CHAPTER 30

Abel's Theorem, part I

Recall the setup from Chapter 21: M is a Riemann surface of genus $g \geq 1$, with closed paths ("1-cycles") γ_i giving a basis $\{[\gamma_i]\}_{i=1}^{2g}$ for $H_1(M, \mathbb{Z})$. We have the *Jacobian* of M, which is the complex g-torus

$$J(M) := \frac{(\Omega^1(M))^\vee}{H_1(M,\mathbb{Z})} \overset{\cong}{\to} \frac{\mathbb{C}^g}{\Lambda_M}.$$

The isomorphism is given by evaluating functionals against a basis $\{\omega_1, \ldots, \omega_g\} \subset \Omega^1(M)$, and Λ_M is called the *period lattice*. The Picard group

$$\mathrm{Pic}^0(M) := \frac{\mathrm{Div}^0(M)}{(\mathcal{K}(M)^*)}$$

of degree-0 divisors modulo rational equivalence is the object we want to understand. To this end, we had shown that the *Abel-Jacobi map*

$$AJ : \mathrm{Pic}^0(M) \to J(M)$$
$$D \mapsto \int_{\partial^{-1}D}$$

is a well-defined homomorphism, where $\partial^{-1}D$ is just shorthand for "some 1-chain Γ with $\partial\Gamma = D$". The important content of this is that $AJ((f)) = 0$ for any $f \in \mathcal{K}(M)^*$.

By *Abel's theorem* we shall always mean the statement that AJ is injective, that is

(30.0.1) $$AJ(D) = 0 \implies D = (f) \text{ for some } f \in \mathcal{K}(M)^*\,;$$

while the surjectivity will be known as *Jacobi inversion*: i.e.,

(30.0.2)

given any point in $J(M)$ (= any functional on $\Omega^1(M)$, up to periods), there exists a divisor D inducing that functional via $\int_{\partial^{-1}D}(\cdot)$.

These statements will be proved in Chapter 31. Our aim here is just to explain how Abel's theorem relates to Riemann-Roch and develop a couple of technical lemmas to be used in the sequel.

Before starting, let's refine one aspect of the above picture just a bit. Intersecting 1-cycles γ on M — or more precisely, intersecting *transverse* representatives[1] of homology classes $[\gamma]$ — gives a perfect pairing[2]

$$\langle\ ,\ \rangle : H_1(M,\mathbb{Z}) \times H_1(M,\mathbb{Z}) \to \mathbb{Z}.$$

There is always a *symplectic basis* of $H_1(M,\mathbb{Z})$, which means a generating subset $\{[\gamma_i]\}_{i=1}^{2g} \subset H_1(M,\mathbb{Z})$ that satisfies

$$\langle [\gamma_i], [\gamma_{g+j}] \rangle = \delta_{ij} = -\langle [\gamma_{g+j}] \cdot [\gamma_i] \rangle$$
$$\langle [\gamma_i], [\gamma_j] \rangle = 0 = \langle [\gamma_{i+g}], [\gamma_{j+g}] \rangle$$

for $1 \leq i, j \leq g$ (where δ_{ij} is the Kronecker delta). This is the situation *pictured* in §21.1.

We should also remark on what the Picard group is really doing here. For an elliptic curve E, in $\mathrm{Pic}^0(E)$ we have $[p+q] - [p] - [q] + [\mathcal{O}] \equiv 0$, where addition inside the brackets is the group law on E and outside the brackets means adding divisors. What this says is: while as divisors (i.e. in the free abelian group on points of E) $[p+q] + [\mathcal{O}] \neq [p] + [q]$, working modulo rational equivalence we *do* have $[p+q] + [\mathcal{O}] \equiv [p] + [q]$. So Pic effectively recovers the group law on E. Now, curves of higher genus have no group law on points; but by "linearizing" points and working modulo divisors of functions, we get a form of generalization of the group law in genus 1. Intriguingly, a more precise form of Jacobi inversion in the next

