CHAPTER 27

Applications of Riemann-Roch, I: special Riemann surfaces

We now focus our attention on Riemann surfaces with a degree-two mapping to $\mathbb{P}^1$, starting with the case of genus 1. (The higher genus cases can be viewed as a generalization of elliptic curves, although there is no group law.) The first section begins with some general claims which will be more thoroughly investigated in the next chapter. In both chapters, the pretense is that we have Riemann-Roch in hand for an arbitrary Riemann surface, although what we have *proved* so far, strictly speaking, is Riemann-Roch for (possibly nodal) plane curves.

27.1. Curves of genus 1

For the proof of Riemann-Roch (Theorem 26.2.7), we needed to invoke the (still unproven) Normalization Theorem 3.2.1(B). A much more analytic route through the material[1] establishes Riemann-Roch first, then uses this to establish to the existence of plane projective immersions (with nodal singularities) for arbitrary Riemann surfaces.

When can we do better? The degree-genus formula tells you that only Riemann surfaces of genera 0, 1, 3, 6, 10, 15, . . . (numbers expressible as $\frac{(d-1)(d-2)}{2}$, $d \in \mathbb{N}$) can ever be embedded as *smooth* curves in $\mathbb{P}^2$. There is *no* reason to believe, from this or from the Normalization Theorem, that an *arbitrary* Riemann surface of one of these genera *can* be so embedded. In fact, *it isn't true* once you get to genera 6, 10, 15, ... That it works for genus 1 and genus 3 ("almost"; see Chapter 28) is a bit of a miracle!

[1] e.g. see Farkas and Kra, *Riemann surfaces*.

So: if you buy that any genus 1 Riemann surface is a complex 1-torus and any torus can be "Weierstrassed" into $\mathbb{P}^2$, the following result isn't surprising. On the other hand, it shows that Riemann-Roch is powerful and gives us a hint of how we might prove similar results in higher genus (e.g., 2 and 3) later.

27.1.1. THEOREM. *Let M be a Riemann surface of genus one. There exists an injective morphism of complex manifolds $\sigma : M \hookrightarrow \mathbb{P}^2$ with image $\sigma(M)$ a smooth algebraic curve of degree* 3.

PROOF. Given $p \in M$, we know that $i(2[p]) = 0 = i([p])$ by Exercise (2) of Chapter 26, so that Riemann-Roch yields

$$\ell(2[p]) = \deg(2[p]) - g + 1 = 2 - 1 + 1 = 2,$$

$$\ell([p]) = \deg([p]) - g + 1 = 1.$$

In terms of the spaces of meromorphic functions, this says that

$$\mathfrak{L}(2[p]) \supsetneq \mathfrak{L}([p]) = \mathfrak{L}(0) = \mathcal{O}(M),$$

where $\dim \mathfrak{L}([p]) = 1$ means $\mathfrak{L}([p])$ consists of constant (or equivalently, holomorphic) functions. Therefore, we have an element

$$x \in \mathfrak{L}(2[p]) \backslash \mathfrak{L}([p]),$$

i.e. a meromorphic function with a double pole at p and no other poles.

Regard x as a morphism $M \to \mathbb{P}^1$, evidently of mapping degree 2 (why?). By Riemann-Hurwitz, the ramification degree

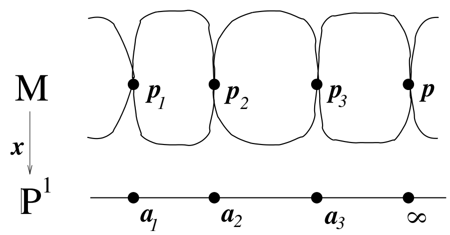
$$\deg(R_x) = 2(\deg(x) + g - 1) = 2(2 + 1 - 1) = 4,$$

whereas the ramification indices $\nu_p(x)$ for a degree two mapping are all ≤ 2. Hence, the ramification divisor is of the form (cf. §14.1 for notation)

$$R_x = [p_1] + [p_2] + [p_3] + [p_4]$$

with $p_1, p_2, p_3, p_4 \in M$ distinct. Set $a_i = x(p_i) \in \mathbb{P}^1$. The $\{a_i\}$ are still distinct points: by the form of R_x, $[p_i]$ must occur with multiplicity two in $x^{-1}([a_i])$; and since $(\deg(x) = 2 \implies) \deg(x^{-1}([a_i])) = 2$, the only possibility is $x^{-1}([a_i]) = 2[p_i]$.

