CHAPTER 28

Applications of Riemann-Roch, II: general Riemann surfaces

Our next aim is to use Riemann-Roch to develop two methods for mapping an arbitrary Riemann surface into a (usually higher-dimensional) projective space, with a nice application to curves of genus three. The second approach behaves differently in the hyperelliptic and nonhyperelliptic cases, so we first will want to convince ourselves that there *are* nonhyperelliptic Riemann surfaces! To see this, we will start with a heuristic argument for the "number of complex parameters" governing Riemann surfaces, and show that the hyperelliptic ones have fewer parameters. But there's much more in this chapter, which should give a glimpse of how rich the correspondence between algebraic curves and Riemann surfaces really is.

28.1. Moduli

In algebraic geometry there is the notion of *moduli spaces*, which parametrize structures of a prescribed sort modulo some equivalence relation, such as "smooth algebraic curves of degree 5 up to projective equivalence" or "Riemann surfaces of genus 4 up to isomorphism." A main point is that these spaces can be given algebraic structure themselves, i.e. turned into algebraic varieties, in many cases. Suitably refined, the structure of these varieties (or more generally, schemes or stacks) is one of the hotter topics of study around.[1]

We shall only be concerned with the notion of moduli as a set of local parameters (on the moduli space), and will say colloquially that some structure has a certain *number of moduli*: e.g. genus

[1]So-called "modular curves" or (more generally) "Shimura [modular] varieties" are a more specialized notion with an arithmetic and group-theoretic flavor.

1 (resp. 0) Riemann surfaces have one modulus (resp. zero moduli) since they can all be expressed as $\mathbb{C}/\mathbb{Z}\langle 1,\tau\rangle$ (resp. $\mathbb{P}^1$) up to isomorphism. That is, the number of moduli is the dimension of the moduli space. Underlying the claim in §27.1 that not all Riemann surfaces of genus 6, 10, 15, etc. can be embedded smoothly in $\mathbb{P}^2$ is a deep calculation of Riemann:

28.1.1. THEOREM. *Riemann surfaces of genus $g \geq 2$ (considered up to isomorphism) have $3g-3$ moduli.*

SKETCH. Consider a genus g Riemann surface M, and any effective divisor D of degree $2g$ on M. By Exercise (2) of Chapter 26, $i(D) = i(D-[p]) = 0$ for any point $p \in M$. Riemann-Roch yields

$$\begin{aligned} &\ell(D) = \deg(D) - g + 1 = g + 1 \\ \text{and} \quad &\ell(D-[p]) = g, \end{aligned}$$

whence there exists $f \in \mathfrak{L}(D)$ and not in any of the finitely many $\mathfrak{L}(D-[p])$ for those p appearing in D. That is, $f^{-1}([\infty]) = D$, and $\deg(f) = \deg(f^{-1}([\infty])) = 2g$. Riemann-Hurwitz tells us about the ramification behavior of f:

$$\begin{aligned} \chi_M &= \deg(f)\cdot\chi_{\mathbb{P}^1} - \deg(R_f) \\ 2-2g &= 2g\cdot 2 - r_f \\ r_f &= 6g-2. \end{aligned}$$

For "almost all" D the points in R_f will have multiplicity one (ramifications of order two) and lie over distinct points in $\mathbb{P}^1$, meaning that the branch locus $B \subset \mathbb{P}^1$ consists of $6g-2$ points. We want to use all of this data to compute the number of "local deformation parameters" of M.

Looking at this in a slightly more formal way, consider the set $\mathfrak{S}_1$ of 2-tuples (M, f) where M has genus g and f has degree $2g$. This maps to the set $\mathfrak{S}_2$ of 2-tuples (M, D) where $D \geq 0$ of degree $2g$ (take $D := f^{-1}([\infty])$). From there you can map to the set $\mathfrak{S}$ of Riemann surfaces of genus g, by forgetting D. It's clear that (fixing M) D has $2g$ parameters, making $\dim(\mathfrak{S}_2) - \dim(\mathfrak{S}) = 2g$. Moreover, given

M and D, there are $\ell(D) = g+1$ choices of parameter for f (to have D as its poles), meaning $\dim(\mathfrak{S}_1) - \dim(\mathfrak{S}_2) = g+1$. Our argument in the first paragraph shows that the first map is surjective (while the second obviously is), and so

$$\dim(\mathfrak{S}) = \underbrace{\{\dim(\mathfrak{S}_1) - g - 1\}}_{\dim(\mathfrak{S}_2)} - 2g = \dim(\mathfrak{S}_1) - 3g - 1. \tag{28.1.2}$$

