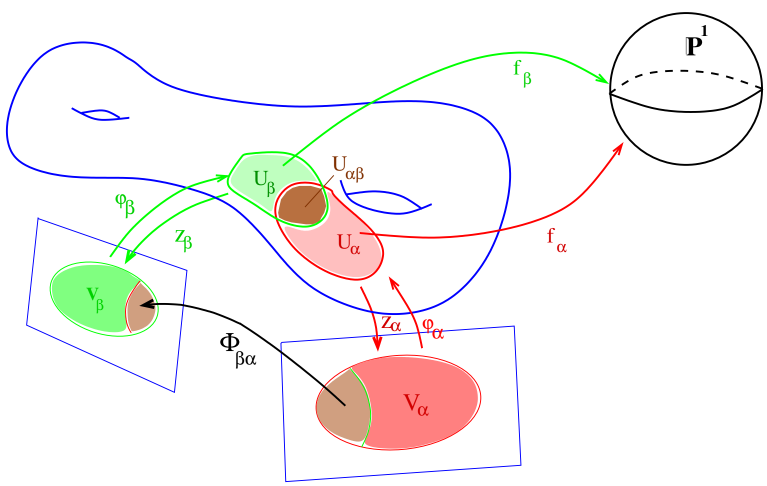
CHAPTER 3

The normalization theorem

We state (but do not yet prove) the promised relationship between algebraic curves and Riemann surfaces, and explain how to work it out directly for conics. To state the general relationship, however, we need the notion of meromorphic functions on a Riemann surface, so we will first define and prove a few results about those.

3.1. Meromorphic functions on a Riemann surface

Let M be a Riemann surface (Definition 2.3.3) with analytic atlas $\{(U_\alpha, z_\alpha)\}$ (Definition 2.2.1), and write $V_\alpha := z_\alpha(U_\alpha) \subseteq \mathbb{C}$. The *local analytic chart* $\varphi_\alpha : V_\alpha \to U_\alpha (\subseteq M)$ is simply defined to be the (composition) inverse of of the local coordinate z_α. (I've avoided writing z^{-1} since in some settings this is easy to confuse with $\frac{1}{z}$.)



3.1.1. DEFINITION. A meromorphic (resp. holomorphic) function $f \in \mathcal{K}(M)$ (resp. $\mathcal{O}(M)$) is a collection of continuous maps $f_\alpha : U_\alpha \to \mathbb{P}^1$ such that

- the $\{f_\alpha\}$ "agree" on overlaps (viz., $f_\alpha = f_\beta$ on $U_{\alpha\beta}$), and

- $f_\alpha \circ \varphi_\alpha$ is a meromorphic (resp. holomorphic) function, in the sense of complex analysis, for all α.

3.1.2. REMARK. (a) One really works with functions of the coordinate z_α, i.e. the function $f_\alpha \circ \varphi_\alpha =: g_\alpha$ (mapping $V_\alpha \to \mathbb{P}^1$), and then the compatibility condition reads

$$g_\alpha \circ \Phi_{\alpha\beta} = g_\beta. \tag{3.1.3}$$

(b) $\mathcal{K}(M)$ is a field, since you can multiply, add, and invert (additively and multiplicatively) meromorphic functions.

For the above Definition and Remark, we could just as well take M to be a *noncompact* complex 1-manifold. In that case $\mathcal{O}(M)$ may be an interesting ring. But in the Riemann surface case it is not:

3.1.4. PROPOSITION. [LIOUVILLE'S THEOREM] *M compact $\implies$ $\mathcal{O}(M) \cong \mathbb{C}$ (constant functions).*

PROOF. On the one hand, $f \in \mathcal{O}(M) \implies f(M) \subset (\mathbb{P}^1 \backslash \{\infty\}) = \mathbb{C}$; while on the other, M compact and f continuous $\implies$ $f(M)$ is compact. Applying absolute value gives a compact subset $|f(M)| \subset \mathbb{R}_{\geq 0}$. This has a maximum element, which is assumed at some point $p \in M$, and this p lies in some U_α. Hence, the absolute value of the holomorphic function $g_\alpha = f_\alpha \circ \varphi_\alpha$ attains a maximum on V_α (at $\varphi_\alpha(p)$), and by the maximum modulus principle, g_α (and thus f_α) is some constant $c \in \mathbb{C}$.

