intersection divisor (of degree d).[3]

26.2.1. LEMMA. *For all sufficiently large $m \in \mathbb{N}$,*

$$\ell(m\mathcal{H} - \mathcal{E}) \geq md - 2\delta - g + 1$$

and

$$i(m\mathcal{H} - \mathcal{E}) = 0,$$

where $g = \frac{(d-1)(d-2)}{2} - \delta$ is the genus of M.

[3] If H passes through an ODP p_i, then $q_i$and r_i will both show up in $\mathcal{H}$, with multiplicities determined by the local intersection multiplicities of H with the two local analytic components of C at p_i. (See Defn. 12.2.2*ff*.)

PROOF. Write $R \in S_3^1$ and $F \in S_3^d$ for the defining homogeneous polynomials of H and C (resp.). Consider the map

$$\begin{aligned} S_3^m(-\mathcal{S}) &\xrightarrow{\theta} \mathfrak{L}(m\mathcal{H} - \mathcal{E}) \\ G &\longmapsto \sigma^* \left(\frac{G}{R^m} \right), \end{aligned}$$

where we note that $\frac{G}{R^m}$ is a well-defined meromorphic function because numerator and denominator have the same degree. By Study's lemma,

$$\sigma^* \left(\frac{G}{R^m} \right) \equiv 0 \quad \Longleftrightarrow \quad G|_C \equiv 0 \quad \Longleftrightarrow \quad F|G,$$

and so $\ker \theta = F \cdot S_3^{m-d}$ $(\subset S_3^m(-\mathcal{S}))$.

Therefore, taking dimensions of

$$\mathfrak{L}(m\mathcal{H} - \mathcal{E}) \supseteq \operatorname{im}(\theta),$$

we find

$$\begin{aligned} \ell(m\mathcal{H} - \mathcal{E}) \geq \dim(\operatorname{im}(\theta)) &= \dim \left(\frac{S_3^m(-\mathcal{S})}{F \cdot S_3^{m-d}} \right) \\ &= \dim S_3^m(-\mathcal{S}) - \dim S_3^{m-d}. \end{aligned}$$

Using (25.2.3) (but with $m + 3$ replacing d), and assuming $m \geq d$, this

$$\begin{aligned} \geq \frac{(m+1)(m+2)}{2} - \delta - \frac{(m-d+1)(m-d+2)}{2} \\ = md - \delta - \frac{d(d-3)}{2} \\ = md - \delta - \frac{(d-1)(d-2)}{2} + 1, \end{aligned}$$

which by Prop. 25.2.4(c)

$$= md - 2\delta - g + 1.$$

Finally, any $\omega \in \mathfrak{I}(m\mathcal{H} - \mathcal{E}) \setminus \{0\}$ has $(\omega) \geq m\mathcal{H} - \mathcal{E}$, and taking degrees gives $2g - 2 = \deg((\omega)) \geq md - 2\delta$. Clearly this is untenable once $m > \frac{2}{d}(g + \delta - 1)$, whence $i(m\mathcal{H} - \mathcal{E}) = 0$. $\square$

26.2.2. LEMMA. *Let $D \in \mathrm{Div}(M)$, $p \in M$. Then*

$$0 \ \leq \ \ell(D+[p]) - \ell(D) - (i(D+[p]) - i(D)) \ \leq \ 1.$$

PROOF. First note that $\mathfrak{L}(D) \subseteq \mathfrak{L}(D+[p]) \implies \ell(D+[p]) - \ell(D) \geq 0$.

Next, writing $D = \sum_{q \in M} n_q[q]$, an element of $\mathfrak{L}(D+[p]) \backslash \mathfrak{L}(D)$ is a function $f \in \mathcal{K}(M)^*$ satisfying

$$(f) + D + [p] \geq 0 \quad \text{and} \quad \nu_p(f) = -(n_p+1). \tag{26.2.3}$$

If f, g are two such functions, then setting $\alpha := \lim_{x \to p} \frac{f(x)}{g(x)}$, we have

$$\mathrm{ord}_p(f - \alpha g) \geq -n_p$$

so that $f - \alpha g \in \mathfrak{L}(D)$. So $\ell(D+[p]) - \ell(D) \leq 1$, and we conclude

$$0 \leq \ell(D+[p]) - \ell(D) \leq 1. \tag{26.2.4}$$

Similarly, writing K for a canonical divisor,

$$0 \leq \ell(K-D) - \ell(K-D-[p]) \leq 1$$

or equivalently

$$0 \leq i(D) - i(D+[p]) \leq 1. \tag{26.2.5}$$

Altogether,

$$0 \leq \ell(D+[p]) - \ell(D) + i(D) - i(D+[p]) \leq 2$$

and we just have to show that "2" is impossible.