---

[1] Any two given homology classes have representative 1-cycles (say, α, β) which intersect transversely. At each intersection point p there is a local holomorphic coordinate $z = x + iy$, and the tangent vectors $\underline{v}_\alpha$ and $\underline{v}_\beta$ to the 1-cycles (which have well-defined directions) can be wedged to produce an element $\underline{v}_\alpha \wedge \underline{v}_\beta = \xi_p \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \in \bigwedge^2 T_p M$ ($\xi_p \neq 0$). The intersection is called positively or negatively oriented depending upon the sign of ξ_p, and the intersection number $[\alpha] \cdot [\beta]$ is the number of positive intersection points minus the number of negative ones.

[2] This means that the (bilinear) pairing is described (with respect to an integral basis of $H_1(M,\mathbb{Z})$) by an integrally invertible, i.e. unimodular, matrix.

chapter will tell us that this may "almost" be seen as a group law on *unordered g-tuples of points on M*.

30.1. From Riemann-Roch to Abel-Jacobi

Let D be a divisor on M; we have been interested in the dimensions of the vector spaces $\mathfrak{L}(D)$ and $\mathfrak{I}(D)$. In the interval

$$0 \leq \deg(D) \leq 2g-2$$

is where anything "of interest" lies: outside this range, either $\ell(D)$ or $i(D)$ is zero. At the extremes, Abel's theorem will tell us:

(i) $\ell(D)$ when $\deg(D) = 0$; and
(ii) $i(D)$ when $\deg(D) = 2g-2$.

In case (i), if there is a meromorphic function $f \in \mathcal{K}(M)^*$ with $(f) + D \geq 0$, then

$$\deg((f)+D) = \deg((f)) + \deg(D) = 0 + 0 = 0$$
$$\implies \quad (f) + D = 0 \quad \implies \quad D \overset{\text{rat}}{\equiv} 0.$$

In this event, there can be only one such f (up to scale), as

$$(f) = -D = (g) \quad \implies \quad (f/g) = 0 \quad \implies \quad f/g \text{ constant.}$$

Together with similar reasoning in case (ii), and assuming Abel, this argument proves

30.1.1. PROPOSITION. (i) *If* $\deg D = 0$, *then* $\ell(D) = 0$ *or* 1; *and*

$$AJ(D) = 0 \iff D \overset{rat}{\equiv} 0 \iff \ell(D) = 1.$$

(ii) *If* $\deg D = 2g-2$, *then* $i(D) = 0$ *or* 1; *and*

$$AJ(K-D) = 0 \iff D \overset{rat}{\equiv} K \iff \ell(D-K) = 1 \iff i(D) = 1.$$

Another point of contact with the last few chapters comes in the context of canonical and hyperelliptic curves. First, fix $q \in M$ and look at the mapping

$$\begin{aligned} u_q :\ & M \longrightarrow J(M) \\ & p \longmapsto AJ([p]-[q]) = \begin{pmatrix} \int_q^p \omega_1 \\ \vdots \\ \int_q^p \omega_g \end{pmatrix} \bmod \Lambda_M. \end{aligned}$$

Assuming Abel's theorem, we have (for genus ≥ 1)

30.1.2. PROPOSITION. (a) u_q *is injective;*

(b) *its differential yields the canonical map; and*

(c) *if M is hyperelliptic and q is a fixed point of $\jmath$, then $u_q(M)$ is symmetric with respect to the involution $\underline{u} \mapsto -\underline{u}$ of $J(M)$.*

PROOF. (a) Assuming $p_1 \neq p_2$ and $u_q(p_1) = u_2(p_2)$, we have

$$\begin{aligned} AJ([p_1]-[p_2]) = 0 & \\ & \overset{\text{Abel}}{\Longrightarrow} \quad \exists\, f \in \mathcal{K}(M)^* \text{ with } (f) = [p_1]-[p_2] \\ & \Longrightarrow \quad f : M \underset{(\cong)}{\longrightarrow} \mathbb{P}^1 \text{ has degree one,} \end{aligned}$$

contradicting $g \geq 1$.