Let U_β be any open set of the atlas meeting U_α. Since $f_\beta = f_\alpha = c$ on $U_{\alpha\beta}$, and $U_{\alpha\beta}$ has accumulation points, $f_\beta = c$ on U_β. One continues this argument now for any open set meeting U_α or U_β, and so forth. By connectedness of M, this shows $f = c$ on all open sets of the atlas, hence on all of M. $\square$

3.1.5. DEFINITION. Let $f \in \mathcal{K}(M)$ be a meromorphic function. For any $p \in M$, f is locally of the form

$$z^m h(z) \tag{3.1.6}$$

with $m \in \mathbb{Z}$, z a local coordinate vanishing at p (i.e. $z(p) = 0$), and $h(z)$ a local holomorphic function of z with $h(0) \neq 0$.[1] We say that the *order* $\nu_p(f)$ of f at p is m.

With this bit of language it is easy to compute the meromorphic function field for Riemann surfaces of genus 0 and 1.

3.1.7. THEOREM. *(a)* $\mathcal{K}(\mathbb{P}^1) \cong \mathbb{C}(z)$ *(z an indeterminate).*
(b) Writing $\Lambda := \{m_1\lambda_1 + m_2\lambda_2 \,|\, m_i \in \mathbb{Z}\}$ *(*$\lambda_1, \lambda_2 \in \mathbb{C}$ *linearly independent over* $\mathbb{R}$*) for a lattice,* $\mathcal{K}(\mathbb{C}/\Lambda) \cong \mathbb{C}(\wp, \wp')$ *where* $\wp(u)$ *is the Weierstrass* $\wp$*-function for* Λ*.*

PROOF. (a) Referring to Example 2.2.4, write $z = z_0$ and $w = z_1$ for the two local coordinates. I am really going to use z as a global coordinate on $\mathbb{P}^1$; the statement we want to prove is that meromorphic functions on $\mathbb{P}^1$ are precisely the rational functions of z.

In one direction, this is easy: if P, Q are polynomials in z (with $Q \neq 0$), clearly $\frac{P(z)}{Q(z)}$ is the restriction to U_0 of a meromorphic function on $\mathbb{P}^1$ (on U_1, it is $\frac{P(\frac{1}{w})}{Q(\frac{1}{w})}$).

Conversely, are all meromorphic functions rational? Given $f \in \mathcal{K}(\mathbb{P}^1)$, $\nu_p(f) < 0$ at finitely many[2] points $z_i (= p)$, and we shall for simplicity assume none of these is the point ∞. Let $\mathsf{P}_i(z) = \sum_{\nu_{z_i}(f) \leq k < 0} \beta_{ik}(z - z_i)^k$ (sum is over k) be the principal part of the Laurent expansion of f at z_i, and consider $G(z) = \sum \mathsf{P}_i(z)$. Then $f - G \in \mathcal{O}(\mathbb{P}^1)$ is constant by Liouville; and since G is rational, we're done.

(b) Next, $f \in \mathcal{K}(\mathbb{C}/\Lambda)$ if and only if f is a doubly-periodic meromorphic function on $\mathbb{C}$: that is, $f(u) = f(u + m_1\lambda_1 + m_2\lambda_2)$ for all $m_1, m_2 \in \mathbb{Z}$ (also known as an elliptic function). We will see later

[1]To be absolutely precise, if z is a local coordinate on $U \ni p$, with $V = z(U)$, then h is a holomorphic function on V. I'll frequently assume things like this to be "understood".

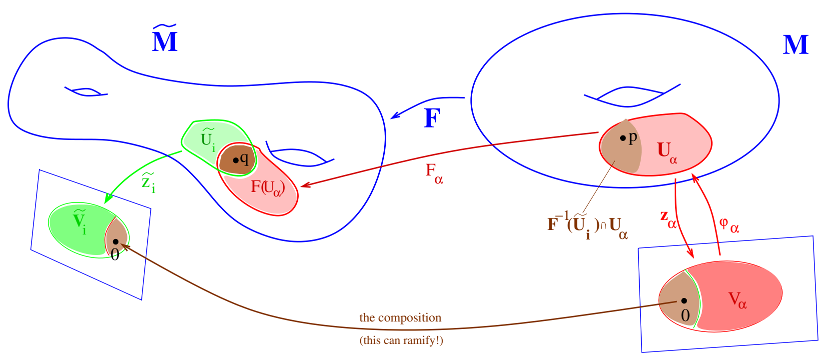
[2]otherwise compactness $\Longrightarrow$ zeroes of $\frac{1}{f}$ have an accumutaion point $\Longrightarrow$ $\frac{1}{f}$ identically 0. (Also, note that I am identifying points by the value of the coordinate z on $\mathbb{P}^1$. If M were not $\mathbb{P}^1$, I would write p_i instead of z_i.)

