CHAPTER 31

Abel's Theorem, part II

As mentioned at the beginning of the previous chapter, on any Riemann surface M, we get a perfect pairing on homology

$$\langle\ ,\ \rangle : H_1(M,\mathbb{Z}) \times H_1(M,\mathbb{Z}) \to \mathbb{Z} \tag{31.0.1}$$

by intersecting 1-cycles. With respect to a *symplectic basis* $\{\gamma_j\}_{j=1}^{2g}$ as described there, this pairing has $(2g \times 2g)$ matrix

$$Q = \begin{pmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{pmatrix}.$$

We can use (31.0.1) to produce an isomorphism of dual spaces

(31.0.2)

$$H_1(M,\mathbb{C}) = H_1(M,\mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\cong} \mathrm{Hom}(H_1(M,\mathbb{Z}),\mathbb{C}) = H^1(M,\mathbb{C})$$

which is a special case of *Poincaré duality*.

Recalling the isomorphisms

$$\Omega^1(M) \oplus \overline{\Omega^1(M)} \xrightarrow{\cong} H^1_{dR}(M,\mathbb{C}) \xrightarrow{\cong} H^1(M,\mathbb{C}),$$

there is also a pairing (the "cup-product")

$$H^1(M,\mathbb{C}) \times H^1(M,\mathbb{C}) \to \mathbb{C}$$

induced on the level of 1-forms by

$$(\omega,\eta) \longmapsto \int_M \omega \wedge \eta.$$

Notice that since two holomorphic forms wedge to zero, this pairing restricts to zero on $\Omega^1(M) \times \Omega^1(M)$ (and $\overline{\Omega^1(M)} \times \overline{\Omega^1(M)}$).

Yet another pairing (the "cap-product")

$$H_1(M,\mathbb{Z}) \times H^1(M,\mathbb{C}) \to \mathbb{C}$$

is induced by

$$(\gamma, \omega) \mapsto \int_\gamma \omega.$$

The restriction of this pairing to $H_1(M,\mathbb{Z}) \times \Omega^1(M)$ is captured by the period matrix of Chapter 21. An important fact is that, under (31.0.2), both of these integration-induced products are nothing but complex-linear extensions of (31.0.1).

Assuming this compatibility, we can quickly derive the *Riemann bilinear relations* as follows. If for any closed 1-form $\varphi \in \Omega^1(M) \oplus \overline{\Omega^1(M)}$, we write

$$\pi_j(\varphi) := \int_{\gamma_j} \varphi,$$

then (31.0.2) identifies

$$[\varphi] \;=\; \sum_{j=1}^{g} \left(\pi_j(\varphi)[\gamma_{j+g}] - \pi_{j+g}(\varphi)[\gamma_j]\right) \tag{31.0.3}$$

in $H^1(M,\mathbb{C})$, i.e. as functionals on homology. One has for $\omega, \varphi \in \Omega^1(M)$

$$0 = \int_M \omega \wedge \varphi = -\sum_{j=1}^{g} \left(\pi_j(\varphi)\pi_{j+g}(\omega) - \pi_{j+g}(\varphi)\pi_j(\omega)\right) \tag{31.0.4}$$

by writing $\int_M \omega \wedge \varphi = \langle [\omega], [\varphi] \rangle$ and expanding both classes as in (31.0.3). Similar reasoning together with the local computation

$$idz \wedge d\bar{z} = i(dx + idy) \wedge (dx - idy) = i(-2idx \wedge dy) = 2dx \wedge dy,$$

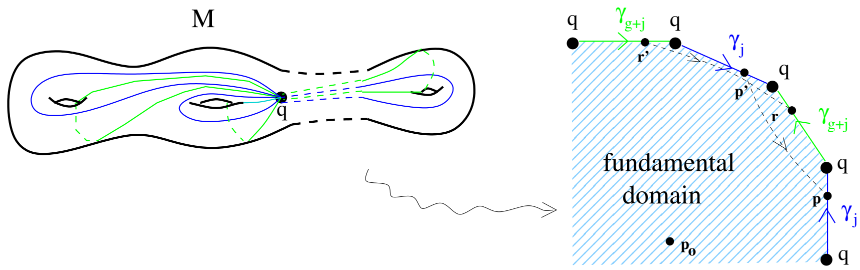
leads to

$$0 < i \int_M \omega \wedge \bar{\omega} = -i \sum_{i=1}^{g} \left(\overline{\pi_j(\omega)}\pi_{j+g}(\omega) - \overline{\pi_{j+g}(\omega)}\pi_j(\omega)\right). \tag{31.0.5}$$

This is all meant as motivation, though it can be made completely rigorous. We'll start the first section with a more concrete, classical proof of (31.0.4)-(31.0.5), without the compatibility assumptions on the three bilinear pairings.