Suppose (for a contradiction) that f satisfies (26.2.3), which is equivalent to "1" in (26.2.4), and $\omega \in \mathcal{K}^1(M)$ satisfies

$$(\omega) \geq D \quad \text{and} \quad \nu_p(\omega) = n_p,$$

which is equivalent to "1" in (26.2.5). Then

$$(f\omega) = (f) + (\omega) \geq -[p]$$

with

$$\nu_p(f\omega) = \nu_p(f) + \nu_p(\omega) = -1.$$

But the sum of residues of a meromorphic form is 0 (Prop. 13.1.10(b)), so $f\omega$ having a single simple pole (and no other poles) is absurd. $\square$

By Lemma 26.2.1, there exists $m_0 \in \mathbb{Z}$ such that $m \geq m_0 \implies$

$$\ell(m\mathcal{H} - \mathcal{E}) - i(m\mathcal{H} - \mathcal{E}) \;\geq\; md - 2\delta - g + 1.$$

Now for any two lines H_1, H_2, we have $\mathcal{H}_1 \overset{\text{rat}}{\equiv} \mathcal{H}_2$; so if $H_1, \ldots, H_m$ are lines in $\mathbb{P}^2$ then by Proposition 26.1.5(iv)

$$\ell(\mathcal{H}_1 + \cdots + \mathcal{H}_m - \mathcal{E}) - i(\mathcal{H}_1 + \cdots + \mathcal{H}_m - \mathcal{E}) \;\geq\; md - 2\delta - g + 1.$$

Taking m large enough and lines through (a) all points of $\mathcal{S}$ and (b) all points in D, we can ensure that $\sum_{i=1}^m \mathcal{H}_i - \mathcal{E} - D$ is effective, so that

$$\mathcal{H}_1 + \cdots + \mathcal{H}_m - \mathcal{E} \;=\; D + [P_1] + \cdots + [P_k]$$

where $k = md - 2\delta - \deg(D)$ (and the P_j are points of M). Therefore we have

$$\ell\left(D + \sum_{j=1}^k [P_j]\right) - i\left(D + \sum_{j=1}^k [P_j]\right) \;\geq\; k + \deg(D) - g + 1.$$

Repeatedly applying the right-hand inequality of Lemma 26.2.2 gives

$$k + \ell(D) - i(D) \;\geq\; \ell\left(D + \sum_{j=1}^k [P_j]\right) - i\left(D + \sum_{j=1}^k [P_j]\right)$$

and we conclude that

$$\ell(D) - i(D) \;\geq\; \deg(D) - g + 1. \tag{26.2.6}$$

Next we show the reverse inequality. Plugging $K - D$ into (26.2.6), we have

$$\ell(K - D) - i(K - D) \;\geq\; \deg(K - D) - g + 1$$

which becomes (using Brill-Noether reciprocity)

$$i(D) - \ell(D) \;\geq\; 2g - 2 - \deg(D) - g + 1 \;=\; -(\deg(D) - g + 1),$$

so that

$$\ell(D) - i(D) \;\leq\; \deg(D) - g + 1.$$

We have thus proved the

26.2.7. THEOREM. [RIEMANN-ROCH] *Let M be a Riemann surface of genus g, D a divisor on M. Then*

$$\ell(D) - i(D) = \deg(D) - g + 1.$$

Amongst the easy corollaries of this important result are the *Riemann inequality*

$$\ell(D) \ \geq \ \deg(D) - g + 1,$$

and (by putting $D = 0$ in the theorem) the formula

$$\dim \Omega^1(M) \ = \ g.$$

Here is another simple application to whet your appetite for the next two chapters.

26.2.8. PROPOSITION. *Up to isomorphism, $\mathbb{P}^1$ is the only Riemann surface of genus* 0.

PROOF. Suppose M has genus 0; then, first of all, the above "corollary of Riemann-Roch" says that $\dim \Omega^1(M) = 0$. If we take (for some $p \in M$) $D = [p]$, then $\mathfrak{I}(D) \subset \Omega^1(M) = \{0\} \implies i(D) = 0$. So by Riemann-Roch itself,

$$\ell(D) = \deg(D) - g + 1 = 1 - 0 + 1 = 2.$$

Now $\mathfrak{L}(D)$ consists of functions with a simple pole allowed at p (and no other poles). The constant function 1 belongs to $\mathfrak{L}(D)$; and since $\dim \mathfrak{L}(D) = 2$ there is also a nonconstant function $f \in \mathfrak{L}(D)$, which by Liouville *must* have the allowed simple pole. Therefore the mapping degree of $f\colon M \to \mathbb{P}^1$ is (cf. §14.1)

$$\deg(f) = \deg(f^{-1}([\infty])) = \deg([p]) = 1;$$

that is, f is an isomorphism. □

Exercises

(1) Check that any two canonical divisors on a Riemann surface are rationally equivalent.

(2) Let $D \in Div(M)$, $g = \text{genus}(M)$. Prove that if $\deg D > 2g - 2$, then $i(D) = 0$. Likewise show that if $\deg D < 0$, then $\ell(D) = 0$.

(3) Let M be a genus g Riemann surface, and $p \in M$. Using Riemann-Roch, find the smallest value of k for which there *must* exist $f \in \mathcal{K}(M)^*$ having a pole at p of order no worse than k (i.e. $\nu_p(f) \geq -k$), and no other poles.

(4) Let M have genus $g \geq 2$. (a) Prove that M has a morphism to $\mathbb{P}^1$ of degree $\leq g + 1$. [Hint: use Exercise (3)] (b) Prove that M has a morphism to $\mathbb{P}^1$ of degree $\leq g$. [Hint: let $p \in M$, and look at $i((g-2)[p])$. This is a bit harder than (a).]

(5) Assume $D > 0$. By Exercise (2), if $g \leq 1$ then $i(D) = 0$, and Riemann-Roch becomes $\ell(D) = \deg(D)$ for $g = 1$ and $\deg(D) + 1$ for $g = 0$. Prove this directly (a) for $M \cong \mathbb{P}^1$ and (b) for $M \cong \mathbb{C}/\Lambda$ (1-torus).