that these are generated (rationally) by

$$\wp(u) := \frac{1}{u^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left(\frac{1}{(u-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

and its derivative. □

3.1.8. DEFINITION. A *morphism* (or holomorphic map) $M \xrightarrow{F} \tilde{M}$ of Riemann surfaces[3] is a collection $F_\alpha : U_\alpha \to \tilde{M}$ of continuous maps (agreeing on the $\{U_{\alpha\beta}\}$) such that the composition[4] $\tilde{z}_i \circ F_\alpha \circ \varphi_\alpha|_{z_\alpha\{F^{-1}(\tilde{U}_i)\cap U_\alpha\}}$ is holomorphic for all α, i. (Note that this definition works more generally for complex 1-manifolds — compactness is inessential.)



Now suppose we have $p \in (U_\alpha \subset)M$ and $q \in (\tilde{U}_i \subset)M$ with $F(p) = q$, $z_\alpha(p) = 0$ and $\tilde{z}_i(q) = 0$, as shown in the above figure. Assuming F is nonconstant, then after "normalizing" the local coordinates,[5] we have $\tilde{z}_i(z_\alpha) = (z_\alpha)^\mu$ for some (unique) $\mu \in \mathbb{Z}_{>0}$. One says that f has *ramification index* μ at p (over q). If this index is > 1, we say that f is *branched* over q (or ramifies at p).

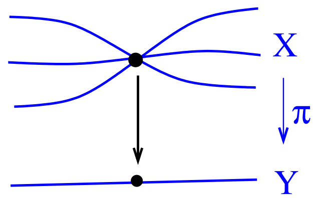
3.1.9. REMARK. For $\mu = 3$, we have already seen this picture in Example 2.3.1. In general, for a holomorphic map of Riemann surfaces $\pi : X \to Y$, for all but finitely many $y \in Y$ the number $|\pi^{-1}(y)|$ is the same, and this is called the *degree* of the mapping

[3] write $\{U_\alpha, z_\alpha\}$ and $\{\tilde{U}_i, \tilde{z}_i\}$ for the atlases.

[4] this composition renders $\tilde{z}_i$ as a function of z_α (and is a local snapshot of F in this sense)

[5] see Exercise 4 below

π. (This will be explained in greater depth in a later chapter.) The branch points of π are just the remaining points of Y. Usually we will just draw a schematic picture like



and it is understood that the picture is really as in Example 2.3.1 — so that going around the point on the "base" Y moves you between *branches* of the "cover" X.

3.1.10. PROPOSITION. *Let M be a Riemann surface (or, more generally, a complex 1-manifold). The holomorphic maps $M \to \mathbb{P}^1$, excluding the constant map sending all points to $\{\infty\}$, are simply the meromorphic functions $\mathcal{K}(M)$.*

PROOF. Again refer to Example 2.2.4: given a morphism $F : M \to \mathbb{P}^1$ (Definition 3.1.8), by definition $z_0 \circ F_\alpha \circ \varphi_\alpha$ is holomorphic on the complement of the preimage of ∞, while $z_1 \circ F_\alpha \circ \varphi_\alpha = \frac{1}{z_0 \circ F_\alpha \circ \varphi_\alpha}$ is holomorphic on the complement of the preimage of 0.[6] Hence, $F_\alpha \circ \varphi_\alpha$ is meromorphic and $\{F_\alpha\}$ defines a meromorphic function (Definition 3.1.1). The converse is even more tautological! □

Later we will discuss morphisms (holomorphic maps) of complex manifolds of any dimension. The following is a special case:

3.1.11. DEFINITION. Write $[Z_0 : Z_1 : \cdots : Z_n]$ for (projective) coordinates on $\mathbb{P}^n$. A map σ from a Riemann surface M to $\mathbb{P}^n$ is called holomorphic if and only if all compositions $[Z_i \circ \sigma : Z_j \circ \sigma]$ are holomorphic as maps to $\mathbb{P}^1$ on the open subsets of M where they are well-defined.

[6]The holomorphicity of $\frac{1}{z_0 \circ F_\alpha \circ \varphi_\alpha}$ guarantees, in particular, that $F_\alpha \circ \varphi_\alpha$ has only poles and not essential singularities.