31.1. Derivation of the Riemann Bilinear Relations

We start by cutting M open to get the "fundamental domain", a simply-connected closed region $\mathfrak{F}$



with boundary $\partial\mathfrak{F}$. (Only a piece of it is shown in the picture.) Let p_0 in the interior of $\mathfrak{F}$ be fixed. Given $\omega \in \Omega^1(M)$,

$$u(p) := \int_{p_0}^{p} \omega$$

then yields a well-defined (single-valued) holomorphic[1] function on $\mathfrak{F}$. If we take a second holomorphic form $\varphi \in \Omega^1(M)$, then

$$d(u\varphi) = \omega \wedge \varphi = 0.$$

That is, $u\varphi$ is a closed holomorphic form on $\mathfrak{F}$ with the consequence that

$$0 \;=\; \int_{\mathfrak{F}} d(u\varphi) \;=\; \int_{\partial\mathfrak{F}} u\varphi$$

by Stokes's theorem. Now, the picture above tells us that $\partial\mathfrak{F}$ is the composition of paths

$$\gamma_{2g}^{-1}\gamma_g^{-1}\gamma_{2g}\gamma_g \cdot \cdots \cdot \gamma_{g+2}^{-1}\gamma_2^{-1}\gamma_{g+2}\gamma_2\gamma_{g+1}^{-1}\gamma_1^{-1}\gamma_{g+1}\gamma_1,$$

written right to left (with inverse meaning the reverse direction). So the last integral becomes

$$= \sum_{j=1}^{g}\Big\{ \int_{\gamma_j} \underbrace{(u(p) - u(p'))}_{\int_{p'}^{q}\omega - \int_{\gamma_{g+j}}\omega + \int_q^p \omega} \varphi \;+\; \int_{\gamma_{j+g}} \underbrace{(u(r) - u(r'))}_{\int_{r'}^{q}\omega + \int_{\gamma_j}\omega + \int_q^r \omega} \varphi \Big\}$$

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[1] To be holomorphic on a closed set means that the function extends to a holomorphic function on a slightly larger open set (which, in this case, would live on the universal cover of M).

and, noting that $\int_{p'}^{q}\omega = \int_p^q \omega = -\int_q^p \omega$, this

$$= \sum_{j=1}^{g}\left(-\int_{\gamma_{g+j}}\omega\int_{\gamma_j}\varphi + \int_{\gamma_{g+j}}\varphi\int_{\gamma_j}\omega\right).$$

This calculation of $\int_{\partial\mathfrak{F}} u\varphi$ is evidently also valid with φ replaced by a more general (antiholomorphic, meromorphic) 1-form. In particular, replacing it by $i\bar{\omega}$ yields

$$0 < i\int_{\mathfrak{F}}\underbrace{\omega\wedge\bar{\omega}}_{d(u\bar{\omega})} = \int_{\partial\mathfrak{F}} u(i\bar{\omega}) = i\sum_{j=1}^{g}\left(-\int_{\gamma_{j+g}}\omega\int_{\gamma_j}\bar{\omega} + \int_{\gamma_{g+j}}\bar{\omega}\int_{\gamma_j}\omega\right).$$

So we have recovered (31.0.4)-(31.0.5).