(b) Given $\omega \in \Omega^1(M)$, we can consider $\omega(p) \in T_p^* M$. By the fundamental theorem of calculus, the differential

$$du_q(p) :\ T_p M \longrightarrow T_{u_q(p)} J(M) \cong \mathbb{C}^g$$

is given by $(\omega_1(p), \ldots, \omega_g(p))$. (That is, if $\omega_i \overset{\text{loc}}{=} f_i(z)dz$, with $z(p) = 0$, then $du_q(p)$ sends $\frac{\partial}{\partial z}|_p \mapsto (f_1(0), \ldots, f_g(0)) \in \mathbb{C}^g$.) This associates a line in $\mathbb{C}^g$ to each $p \in M$; projectivizing clearly recovers $\varphi_K :\ M \to \mathbb{P}^{g-1}$ from §28.3.

(c) Using $\jmath((x,y)) = (x,-y)$, we have

$$u_q(\jmath(p)) = \begin{pmatrix} \int_{q=\jmath(q)}^{\jmath(p)} \frac{dx}{y} \\ \vdots \\ \int_{\jmath(q)}^{\jmath(p)} \frac{x^{g-1}dx}{y} \end{pmatrix} = \begin{pmatrix} \int_q^p \jmath^* \frac{dx}{y} \\ \vdots \\ \int_q^p \jmath^* \frac{x^{g-1}dx}{y} \end{pmatrix} = \begin{pmatrix} -\int_q^p \frac{dx}{y} \\ \vdots \\ -\int_q^p \frac{x^{g-1}dx}{y} \end{pmatrix}$$

$= -u_q(p)$. $\square$

In fact, in the hyperelliptic case it is clear from (c) that the fixed points of $\jmath$ map to 2-torsion points of $J(M)$.

30.2. Differential forms of the third kind

There is a classical (and passé) terminology for meromorphic differential forms on a Riemann surface: "first kind" refers to holomorphic forms; "second kind" to meromorphic forms with trivial residues (and hence no *simple* poles); and "third kind" to everything else. In this section we'll pursue a method for constructing functions with a given divisor (if possible). The title refers to the essential use we shall make of meromorphic forms with prescribed (nonzero) residues.

Given $p, q \in M$

$$i(-[p]-[q]) \;=\; g-(-2)-1+\underbrace{\ell(-[p]-[q])}_{0} \;=\; g+1\ (>g),$$

so there exists $\omega \in \mathfrak{I}(-[p]-[q])\backslash\Omega^1(M)$. By the residue theorem,

$$0 \;=\; \underbrace{Res_p(\omega)+Res_q(\omega)}_{\text{both nonzero since poles simple}}$$

and we can normalize ω so that

$$Res_p\omega = \frac{1}{2\pi\sqrt{-1}}\,, \quad Res_q\omega = \frac{-1}{2\pi\sqrt{-1}}.$$

For any meromorphic form η, we write $(\eta) = (\eta)_0 - (\eta)_\infty$ where $(\eta)_0, (\eta)_\infty \geq 0$ are the zero- and polar-divisors.

30.2.1. LEMMA. *Given $D \in \mathrm{Div}^0(M)$, there exists*[3]

$$\eta_D \in \mathfrak{I}\left(-\textstyle\sum_{p\in|D|}[p]\right)$$

such that

$$(\eta_D)_\infty = \sum_{p\in|D|}[p] \quad \textit{and} \quad Res_p\eta_D = \frac{\mathrm{ord}_p(D)}{2\pi\sqrt{-1}}.$$

[3] As before, $|D|$ denotes the *support* of the divisor D, i.e. the finite subset of M comprising points that appear in D with nonzero multiplicity. Similarly, the support $|\gamma|$ of a 1-cycle γ means that we just consider it as a subset of M.