Now clearly one of the p_i, say p_4, has to be p (as x has a double pole there). So also $a_4 = \infty$, and we have the picture:



In the following, $\{p_i\}$ resp. $\{a_i\}$ means $i = 1, 2, 3$.

Next, notice that $x - a_i$ is a local coordinate about a_i on $\mathbb{P}^1$. The meaning of a degree-2 ramification at p_i is simply that there is a local (holomorphic) coordinate about p_i on M such that[2] $z_i^2 = x - a_i$. Differentiating gives $dx \overset{\text{loc}}{=} 2z_i dz_i$. Again using Riemann-Roch, we have a nonvanishing holomorphic form $\omega \in \Omega^1(M)$, and by the Residue Theorem

$$0 = \sum_{q \in M} Res_q(x \cdot \omega) = Res_p(x \cdot \omega).$$

Writing $z := \int_p \omega$ for a local holomorphic coordinate at p, we have (locally) $\omega = dz$; since the residue vanishes, $x \overset{\text{loc}}{=} \frac{1}{z^2} + h(z)$ (h holomorphic) has no $\frac{1}{z}$ term.[3] Taking differentials, $dx \overset{\text{loc}}{=} \left(\frac{-2}{z^3} + h'(z)\right) dz$. Put together with the previous local computation, this tells us that dx has divisor

$$(dx) = \left(\sum_{i=1}^{3} [p_i]\right) - 3[p].$$

Set $y_0 := \frac{dx}{\omega} \in \mathcal{K}(M)^*$. In light of the fact that $(\omega) = 0$, we have that $(y_0) = (dx)$. If we put

$$g(x) := \prod_{i=1}^{3} (x - a_i),$$

[2] For an arbitrary choice of local coordinate $\mathfrak{z}_i$, it means that $x - a_i = \mathfrak{z}_i^2 h_i(\mathfrak{z}_i)$ where h doesn't vanish at 0; and then we can put $z_i := \mathfrak{z}_i \sqrt{h_i(\mathfrak{z}_i)}$.

[3] The coefficient of $\frac{1}{z^2}$ can be achieved by rescaling ω if needed.

then $(g(x)) = \sum_{i=1}^{3}(x-a_i) = (\sum_{i=1}^{3} 2[p_i]) - 6[p] = 2(y_0) = (y_0^2)$. We conclude that $\frac{g(x)}{y_0^2}$ has trivial divisor and so is some constant C, and define $y := y_0\sqrt{C}$ so as to have

$$y^2 - g(x) = 0$$

on M.

Now for the embedding. Write $\sigma \colon M \to \mathbb{P}^2$ for the morphism defined by sending $p \mapsto [0:0:1]$ and all other points $q \mapsto [1 : x(q) : y(q)]$. The image $\sigma(M)$ is contained in the projective closure E of $\{y^2 - g(x) = 0\}$ (in $\mathbb{P}^2$), which is smooth due to distinctness of the $\{a_i\}$, and connected due to its irreducibility. By the usual arguments, $\sigma(M)$ is open and closed in E, hence equals E. At this point we have a diagram

$$\begin{array}{ccccc} C\setminus\{p\} & \xrightarrow{\ \sigma\ } & E\setminus\{[0:0:1]\} & \hookrightarrow & \mathbb{P}^2\setminus\{[0:0:1]\} \\ & {\scriptstyle x}\searrow & \downarrow{\scriptstyle \pi|_E} & \swarrow{\scriptstyle \pi} & \\ & & \mathbb{P}^1 & & \end{array} \tag{27.1.2}$$

where $\pi([Z:X:Y]) := [Z:X]$.