To reformulate this in matrix terms for any symplectic basis $\{\gamma_j\}_{j=1}^{2g}$ of $H_1(M,\mathbb{Z})$ and any basis $\{\omega_i\}_{i=1}^{g}$ of $\Omega^1(M)$, notice that the $(k,\ell)^{\text{th}}$ entry of[2]

$$\Pi\cdot Q\cdot {}^t\Pi = \begin{pmatrix} \uparrow & & \uparrow \\ \pi_1 & \cdots & \pi_{2g} \\ \downarrow & & \downarrow \end{pmatrix}\begin{pmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{pmatrix}\begin{pmatrix} \leftarrow & \pi_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \pi_{2g} & \rightarrow \end{pmatrix}$$

$$= \begin{pmatrix} \uparrow & & \uparrow & \uparrow & & \uparrow \\ -\pi_{g+1} & \cdots & -\pi_{2g} & \pi_1 & \cdots & \pi_g \\ \downarrow & & \downarrow & \downarrow & & \downarrow \end{pmatrix}\begin{pmatrix} \leftarrow & \pi_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \pi_{2g} & \rightarrow \end{pmatrix}$$

is

$$\sum_{j=1}^{g}\left(\pi_j(\omega_k)\pi_{g+j}(\omega_\ell) - \pi_j(\omega_\ell)\pi_{g+j}(\omega_k)\right),$$

which is zero by (31.0.4); so

$$\Pi\cdot Q\cdot {}^t\Pi = 0. \tag{31.1.1}$$

Similarly,

$$\sqrt{-1}\Pi\cdot Q\cdot {}^t\overline{\Pi} > 0 \tag{31.1.2}$$

in the sense that ${}^t\underline{x}(\sqrt{-1}\Pi\cdot Q\cdot {}^t\overline{\Pi})\bar{\underline{x}} \in \mathbb{R}_{>0}$ for any $\underline{x}\in\mathbb{C}^g$. In particular, the diagonal entries of (31.1.2) are positive real.

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[2] Recall from Chapter 21 that π_j is the complex g-vector with i^{th} entry $\pi_j(\omega_i)$

31.1.3. REMARK. Consider any two symplectic integral bases $\Gamma = \{\gamma_j\}$ and $\Gamma = \{\gamma_j'\}$ (thought of as row-vectors), so that

$$\Gamma' = \Gamma A$$

for some $A \in \mathrm{SL}_{2g}(\mathbb{Z})$. Applying the basis $\{\omega_i\}$ (viewed as a column-vector of 1-forms) on the left yields

$$\Pi' = \Pi A.$$

Furthermore, since both bases are symplectic we have $Q = {}^t\Gamma \cdot \Gamma$ and

$$Q = {}^t\Gamma' \cdot \Gamma' = {}^tA^t\Gamma\Gamma A = {}^tAQA;$$

that is, A belongs to the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$. It is for this reason that (31.1.1)-(31.1.2) are compatible with change of symplectic basis: e.g., assuming (31.1.1), we have

$$\Pi' Q^t\Pi' = \Pi A Q^t A^t \Pi = \Pi Q^t \Pi = 0.$$

Now thinking of the $g \times 2g$ period matrix as two $g \times g$ blocks, viz.

$$\Pi = \begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix}, \tag{31.1.4}$$

we have

$$\Pi Q^t \Pi = \begin{pmatrix} \mathcal{A} & \mathcal{B} \end{pmatrix} \begin{pmatrix} & \mathbb{I}_g \\ -\mathbb{I}_g & \end{pmatrix} \begin{pmatrix} {}^t\mathcal{A} \\ {}^t\mathcal{B} \end{pmatrix} = \mathcal{A} \cdot {}^t\mathcal{B} - \mathcal{B} \cdot {}^t\mathcal{A}$$

and

$$\Pi Q^t \overline{\Pi} = \mathcal{A} \cdot {}^t\overline{\mathcal{B}} - \mathcal{B} \cdot {}^t\overline{\mathcal{A}}.$$

In these terms, (31.1.1) reads

$$\mathcal{A} \cdot {}^t\mathcal{B} = \mathcal{B} \cdot {}^t\mathcal{A} \tag{31.1.5}$$

while (31.1.2) becomes

$$\sqrt{-1}\,{}^t\underline{v}(\mathcal{A}^t\overline{\mathcal{B}} - \mathcal{B}^t\overline{\mathcal{A}})\underline{\bar{v}} > 0 \quad (\forall \underline{v} \in \mathbb{C}^g). \tag{31.1.6}$$

If ${}^t\mathcal{A}$ has nonzero kernel, then there exists $\underline{v} \in \mathbb{C}^g$ satisfying ${}^t\mathcal{A}\underline{v} = 0$, hence ${}^t\underline{v}\mathcal{A} = 0$ and ${}^t\overline{\mathcal{A}}\underline{\bar{v}} = 0$, contradicting (31.1.6). It follows that

$\mathcal{A}$ is invertible, and so we have proved the statement on $\mathbb{C}$-linear independence asserted at the end of §30.2.