PROOF. See Exercise (3). □

Next let $D = \sum n_j[P_j]$ and η_D be as in Lemma 30.2.1 (in particular, $\sum n_j = 0$), and consider a collection $\{\gamma_i\}_{i=1}^{2g}$ of closed paths with support $|\gamma_i| \subset M \backslash |D|$, such that their classes $\{[\gamma_i]\}_{i=1}^{2g} \subset H_1(M, \mathbb{Z})$ yield a basis.

30.2.2. LEMMA. *If*

$$\int_{\gamma_i} \eta_D \in \mathbb{Z} \quad (\forall i), \tag{30.2.3}$$

then (fixing $Q \in M$)

$$f(P) := \exp\left(2\pi\sqrt{-1}\int_Q^P \eta_D\right)$$

yields a well-defined function $f \in \mathcal{K}(M)^*$ with $(f) = D$.

PROOF. We first check independence of path. Let C_j denote circular paths around the P_j. Given two paths $\overrightarrow{Q.P}$ and $\overrightarrow{Q.P}'$,

$$\overrightarrow{Q.P} - \overrightarrow{Q.P}' \;=\; \partial\Delta + \sum m_j C_j + \sum \ell_i \gamma_i$$

where Δ is a (real-2-dimensional) closed region in $M \backslash |D|$. Now

$$\int_{\partial\Delta} \eta_D = \int_\Delta d\eta_D = \int_\Delta 0 = 0,$$

$$\sum m_j \int_{C_j} \eta_D = \sum m_j n_j \in \mathbb{Z}$$

since $Res_{P_k}\eta_D = \frac{n_k}{2\pi\sqrt{-1}}$, and

$$\sum \ell_i \int_{\gamma_i} \eta_D \in \mathbb{Z}$$

by assumption (30.2.3). So for some $\mu \in \mathbb{Z}$

$$\frac{\exp\left(2\pi\sqrt{-1}\int_{\overrightarrow{Q.P}} \eta_D\right)}{\exp\left(2\pi\sqrt{-1}\int_{\overrightarrow{Q.P}'} \eta_D\right)} = \exp\left(2\pi\sqrt{-1}\mu\right) = 1,$$

and f is well-defined (and holomorphic) on $M \backslash |D|$.

For the divisor, let z be a holomorphic coordinate defined in a neighborhood of P_j (with $z(P_j) = 0$), and write

$$\eta \overset{\text{loc}}{=} \frac{n_j}{2\pi\sqrt{-1}}\frac{dz}{z} + h(z)dz$$

with h holomorphic. Without loss of generality, we can assume that Q lies in the neighborhood, with $z(Q) =: z_0$ (fixed) and $z(P) =: z$ (variable). Locally

$$\begin{aligned} f(z) &= \exp\left(2\pi\sqrt{-1}\int_Q^P \eta_D\right) \\ &= \exp\left(2\pi\sqrt{-1}\int_{z_0}^z h(w)dw\right) \cdot \exp\left(n_j \int_{z_0}^z \frac{dw}{w}\right) \\ &= H(z) \cdot \frac{\exp\left(n_j \log z\right)}{\exp\left(n_j \log z_0\right)} \end{aligned}$$

where H is holomorphic and nonvanishing in our neighborhood (being the exponential of something holomorphic). Writing $H_0(z) = z_0^{-n_j} H(z)$, the above

$$= H_0(z) \cdot z^{n_j}.$$

This makes it clear that f is meromorphic at P_j with

$$\nu_{P_j}(f) = n_j.$$

Doing this for each j, we conclude that

$$(f) = \sum n_j [P_j] = D. \qquad \square$$

In the next chapter we will take $\gamma_1, \ldots, \gamma_{2g}$ to yield a symplectic basis for H_1. It turns out that the period vectors $\pi_1, \ldots, \pi_g$ associated to $\gamma_1, \ldots, \gamma_g$ are actually linearly independent over $\mathbb{C}$,[4] and so the $g \times g$ matrix they form is invertible. Applying the inverse matrix to $\omega_1, \ldots, \omega_g$, we may replace them by $\{\omega_j\}$ satisfying

$$\int_{\gamma_i} \omega_j = \delta_{ij}.$$

[4] to be proved in §31.1. Since the vectors are non-real, this doesn't follow from independence over $\mathbb{R}$ (which we already have from Prop. 25.2.5).