3.1.12. REMARK. If the image $\sigma(M)$ is not contained in the "hyperplane at infinity" $Z_0 = 0$, this is the same as saying that *composing σ with each affine coordinate on $\mathbb{C}^n$ gives a meromorphic function*.

To see this, first write $\mathcal{M}_{ij}$ for the subsets of M where (under σ) Z_i and Z_j are not both zero; these are the open subsets in the last definition.[7] By Prop. 3.1.10, the conditions of Defn. 3.1.11 mean that $\frac{Z_j}{Z_i}$ are meromorphic functions on the $\mathcal{M}_{ij}$. We need to show that the $z_j = \frac{Z_j}{Z_0}$ extend to meromorphic functions on all of M. First, M is covered by the open sets $\{Z_i \neq 0\}$. Hence, for $p \notin \mathcal{M}_{0j}$ (i.e. Z_j and Z_0 vanish at p), we have a neighborhood U containing p where some other Z_i does not vanish, so that $U \subset \mathcal{M}_{ij}, \mathcal{M}_{i0}$. Now, on $U \cap \mathcal{M}_{0j}$ we can write $z_j = \frac{Z_j}{Z_i} \cdot \left(\frac{Z_0}{Z_i}\right)^{-1}$ as a product of functions which are meromorphic on all of U, hence showing that z_j extends as desired.

3.2. Riemann surfaces parametrize algebraic curves

Here is the Normalization Theorem. We will prove part (A) in this course.

3.2.1. THEOREM. *(A) Given an irreducible algebraic curve $C \subset \mathbb{P}^2$, there exists a Riemann surface M and a holomorphic map $\sigma : M \to \mathbb{P}^2$ with C as its image which is 1-to-1 on $\sigma^{-1}(C \backslash sing(C))$.*

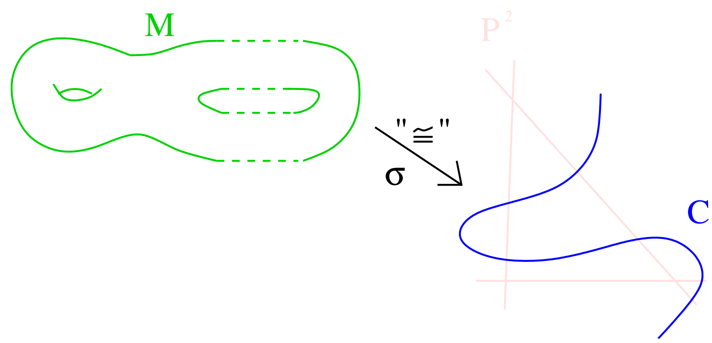
(B) Given a Riemann surface M, there exists a holomorphic map $\sigma : M \to \mathbb{P}^2$ such that

- *$\sigma(M)$ is an irreducible algebraic curve with $sing(\sigma(M))$ consisting of ordinary double points (or empty), and*
- *σ is 1-to-1 off the preimage of these ordinary double points.*

In this sense, irreducible smooth projective algebraic plane curves (over $\mathbb{C}$) are equivalent to, and are isomorphically parametrized by,

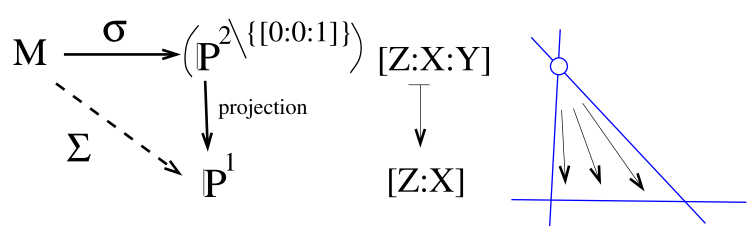
[7]Some, but not all, of these $\mathcal{M}_{ij}$ (with $i, j \neq 0$) will be empty if $\sigma(M)$ is contained in an intersection of coordinate hyperplanes; this doesn't matter.

Riemann surfaces.



If a curve C is not smooth, then the normalization "desingularizes" it (and we shall see this quite explicitly later on). In either case, we say that M is the *normalization* of C.