Applying $\mathcal{A}^{-1}$ to the left of Π amounts to a change of the basis $\{\omega_i\}$ for $\Omega^1(M)$, viz.[3]

$$\mathcal{A}^{-1}\Pi = \mathcal{A}^{-1}\begin{pmatrix}\omega_1\\ \vdots \\ \omega_g\end{pmatrix}\begin{pmatrix}\gamma_1 & \cdots & \gamma_{2g}\end{pmatrix} = \begin{pmatrix}\omega_1'\\ \vdots \\ \omega_g'\end{pmatrix}\begin{pmatrix}\gamma_1 & \cdots & \gamma_{2g}\end{pmatrix}.$$

If we apply it to (31.1.4), then we get

$$\Pi' := \mathcal{A}^{-1}\Pi = \begin{pmatrix} \mathbb{I}_g & \mathcal{A}^{-1}\mathcal{B} \end{pmatrix}.$$

We can therefore always assume that $\{\omega_i\}$ is chosen so that

$$\Pi = \begin{pmatrix} \mathbb{I}_g & Z \end{pmatrix},$$

again as claimed in §30.2. The bilinear relations (31.1.5)-(31.1.6) simplify to

$$\left\{ \begin{array}{c} Z = {}^tZ \\ \sqrt{-1}(\overline{Z} - Z) > 0 \end{array} \right. , \tag{31.1.7}$$

which in particular tell us that the imaginary part $Im(Z)$ is a positive-definite, real symmetric matrix.

31.2. Proof of Abel's Theorem

With the holomorphic basis as normalized above, we can now quickly establish (30.2.4) and hence (30.0.1). Write $D = \sum n_i[P_i]$ (with $\sum n_i = 0$) and let $\varphi := \widetilde{\eta_D}$ be as in §30.2, so that

$$Res_{P_i}(\varphi) = \frac{n_i}{2\pi\sqrt{-1}} \quad (\forall i) \tag{31.2.1}$$

and

$$\int_{\gamma_j} \varphi = 0 \quad (j = 1, \ldots, g).$$

[3]Here the "product" of ω_i and γ_j is just the integral $\int_{\gamma_j} \omega_i$.

For each $k = 1, \ldots, g$ set

$$u_k(p) := \int_{p_0}^{p} \omega_k$$

on $\mathfrak{F}$, and let Γ be a 1-chain (sum of paths) on $\mathfrak{F}$ with $\partial\Gamma = D$. Then noting $D = \sum_i n_i([P_i] - [p_0])$, we have

$$\int_\Gamma \omega_k = \sum_i n_i u_k(P_i) = 2\pi\sqrt{-1} \sum_{p \in |D|} Res_p(u_k \varphi)$$

which by the Residue Theorem

$$= \int_{\partial\mathfrak{F}} u_k \varphi \overset{\S 31.1}{=} \sum_j (\underbrace{\pi_j(\omega_k)}_{\delta_{jk}} \pi_{g+j}(\varphi) - \pi_{g+j}(\omega_k) \underbrace{\pi_j(\varphi)}_{0}) = \pi_{g+k}(\varphi).$$

If $AJ(D) = 0$ then there are integers m_j ($j = 1, \ldots, 2g$) such that for every k

$$\int_\Gamma \omega_k = \sum_{j=1}^{2g} m_j \int_{\gamma_j} \omega_k.$$

Using $\int_{\gamma_j} \omega_k = \delta_{jk}$ and $Z = {}^tZ$ (from (31.1.7)), this

$$= m_k + \sum_{j=1}^{g} m_{j+g} \pi_{j+g}(\omega_k) = m_k + \sum_{j=1}^{g} m_{j+g} \pi_{k+g}(\omega_j).$$