If σ is *not* injective, there exist distinct points $q_1, q_2 \in M\backslash\{p\}$ such that

$$\sigma(q_1) = \sigma(q_2) =: Q;$$

applying π to this gives

$$x(q_1) = x(q_2) = \pi(Q) =: \xi,$$

in which ξ is not ∞ or one of the $\{a_i\}$. Since $\deg(x) = 2$, we must have $\deg(x^{-1}([\xi])) = 2$ hence $x^{-1}(\xi) = \{q_1, q_2\}$. From the equation for E it is evident that $\deg(\pi|_E) = 2$ also, with $(\pi|_E)^{-1}(\xi)$ consisting of $(\xi, \sqrt{g(\xi)})$ and $(\xi, -\sqrt{g(\xi)})$. Clearly one of these points has to be Q. From (27.1.2), it is also clear that q_1, q_2 are the only points of M that can go to these points. So whichever is not Q cannot get hit and σ fails to be surjective, a contradiction. □

27.2. Hyperelliptic curves

Above we used the fact, for a genus one Riemann surface M, that $\ell(2[p]) = 2 > 1 = \ell([p])$ for $p \in M$, to construct a degree-two mapping $x\colon M \to \mathbb{P}^1$. Now suppose M has genus 2: how to map it to $\mathbb{P}^1$? Well, we have a basis $\{\omega_1, \omega_2\} \subset \Omega^1(M)$, and $\frac{\omega_2}{\omega_1}$ produces a (nonconstant) meromorphic function, which does the job. By Poincaré-Hopf, $\deg((\omega_1)) = 2g - 2 = 2$, and so this map has two simple poles (or one double pole), hence has degree two.

In terms of homogeneous coordinates, we might write

$$p \mapsto [\omega_1(p) : \omega_2(p)],$$

where the meaning of the right-hand side is (expressing $\omega_i \overset{\text{loc}}{=} f_i(z)dz$ in terms of a local coordinate vanishing at p) simply $[f_1(0) : f_2(0)]$. If both f_i could simultaneously equal zero we would have a well-definedness problem (which could be gotten around by taking a limit), but this does not happen: we would have to have $i([p]) \geq 2$. By Riemann-Roch this yields $\ell([p]) = \deg([p]) - g + 1 + i([p]) \geq 2$, thereby producing an isomorphism $M \to \mathbb{P}^1$ as in the proof of Prop. 26.2.8, and contradicting $g = 2$.

This discussion hopefully motivates

27.2.1. DEFINITION. A Riemann surface M is *hyperelliptic* iff there exists a (nonconstant) degree-two morphism $x : M \to \mathbb{P}^1$.

Clearly, any genus 2 Riemann surface is hyperelliptic.

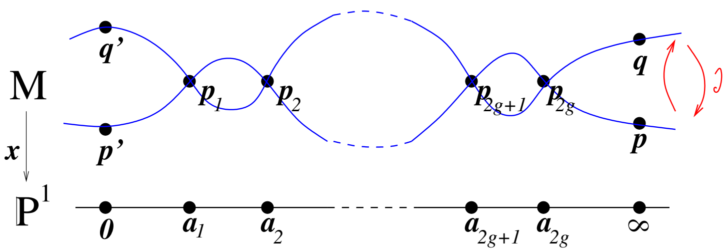
Now, let M be hyperelliptic of any genus and consider what the Riemann-Hurwitz formula has to say when applied to x:

$$\chi_M = 2\chi_{\mathbb{P}^1} - r_x$$
$$2 - 2g = 2 \cdot 2 - \sum_{p \in M} (\nu_p(x) - 1),$$

where $\deg(x) = 2 \implies \nu_p(x) \leq 2$. So the sum equals the number of ramification points, and this is just $2g + 2$:

$$R_x = [p_1] + \cdots + [p_{2g+2}].$$

By composing x with an automorphism of $\mathbb{P}^1$ if necessary, we may assume that none of the $x(p_i) =: a_i$ are 0 or ∞. Put $x^{-1}([\infty]) =: [p] + [q]$ and $x^{-1}([0]) =: [p'] + [q']$. We have the picture



in which $\jmath : M \to M$ denotes the involution exchanging the branches of M over $\mathbb{P}^1$ (cf. Exercise (1)).

27.2.2. LEMMA. *Let V be a finite-dimensional vector space, $J : V \to V$ an involution. Then we have a decomposition $V = V^+ \oplus V^-$ into the $(+1)$- and (-1)- eigenspaces of J.*

PROOF. With respect to any basis for V, J is a matrix with minimal polynomial $m(t) = t^2 - 1$. This has no repeated roots, and so J is diagonalizable. Moreover, since $J^2 = \mathrm{id}_V$, any eigenvalue λ satisfies $\lambda^2 = 1$. □

We apply this to the pullback map $\jmath^* : \Omega^1(M) \to \Omega^1(M)$. Notice that $\Omega^1(M)^+ = \{0\}$ since such forms would be pullbacks of holomorphic forms from $\mathbb{P}^1$ (cf. Exercise (2)). Hence $\Omega^1(M) = \Omega^1(M)^-$ and

$$\jmath^*\omega = -\omega$$

for all $\omega \in \Omega^1(M)$.