Let's look briefly at the meaning of (B), which we will not prove in this course. For a given Riemann surface (i.e. compact complex 1-manifold) M, it guarantees a holomorphic map to $\mathbb{P}^2$, with image $\sigma(M) = C =$ projective closure of $\{f(x,y) = 0\}$. Changing coordinates on $\mathbb{P}^2$ if necessary, we may assume that C does not pass through $[0:0:1]$. So it makes sense to consider the composition



which exhibits M as a branched cover of $\mathbb{P}^1$ — or more precisely, as the existence domain of the algebraic function $g(x)$ obtained by solving

$$f(x, g(x)) = 0.$$

So Theorem 3.2.1(B) contains the statement that every compact complex 1-manifold is an existence domain in the sense of §2.3.

We should also note that any Riemann surface admits a holomorphic *embedding* $\sigma : M \hookrightarrow \mathbb{P}^3$, an even nicer result than part (B) above!

3.3. Stereographic projection

As a plausibility check on Theorem 3.2.1(A), we'd like a recipe for normalizing conics — i.e. degree-2 (conic) curves $C \subset \mathbb{P}^2$. Given a point $p \in C$, and any line line ℓ through p, by Proposition 2.1.15 ℓ either meets C in two points with mutliplicity 1 or in 1 point with multiplicity 2. Put differently, we have either

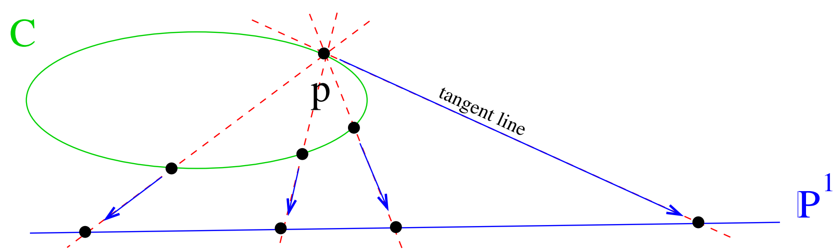
- $\ell \cap C = \{p, q\}$

or

- $\ell \cap C = 2p$, i.e. $\ell = T_pC$ is the *tangent line* to C at p.

(We will give a systematic treatment of tangent lines below.) Conversely, given p and any other point q on C, there is a unique line through them (and it doesn't meet C anywhere else).

There are two ways to think of why this gives a parametrization of C. One possibility is to take a fixed line ($\cong \mathbb{P}^1$) and use lines through p to project C onto it:



This is where the term "stereographic projection" comes from.

But this auxiliary projective line is superfluous, because the family of lines through p already gives a $\mathbb{P}^1$. (Indeed this is close to the original definition of what $\mathbb{P}^1$ is.) We can parametrize this $\mathbb{P}^1$ by the slope of the line with respect to suitable coordinates (usually $(x, y) = (\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0})$). The upshot is that we get a 1-1 correspondence between lines through p and points on C, so that we are in the situation of §3.2 with $M \cong \mathbb{P}^1$.

3.3.1. EXAMPLE. Suppose we wish to find a parametrization $\mathbb{P}^1 \xrightarrow{\sigma} C$ of the conic $\{X^2 + Y^2 = Z^2\} \subset \mathbb{P}^2$, which in affine coordinates is $x^2 + y^2 = 1$. We choose a point on C, say $p = (1, 0)$, and draw lines

$y = \mu(x-1)$ through p. (The slope here is μ, and this should be viewed as a choice of coordinate on $\mathbb{P}^1$.) Substituting into $x^2+y^2=1$ and solving for x in terms of μ, we have

$$\begin{aligned} & x^2+\mu^2(x-1)^2 = 1 \\ \Longrightarrow \quad & (\mu^2+1)x^2 - 2\mu^2 x + (\mu^2-1) = 0 \\ \Longrightarrow \quad & (x-1)\{(1+\mu^2)x + (1-\mu^2)\} = 0. \end{aligned}$$

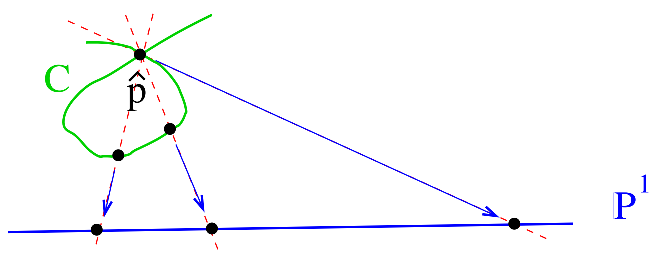
Ignoring the solution $x=1$ (which corresponds to p), we have

$$x = \frac{\mu^2-1}{\mu^2+1}, \quad y = \mu\left(\frac{\mu^2-1}{\mu^2+1} - 1\right) = \frac{-2\mu}{\mu^2+1}.$$

Hence, we find

$$\sigma(\mu) = \left(\frac{\mu^2-1}{\mu^2+1}, \frac{-2\mu}{\mu^2+1}\right).$$

One can also do stereographic projection to construct normalizations of *singular cubic* curves:



The idea here is to consider lines through the singular point $\hat{p}$; since any such ℓ already meets C "twice", it will only hit C in one additional point (by Proposition 2.1.15). You'll work an example in the exercises below. This will not work for a smooth cubic.