On the other hand, you can map $\mathfrak{S}_1$ to $\mathfrak{S}_3$, the set of $(6g-2)$-tuples of (unordered) points on $\mathbb{P}^1$, by taking $f(R_f) \in \mathrm{Div}(\mathbb{P}^1)$. This map is surjective since given a branch-point set in $\mathbb{P}^1$ you can construct an existence domain for an appropriate algebraic function,[2] and in fact the construction shows that there are only finitely many possibilities for M. Moreover, it shows that a continuous family of degree-$2g$ functions on M with the same branch-point set gives rise to a continuous family of automorphisms of M. But for $g \geq 2$ M has only finitely many automorphisms. So we see that this map is finite-to-1, and thus $\dim(\mathfrak{S}_1) = \dim(\mathfrak{S}_3) = 6g-2$. Plugging this in to (28.1.2), we get the desired result. □

It's much easier to count moduli for hyperelliptic and algebraic plane curves.

28.1.3. PROPOSITION. *Hyperelliptic Riemann surfaces of genus $g \geq 1$ (considered up to isomorphism) have $2g-1$ moduli.*

PROOF. They are essentially just the existence domains of the algebraic functions $\sqrt{\prod_{i=1}^{2g+2}(z-\alpha_i)}$, and so are completely determined by the branch locus $\{\alpha_i\}_{i=1}^{2g+2}$. This has $2g+2$ parameters, but we have to account for change of coordinate on $\mathbb{P}^1$, which is by $\mathrm{PGL}_2(\mathbb{C})$, by subtracting $\dim(\mathrm{PGL}_2) = 3$. □

28.1.4. PROPOSITION. *Smooth algebraic curves of degree d (considered up to projective equivalence) have $\binom{d+2}{2} - 9$ moduli.*

[2]Think $\sqrt{(z-a)(z-b)(z-c)(z-d)}$ (for $g=1$), but more complicated (since $g \geq 2$); cf. [Griffiths and Harris], pp. 255-257.

PROOF. A curve is determined by a polynomial in S_3^d, which has dimension $\binom{d+2}{2}$. We have only to account for changing projective coordinates by $\mathrm{GL}_3(\mathbb{C})$, which has dimension 9. (Here PGL_3 is *not* what we want, as we do want to quotient out the rescalings of the equation.) $\square$

Now we can compare moduli, with two very interesting results. First, consider the numbers you get for general Riemann surfaces of genera 1, 2, 3, 4, 5, 6: the numbers of moduli are 1, 3, 6, 9, 12, 15. For hyperelliptic ones, we have instead 1, 3, 5, 7, 9, 11. So while all genus 2 Riemann surfaces are hyperelliptic, we have:

28.1.5. PROPOSITION. *A general Riemann surface of genus $g \geq 3$ is non-hyperelliptic.*

So we will need to find more general methods of realizing Riemann surfaces as algebraic curves than what was discussed in §27.2, and that is what we endeavor to do in the remainder of this chapter.

Finally, look at those genera which correspond to nonsingular algebraic curves in $\mathbb{P}^2$ of degrees 3, 4, 5, 6, 7, . . . : namely, 1, 3, 6, 10, 15, and so on. The Riemann surfaces of these genera have (by Theorem 28.1.1) numbers of moduli 1, 6, 15, 27, 42, etc. But now look at the smooth algebraic plane curves of the corresponding degrees (via Prop. 28.1.4): we get 1, 6, 12, 19, 27. The case of genus 3 will be treated in §28.3. Beyond that, we have immediately:

28.1.6. PROPOSITION. *Smooth algebraic plane curves of degree $d \geq 5$ do* not *yield all the Riemann surfaces of genus $\frac{(d-1)(d-2)}{2}$ — only a special subset.*

28.2. Projective embeddings

Let M be a Riemann surface of genus g. For any $\mathfrak{D} \in \mathrm{Div}(M)$ of degree $> 2g-2$, recall from Exercise (2) of Ch. 26 that $i(\mathfrak{D}) = 0$. By Riemann-Roch, we then have

$$\ell(\mathfrak{D}) = \deg(\mathfrak{D}) - g + 1.$$

This will be used repeatedly in the argument below.[3]

Now fix a divisor $D = \sum_{p\in M} n_p[p] \in \mathrm{Div}(M)$ of degree $d \geq 2g+1$. We will define an embedding (injective morphism of complex manifolds)

$$\varphi : M \hookrightarrow \mathbb{P}^{d-g}.$$

(Since $d - g \geq g+1$, this can only give an embedding in $\mathbb{P}^2$ for $g = 1$.) The *support* of D, which is the subset of M consisting of the points appearing in D (i.e. those p with nonzero n_p), is written $|D|$.