Put $D = (g+1)[p] + (g+1)[q] \in Div(M)$. We have[4] $i(D) = 0$, so that Riemann-Roch gives

$$\ell(D) = 2g + 2 - g + 1 = g + 3.$$

Now apply the Lemma again, this time to $\jmath^* : \mathfrak{L}(D) \to \mathfrak{L}(D)$, noting that $\mathfrak{L}(D)^+$ contains the linearly independent set

$$\{1, x, x^2, \ldots, x^{g+1}\}.$$

[4]See Exercise (2) of Ch. 26.

In fact,[5]

$$\mathfrak{L}_M(D)^+ \;=\; x^*\underbrace{\mathfrak{L}_{\mathbb{P}^1}((g+1)[\infty])}_{\substack{\text{polynomials of}\\ \text{degree } \le g+1}},$$

and so the above set is a basis. Therefore

$$\begin{aligned}\dim(\mathfrak{L}(D)^-) &= \ell(D) - \dim(\mathfrak{L}(D)^+)\\ &= (g+3)-(g+2) = 1,\end{aligned}$$

and there exists a nonzero $y \in \mathfrak{L}(D)$ such that $\jmath^* y = -y$.

27.2.3. CLAIM. $(y) = \sum_{i=1}^{2g+2}[p_i] - D$.

PROOF. Since the p_i are ramification points, $\jmath(p_i) = p_i$. But then

$$-y(p_i) = (\jmath^* y_i)(p_i) = y(\jmath(p_i)) = y(p_i)$$

and so $y(p_i) = 0$. That is, $y^{-1}([0]) \geq \sum_{i=1}^{2g+2}[p_i]$, which implies

$$\deg(y) = \deg(y^{-1}([0])) \geq \deg\left(\sum [p_i]\right) = 2g+2.$$

On the other hand, $y \in \mathfrak{L}(D) \implies y^{-1}([\infty]) \leq D \implies$

$$\deg(y) = \deg(y^{-1}([\infty])) \leq \deg(D) = 2g+2.$$

So $\deg(y)$ is forced to equal $2g+2$, which means also that $y^{-1}([0]) = \sum[p_i]$ and $y^{-1}([\infty]) = D$. □

Set

$$g(x) := \prod_{i=1}^{2g+2}(x - a_i) \in \mathcal{K}(M)^*,$$

and compute (in $Div(M)$)

$$\begin{aligned}(g(x)) &= \sum((x-a_i))\\ &= 2\sum[p_i] - (2g+2)x^{-1}([\infty])\\ &= 2\sum[p_i] - 2D = (y^2).\end{aligned}$$

[5] using subscripts to denote which Riemann surface we are considering functions on (e.g. $\mathfrak{L}_M(D)$ just means $\mathfrak{L}(D)$)

But then $y^2/g(x)$ has trivial divisor, and so is a constant. Rescaling y, we have that (in $\mathcal{K}(M)$)

$$y^2 - g(x) = 0.$$

By considering the image of

$$\sigma : M \to \mathbb{P}^2$$

given by

$$m(\neq p,q) \;\mapsto\; [1 : x(m) : y(m)]$$

and

$$p,q \;\mapsto\; [0:0:1],$$

we arrive at:

27.2.4. THEOREM. *Hyperelliptic Riemann surfaces are precisely the normalizations of (plane) algebraic curves of the form*[6]

$$\left\{ Y^2 Z^{2g} = \prod_{i=1}^{2g+2} (X - a_i Z) \right\} \subset \mathbb{P}^2.$$

A basis of $\Omega^1(M)$ *is given by* $\omega_j := \frac{x^{j-1}dx}{y}$, $j = 1, \ldots, g$.