Exercises

(1) Give a parametrization $m \mapsto (x(m), y(m))$ (hence an isomorphism $\mathbb{P}^1 \to C$) of the smooth conic curve C that is the projective closure of $3x^2 - y^2 = 1$. (You may work in affine coordinates.)

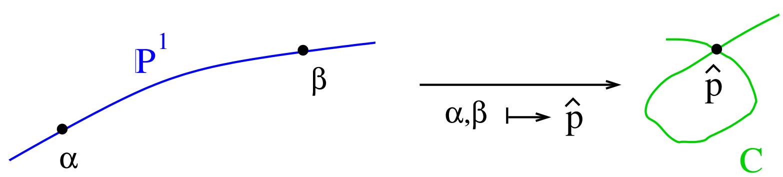
(2) Show that for any Riemann surface M and meromorphic function $(0 \neq) f \in \mathcal{K}(M)$, one has $\sum_{p \in M} \nu_p(f) = 0$. [Hint: Use the

residue theorem from complex analysis. Cut open the RS as in Chapter 2, and integrate $\frac{df}{f}$ along the "boundary".]

(3) Convince yourself that the order $\nu_p(f)$ of a meromorphic function on a Riemann surface M (Definition 3.1.5) is independent of the choice of local coordinate.

(4) Prove the following, which was claimed in Definition 3.1.8: Given M, M' Riemann surfaces with a holomorphic map $f : M \to M'$ (and $f(p) = q$). Then there exist (U, z) on M and (V, w) on M' satisfying $z(p) = 0 = w(q)$, such that $w = z^{\mu}$ (for some $\mu \in \mathbb{N}$) is the local form taken by f near p. [Here for example "(U, z)" means an open disk $U \subset M$ with local coordinate $z : U \to \mathbb{C}$.]

(5) Find a parametrization $\mathbb{P}^1 \to C$ of the singular cubic $Y^2Z - X^2Z + X^3 = 0$ in $\mathbb{P}^2$. (C has an ordinary double point $\hat{p}$ at $[Z : X : Y] = [1 : 0 : 0]$. Check that this point is indeed a singularity of C.) To do this, convert to affine coordinates, substitute in $y = mx$, and solve for the other intersection point's coordinates as a function of m. Two points will go to $\hat{p} = (0, 0)$. Picture:



What are α and β? Change coordinates on $\mathbb{P}^1$ (fractional linear transformation) so that in your new coordinate, 0 and ∞ are sent to $\hat{p}$. Your parametrization should read now $\varphi : \mathbb{P}^1 \to C$ sending $z \mapsto (x(z), y(z))$ with $0, \infty \mapsto \hat{p}$. This will be used in a later exercise.

(6) Let $\Lambda, \Lambda' \subset \mathbb{C}$ be two full lattices (free abelian subgroups of rank 2 whose generators are independent over $\mathbb{R}$), and consider the complex 1-tori $T = \mathbb{C}/\Lambda$ and $T' = \mathbb{C}/\Lambda'$. These are (compact) Riemann surfaces of genus 1. (a) Show that there exists an isomorphism between them iff Λ is a multiple $\mu\Lambda'$ ($\mu \in \mathbb{C}^*$). [Hint: if there exists an isomorphism, then there is one sending $0 \mapsto 0$;

lift this to a biholomorphism of the universal covers.] (b) Replacing Λ, Λ' by multiples, we may assume they are of the form $\mathbb{Z} + \mathbb{Z}\tau$ and $\mathbb{Z} + \mathbb{Z}\tau'$, with $\tau, \tau' \in \mathfrak{H}$ (upper half-plane). Show that $T \cong T'$ (i.e. there exists an isomorphism) iff τ and τ' are related by a fractional linear transformation (i.e. $\tau = \frac{a\tau'+b}{c\tau'+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$). Conclude that $\mathfrak{H}/\mathrm{SL}_2(\mathbb{Z})$ parametrizes equivalence classes of complex 1-tori.