Given D and η_D as in Lemma 30.2.1, then, we can modify η_D to

$$\widetilde{\eta_D} := \eta_D - \sum_{i=1}^{g} \left(\int_{\gamma_i} \eta_D \right) \omega_i$$

so that

$$\int_{\gamma_i} \widetilde{\eta_D} = 0$$

for $i = 1, \ldots, g$. We will prove that

$$\text{(30.2.4)} \qquad AJ(D) = 0 \implies \begin{array}{c} \text{there exists a further modification} \\ \widehat{\eta_D} := \widetilde{\eta_D} + \sum_{j=1}^{g} \mu_j \omega_j \\ \text{with } \int_{\gamma_i} \widehat{\eta_D} \in \mathbb{Z} \ \ (i = 1, \ldots, 2g), \end{array}$$

so as to affirm condition (30.2.3) (for $\widehat{\eta_D}$). To attack (30.2.4), we need the *Riemann bilinear relations*, our next topic.

The first two problems below are only loosely related to the material of his chapter. The second one is rather open ended!

Exercises

(1) In this problem, you will prove a version of Abel's theorem for a "singular" cubic (not its normalization). Think of the cubic C as $\mathbb{P}^1$ with 0 identified to ∞ and coordinate z. We consider $\Omega^1(C)$ to be spanned by $\frac{dz}{z}$ (even though it isn't holomorphic) and $H_1(C, \mathbb{Z})$ by the unit circle S^1. Divisors must avoid the singularity, and meromorphic functions f must have $f(0) = f(\infty) \neq 0, \infty$. (a) What is $J(C)$? (b) Compute $AJ(D)$ for $D = \sum n_i [z_i]$, $\sum n_i = 0$. (c) Show $0 = AJ((f)) \in J(C)$. (d) Formulate and prove the injectivity statement (Abel's theorem). [Hint: the proof will use what you did in (c), even though it's a "converse", and so needn't be long.]

(2) Let M be a Riemann surface and $\Sigma \subset M$ a (nonempty) finite set of points.
(a) Define divisors on the relative variety (M, Σ) to be formal sums $\sum n_i [p_i]$ where no p_i lies in Σ; two of these are rationally equivalent if their difference is the divisor of $f \in \mathcal{K}(M)^*$ which

is 1 on all points of Σ. Construct an AJ map and Jacobian for (M, Σ). [Hint: the case $M = \mathbb{P}^1$, $\Sigma = \{0, \infty\}$ should recover the results of Exercise (1).]

(b) Next consider the complement $M \backslash \Sigma$. We define divisors by $\mathrm{Div}(M)/\mathrm{Div}(\Sigma)$, and rational equivalence by taking divisors of meromorphic functions on M (and ignoring any poles/zeroes on Σ, since that information is quotiented out). Construct an AJ map and Jacobian for $M \backslash \Sigma$. [Hint: note that there is no such thing as degree of a divisor, since the points in Σ effectively have arbitrary coefficients. Or rather, using these points, you can make the degree of any divisor zero! This should have some bearing on your choice of path.]

(3) Prove Lemma 30.2.1. [Hint: write $D := \sum_{k=1}^{r} [p_k] - \sum_{k=1}^{r} [q_k]$, and pick $\omega_k \in \mathfrak{I}(-[p_k] - [q_k])$ as at the beginning of §30.2.]