PROOF. We just need to show ω_j is holomorphic:

$$\begin{aligned}
(\omega_j) &= (j-1)(x) + (dx) - (y) \\
&= \{(j-1)([p'] + [q']) - (j-1)([p]+[q])\} \\
&\qquad + \{\textstyle\sum [p_i] \;-\; 2([p]+[q])\} - \{\textstyle\sum [p_i] \;-\; (g+1)([p]+[q])\} \\
&= (j-1)\left([p'] + [q']\right) + (g-j)\left([p]+[q]\right) \;\geq\; 0.
\end{aligned}$$

□

A *hyperelliptic curve*, by the way, is just an irredicuble projective curve whose normalization is a hyperelliptic Riemann surface!

The first two exercises below are ones you could have done long ago, but fill in (very) small gaps in the proofs above. The same goes for the third, if you had known the definition of hyperelliptic! The fourth does make heavy use of Riemann-Roch.

[6]Note: the singular point $[0:0:1]$ is not an ODP, so the construction of g holomorphic differentials that follows shouldn't be compared with the formulas you know in that case. Also, it should be emphasized that the a_i are distinct.

Exercises

(1) Given a degree 2 holomorphic map $\varphi : M \to M'$ of compact Riemann surfaces, with corresponding involution $\jmath$ defined as follows: if $p \in M$ is a ramification point of φ, $\jmath(p) := p$; otherwise, $\varphi^{-1}(\varphi(p)) = \{p, \tilde{p}\}$ and $\jmath(p) := \tilde{p}$. Clearly $\jmath \circ \jmath = Id_M$ (i.e. $\jmath$ is an involution) and $\varphi \circ \jmath = \varphi$. Prove that $\jmath : M \to M$ is a holomorphic map of Riemann surfaces. Since $\jmath$ is injective and surjective (why?), it follows that $\jmath \in Aut(M)$.

(2) Continuing Exercise (1), let $\omega \in \Omega^1(M)$ satisfy $\jmath^*\omega = \omega$. Prove that $\omega = \varphi^*\eta$ for some $\eta \in \Omega^1(M')$.

(3) Suppose that a dth-degree irreducible algebraic curve $C \subset \mathbb{P}^2$ has a point of multiplicity $(d-2)$. Show that C is hyperelliptic.

(4) Let M be a Riemann surface of genus two. In this problem you will construct a realization of M as an algebraic curve, different to that produced above. You will need to use that M is hyperelliptic, with $x : M \to \mathbb{P}^1$ its degree-two mapping and $\jmath$ the associated involution. Take p and q (in contrast to the notation above) fixed non-ramification points on M with distinct images under x; let α and β denote arbitrary points of M.
(a) Prove that $\ell([\alpha] + [\beta]) = 1$ unless $\jmath(\alpha) = \beta$. [Hint: otherwise you get a different involution (why?). To see why this is a problem you might consider the fact that $\jmath^*\omega = -\omega$ for all holomorphic forms implies their divisors are $\jmath$-symmetric.]
(b) For any points α, β on M, show $i([\alpha] + [\beta] - [p] - [q]) = 1$ (as opposed to 2) $\iff \{\alpha, \beta\} \neq \{p, q\}$. [Hint: use (a), and consider $\ell([\alpha] + [\beta] - [p] - [q])$.]
(c) Use $\mathfrak{I}(-[p] - [q])$ to construct a map $\varphi : M \to \mathbb{P}^2$. [Hint: compute $i(-[p] - [q])$.] You will need to check that φ is well-defined. [Hint: compute $i(-[p] - [q] + [\alpha])$, using Exercise (2) of Ch. 26.]
(d) Show that φ is injective off $\{p, q\}$, but that $\varphi(p) = \varphi(q)$. [Hint: using part (b), compute $i([\alpha] + [\beta] - [p] - [q])$.]
(e) Show that there exists a meromorphic form $\omega \in \mathfrak{I}(-[p] - [q])$ with poles at *both* p and q.
(f) Explain why the zero-divisor $(\omega)_0$ (the effective "part" of (ω))

is φ^{-1} of the intersection [divisor] of a line in $\mathbb{P}^2$ with $C := \varphi(M)$. Prove that $\deg((\omega)_0) = 4$ (easy). Assuming C is an algebraic curve (which actually follows from Exercise (2) of Ch. 25), conclude that $\deg(C) = 4$.

(g) Clearly $\varphi(p) = \varphi(q)$ is a singularity of C. Prove it is the only one, and a double point. [Hint: assume otherwise, and produce a genus zero normalization or similar.]