First off, certainly $d > 2g-2$ and so $\ell(D) = d - g + 1$. Write $\{f_0, \ldots, f_{d-g}\}$ for a basis of $\mathfrak{L}(D)$, and define for $p \notin |D|$

$$\varphi(p) := [f_0(p) : \cdots : f_{d-g}(p)]. \tag{28.2.1}$$

If $p \in |D|$, this is unsuitable since some functions may blow up (or all functions may be required to vanish). Therefore if z is a local coordinate (vanishing at p to first order), we put

$$\varphi(p) := [(z^{n_p} f_0)(p) : \cdots : (z^{n_p} f_{d-g})(p)]. \tag{28.2.2}$$

For points q in a neighborhood of p, $[(z^{n_p} f_0)(q) : \cdots : (z^{n_p} f_{d-g})(q)]$ gives the same result as $[f_0(q) : \cdots : f_{d-g}(q)]$, and so we have constructed an analytic map ... provided that (28.2.1)-(28.2.2) do not yield $[0 : \cdots : 0]$ at any point. That is the central well-definedness issue, and we must check it.[4]

Now for $p, q \in M$, notice that $D - [p]$, $D - 2[p]$ and $D - [p] - [q]$ each still have degree $> 2g - 2$. Therefore we have

$$\ell(D - [p]) = d - g$$

and

$$\ell(D - 2[p]) = d - g - 1 = \ell(D - [p] - [q]),$$

[3]It's very important to understand the argument in this section. Try slimming it down (I've expressed it in a somewhat bloated manner) and writing it out for a specific choice of d and g (> 1, say). (Also, if you are stuck on Exercise (4) of Ch. 27, some of the steps are similar.)

[4]Note that (28.2.2) isn't just a special formula for $p \in |D|$; it contains (is more general than) (28.2.1) since for $p \notin |D|$ we have $n_p = 0$.

with the immediate consequences

$$\mathfrak{L}(D-[p]) \subsetneq \mathfrak{L}(D), \tag{28.2.3}$$

$$\mathfrak{L}(D-[p]-[q]) \subsetneq \mathfrak{L}(D-[p]), \tag{28.2.4}$$

$$\mathfrak{L}(D-2[p]) \subsetneq \mathfrak{L}(D-[p]). \tag{28.2.5}$$

To interpret these, for simplicity first assume $p, q \notin |D|$. Then (28.2.3) says that there exists $f \in \mathfrak{L}(D)$ not vanishing at p, meaning that the $\{f_i(p)\}$ are not all zero; this makes φ well-defined on $M \backslash |D|$. Next, (28.2.4) gives us $g \in \mathfrak{L}(D-[p]) \backslash \mathfrak{L}(D-[p]-[q])$, a function vanishing at p but not q, forcing φ to take different values at p and q; hence φ is injective on $M \backslash |D|$. Finally, (28.2.5) provides $h \in \mathfrak{L}(D-[p]) \backslash \mathfrak{L}(D-2[p])$, i.e. vanishing to exactly first order at p, so that the derivative of h hence that of φ is nonzero there; together with the injectivity result, this proves that the image of $M \backslash |D|$ is smooth.

In order to extend these statements to all of M, we have to refine the argument just a bit. For general points $p, q \in M$, (28.2.3) tells us that there exists a function $f \in \mathfrak{L}(D)$ with $\nu_p(f) = -n_p$ exactly; (28.2.4) that some $g \in \mathfrak{L}(D)$ has $\nu_p(g) > -n_p$ but $\nu_q(g) = -n_q$; and (28.2.5) that there exists an $h \in \mathfrak{L}(D)$ with $\nu_p(h) = -n_p + 1$ exactly. These give precisely the well-definedness, injectivity, and smoothness of image for the map described by (28.2.2). So the image is a compact complex analytic curve $\mathbb{P}^{d-g}$, which is algebraic by GAGA.

28.3. Canonical maps

Once again we consider a Riemann surface M, this time of genus $g \geq 2$, and let

$$\{\omega_1, \ldots, \omega_g\} \subset \Omega^1(M)$$

be a basis. Instead of choosing a divisor and going through Riemann-Roch to get a projective embedding from meromorphic functions, why not just use these? Define the *canonical map*

$$\varphi_K : M \to \mathbb{P}^{g-1}$$

by

$$p \mapsto [\omega_1(p) : \cdots : \omega_g(p)].$$

The meaning of this, as you would expect, is locally writing each ω_i as $f_i(z)dz$, and taking $[f_1(p) : \cdots : f_g(p)]$. This is well-defined, i.e. the $\{\omega_i\}$ do not all have a zero at p. Otherwise we would have $\mathfrak{I}([p]) = \mathfrak{I}(0) = g$ hence (by Riemann-Roch) $\mathfrak{L}([p]) = 2$, which we know to be false for M not isomorphic to $\mathbb{P}^1$.

Bottom line: this looks quite promising, from the standpoint of getting a convenient projective embedding. Or does it?

28.3.1. EXAMPLE. For M hyperelliptic, consider the setting of Theorem 27.2.4; we have

$$\varphi_K(p) = \left[\frac{dx}{y} : \frac{x\,dx}{y} : \cdots : \frac{x^{g-1}dx}{y}\right] = [1 : x : \cdots : x^{g-1}].$$

Notice that this looks a lot like the rational canonical map f from Example 7.3.7. In fact, it factors

(28.3.2)
$$\begin{tikzcd} C \arrow[rr, "\varphi_K"'] \arrow[dr, "x"'] & & \mathbb{P}^{g-1} \\ & \mathbb{P}^1 \arrow[ur, "f"'] & \end{tikzcd}$$

with $\deg(x) = 2$, and so does *not* give an embedding of M in $\mathbb{P}^{g-1}$.

All is not lost: the hyperelliptic case is very special (in a bad way), and for $g \geq 3$ we know that there are (lots of) nonhyperelliptic curves.

28.3.3. THEOREM. *Let φ_K be the canonical map for an arbitrary Riemann surface of genus $g \geq 2$. Then*

(a) φ_K is nondegenerate;[5]

(b) $\varphi_K(M)$ is smooth; and

(c) φ_K is injective $\iff$ M is nonhyperelliptic.

[5]cf. the beginning of Chapter 7, and also Prop. 7.3.1.

PROOF. (a) Were $\varphi_K(M)$ contained in a proper linear subspace of $\mathbb{P}^{g-1}$, this would produce a linear relation on the $\{\omega_i\}$. But they are linearly independent by construction, being a basis!

(b) This is clear in the hyperelliptic case, by observing that the derivative of the (injective) rational canonical map is nowhere vanishing. (We will return to this in the nonhyperelliptic case.)

(c) The implication " $\implies$ " is already done (by contrapositive) in Example 28.3.1.

Now let z be a local coordinate vanishing to first order at a point $p \in M$, and consider the linear functionals on $\Omega^1(M)$ $(= \mathfrak{I}(0))$ given by

$$\begin{aligned}
\omega &\mapsto \left(\frac{\omega}{dz}\right)(p) \\
\omega &\mapsto \left(\frac{\omega}{dz}\right)'(p) \\
&\vdots \\
\omega &\mapsto \left(\frac{\omega}{dz}\right)^{(k-1)}(p).
\end{aligned}$$

If the first is zero on some given ω, then $\omega \in \mathfrak{I}([p])$. If the first and second are zero, then $\omega \in \mathfrak{I}(2[p])$. If all are zero, then $\omega \in \mathfrak{I}(k[p])$. Since k linear conditions cut out a subspace of codimension at most k, we have

$$i(k[p]) = \dim \mathfrak{I}(k[p]) \geq g - k.$$

More precisely, we have $i(k[p]) = g - k + a$ and by Riemann-Roch

$$\ell(k[p]) = k - g + 1 + i(k[p]) = 1 + a.$$

For the special case $k = 1$, there can be no redundancies in *one* linear condition (recall that the $\{\omega_i\}$ have no common zeroes) and we have $a = 0$.

Suppose $\varphi_K([p]) = \varphi_K([q])$ for $p \neq q$. Then $\omega \mapsto \left(\frac{\omega}{dz}\right)(p)$ and $\omega \mapsto \left(\frac{\omega}{dz}\right)(q)$ yield the same functional on $\Omega^1(M)$, up to a constant multiple; in particular they vanish on the same ω's. So $\mathfrak{I}([p]) = \mathfrak{I}([p] + [q])$, which yields $i([p] + [q]) = i([p]) = g - 1$ and via Riemann-Roch

$$\ell([p]+[q]) = 2 - g + 1 + i([p]+[q]) = 2.$$

Thus there exists a nonconstant meromorphic function $\mathcal{F} \in \mathfrak{L}([p]+[q])$. We know that $\mathfrak{L}([p])$ and $\mathfrak{L}([q])$ have only constant functions ($a = 0$ when $k = 1$), and so $\mathcal{F}$ has to have the allowed simple pole at both of p and q. Thus $\deg(\mathcal{F}) = 2$, and so M is hyperelliptic. This completes the proof of (c).

(b, cont'd.) Now assume M is *non*hyperelliptic. Then we must have $\ell(2[p]) = 1$ (why?), i.e. $a = 0$ for $k = 2$. Consequently

$$i(2[p]) = g - 2$$

(whilst $i([p]) = g - 1$), and we can arrange a basis of $\Omega^1(M)$ so that in local coordinates at p,

$$\omega_1 \overset{\text{loc}}{=} dz\,, \quad \omega_2 \overset{\text{loc}}{=} zh_2(z)dz\,,$$
$$\omega_j \overset{\text{loc}}{=} z^2 h_j(z)dz \quad (3 \le j \le g).$$

(Here the $h_i(z)$ are holomorphic, and h_2 doesn't vanish at $z = 0$.) The canonical map takes the local form

$$\varphi_K(z) = [1 : zh_2(z) : z^2 h_3(z) : \cdots : z^2 h_g(z)]$$

with derivative

$$\varphi_K(z) = [0 : \; h_2(z) + zh_2'(z) : \; * : \cdots : \; *]$$

which does not vanish at p. This gives the desired smoothness. $\square$

Consider a smooth, irreducible algebraic curve C and hyperplane $H = \{W(\underline{Z}) = 0\}$, both in $\mathbb{P}^n$. (Here W is a homogeneous polynomial in $Z_1, \ldots, Z_{n+1}$ of degree one, with affine form w.) One can define an intersection divisor $C \cdot H$ on C in a way which extends what we have done in $\mathbb{P}^2$. If C is not necessarily smooth, then the divisor lives on a normalization $M \,(\overset{\sigma}{\to} \mathbb{P}^2)$ of C and is denoted $\sigma^* H$; it is given simply by $\sum_{p \in M} \mathrm{ord}_p(\sigma^* w)[p]$. We define the degree of the curve to be the degree of this divisor, called a *hyperplane section*:

$$\deg(C) := \deg(\sigma^* H).$$

Since any two hyperplane sections are rationally equivalent (why?), any two hyperplane sections have the same degree, making $\deg(C)$ well-defined.

In the case at hand, σ is φ_K and $n = g - 1$. Hyperplane sections are particularly interesting because if we write $W(\underline{Z}) = \sum_{i=1}^{g} \alpha_i Z_i$, then

$$\sigma^* H \ = \ (\alpha_1 \omega_1 + \cdots + \alpha_g \omega_g)$$

is a canonical divisor on M! That's why φ_K is called the canonical map, and its image $\varphi_K(M)$ a *canonical curve*.

28.3.4. PROPOSITION. *Assume M is nonhyperelliptic of genus g. Then the degree of the canonical curve $\varphi_K(M) \subset \mathbb{P}^{g-1}$ is $2g - 2$.*

PROOF. The assumption is necessary in order that M normalize $\varphi_K(M)$. (In the hyperelliptic case, it is normalized by the rational canonical map.) We then compute $\deg(\varphi_K(M)) = \deg(\varphi_K^* H) = \deg(K) = 2g - 2$ by Poincaré-Hopf, and that's it. $\square$

And so, we find that "nearly all" genus 3 curves have a nice embedding into the projective plane.

28.3.5. COROLLARY. *Every nonhyperelliptic genus 3 curve is the normalization of a smooth quartic curve in $\mathbb{P}^2$.*

28.4. Weierstrass points

We began our discussion of Riemann-Roch with a naive analysis, for a fixed point p on a Riemann surface M, of what orders of pole are possible if we are after a meromorphic function with its only pole at p. To conclude, I will now briefly explain the sense in which this can depend on the choice of p and not just the genus g of M. Assume $g > 0$ for what follows.

First note that $\ell(0) = 1$ (constant functions), and $\ell([p]) = 1$ by the argument at the beginning of Ch. 26. By equation (26.2.4), we know for each k that

$$0 \leq \ell((k+1)[p]) - \ell(k[p]) \leq 1.$$

On the other hand, since the degree of $(2g-1)[p]$ exceeds $2g-2$, we have (Ch. 26 Exercise (2)) that $i((2g-1)[p]) = 0$, and so (by Riemann-Roch)

$$\ell((2g-1)[p]) = (2g-1) - g + 1 = g.$$

More generally, for $k \geq 2g-1$, the fact that $i(k[p]) = 0$ yields

$$\ell(k[p]) = k - g + 1.$$

So the scenario is that $\ell(k[p])$ starts (at $k = 0$) at 1 and works its way up to g in increments of 0 or 1, as k rises to $2g-1$; thereafter it increases by 1 whenever k does.

The situation with $i(k[p])$ is "dual": it starts at g and works its way down to 0 in decrements of 0 or 1, as k rises to $2g-1$; and then it stays at 0.

Now it turns out that at all but finitely many points, $\ell(g[p]) = 1$; that is, all the increments are postponed as far as possible and the sequence $\ell(k[p])$ looks like $1, 1, 1, \ldots, 1, 2, 3, \ldots, g$, and so on. Those points where this is not the case are called the *Weierstrass points* of M. The simplest example I am aware of is, for a hyperelliptic curve, the $2g+2$ fixed points of the involution $\jmath$. For these the sequence looks like $1, 1, 2, 2, 3, 3$, etc.

Exercises

(1) Check that the definition of $\varphi_K(p)$ in §28.3 is independent of the choice of local coordinate near p.

(2) Show that any smooth quartic curve in $\mathbb{P}^2$ is a canonical curve (of genus 3), and hence also nonhyperelliptic.

(3) In this exercise you will prove a new Cayley-Bacharach type result (in $\mathbb{P}^2$): *if C_1 and C_2 (of degrees m and n respectively, with C_1 assumed smooth and irreducible) meet in mn distinct points, and C_3 (of degree $m+n-3$) passes through $p_1, \ldots, p_{mn-1} \in C_1 \cap C_2$, then it*

passes through the remaining point $p := p_{mn}$. (We shall write f_1, f_2, f_3 for the resp. defining homogeneous polynomials.) Start out by assuming C_3 does not contain p, and follow these steps:

(a) Let g denote the genus of C_1, and show that $(m-3)m = 2g-2$ and $g = \dim(S_3^{m-3})$.

(b) Let $h \in S_3^{m-3}$, set $F_h := \frac{f_2 \cdot h}{f_3}|_{C_1} \in \mathcal{K}(C_1)^*$, and write $(F_h) = [p] + (h) - D$ (this defines $D \in \mathrm{Div}(C_1)$). Show that the map $S_3^{m-3} \to \mathfrak{L}(D)/\mathbb{C}$ (here $\mathbb{C}$ =constant functions) given by $h \mapsto F_h$ is injective, and use this to put a lower bound on $\ell(D)$.

(c) Find $\deg(D)$, $i(D)$, and obtain a contradiction.

(4) A problem on automorphisms of canonical curves:

(a) Let $\alpha : C \to C$ be an automorphism of a canonical curve of genus g. Prove that α is the restriction to C of a linear automorphism of $\mathbb{P}^{g-1}$. [Hint: consider the action of α^* on $\Omega^1(C)$.]

(b) Let M be a *non*hyperelliptic Riemann surface of genus 3 with an involution $\jmath$. How many fixed points does it have? What is the genus of the quotient Riemann surface? [Hint: consider the canonical embedding and apply (a); $\jmath$ is the restriction of what sort of linear automorphism on $\mathbb{P}^2$?]

(5) Using the embedding of §28.2, try to prove: (a) there exists, for an arbitrary Riemann surface, an embedding onto a smooth curve in $\mathbb{P}^3$ and (b) an immersion onto a curve with only nodal singularities in $\mathbb{P}^2$. [Hint: use sufficiently general projections of the complement of a linear subspace in $\mathbb{P}^n$ onto $\mathbb{P}^2$ and $\mathbb{P}^3$.] Also, (c) what degrees do these curves have?