Now

$$\hat{\varphi} := \varphi - \sum_{j=1}^{g} m_{j+g} \omega_j$$

is still an element of $\mathfrak{I}(-\sum_{p \in |D|}[p])$ satisfying (31.2.1). Moreover, for $k \in \{1, \ldots, g\}$

$$\pi_{k+g}(\hat{\varphi}) = \pi_{k+g}(\varphi) - \sum_{j=1}^{g} m_{j+g} \pi_{k+g}(\omega_j)$$

$$= \int_\Gamma \omega_k - \sum_{j=1}^{g} m_{j+g} \pi_{k+g}(\omega_j) = m_k \in \mathbb{Z}$$

and

$$\pi_k(\hat{\varphi}) \;=\; \underbrace{\pi_k(\varphi)}_{0} - \sum_{j=1}^{g} m_{j+g} \underbrace{\pi_k(\omega_j)}_{\delta_{kj}} \;=\; -m_{k+g} \;\in \mathbb{Z}.$$

By Lemma 30.2.2, $\exp\left(2\pi\sqrt{-1}\int\hat{\varphi}\right)$ now gives a meromorphic function with $(f) = D$.

31.3. Proof of Jacobi Inversion

To show that AJ is surjective, we will study the image of a certain class of (degree zero) divisor on M, namely those of the form

$$[p_1] + \cdots + [p_d] - d[q]$$

given some fixed point $q \in M$ and natural number d. Such divisors are obviously in 1-to-1 correspondence with unordered d-tuples of points on M, in other words with elements of the d^{th} symmetric power

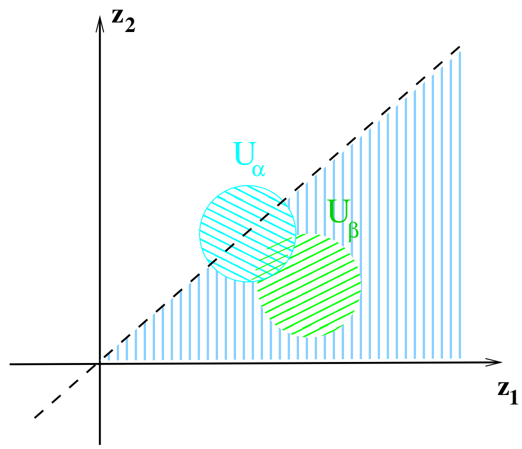
$$Sym^d M \;:=\; \frac{\overbrace{M \times \cdots \times M}^{d \text{ copies}}}{(p_1, \ldots, p_d) \underset{\forall\, \sigma \in \mathfrak{S}_d}{\sim} \left(p_{\sigma(1)}, \ldots, p_{\sigma(d)}\right)}.$$

(These elements are written either $p_1 + \cdots + p_d$ or $\{p_1, \ldots, p_d\}$.) In order to be able to use complex analytic techniques we need to put the structure of a d-dimensional complex manifold on this.[4]

To get a feel for how this works, take $d = 2$ and consider $\mathbb{C}$ instead of a (compact) Riemann surface. The symmetric square $Sym^2\mathbb{C}$ is the quotient of $\mathbb{C} \times \mathbb{C}$ by the involution $(z_1, z_2) \mapsto (z_2, z_1)$. What causes difficulty is the locus consisting of its fixed points, i.e. the

[4] Had we started with M *itself* of dimension > 1, its symmetric powers would be *singular* complex analytic spaces, hence *not* manifolds. So what happens next is special for $\dim(M) = 1$.

diagonal line. Take two small open sets in $Sym^2 M$, one which intersects the diagonal and one which does not:



Clearly (z_1, z_2) give local holomorphic coordinates on U_β. On U_α, they are not well-defined, but their elementary symmetric polynomials $\sigma_1(z_1, z_2) = z_1 + z_2$ and $\sigma_2(z_1, z_2) = z_1 z_2$ are. Moreover, these functions generate all polynomials in z_1, z_2 which are invariant under the involution and hence well-defined on $U_\alpha \subset Sym^2\mathbb{C}$. Taking $(w_1, w_2) := (z_1 + z_2, z_1 z_2)$ as the holomorphic coordinates there,[5] the transition function $\Phi_{\alpha\beta}$ is then just (σ_1, σ_2). To see that this is invertible on $U_{\alpha\beta}$, notice that in U_α the diagonal is defined by $w_1^2 = 4w_2$ (since $z_1 = z_2 \iff (z_1 + z_2)^2 = 4z_1 z_2$). Since $U_{\alpha\beta}$ avoids this locus (and is simply connected), $\sqrt{w_1^2 - 4w_2}$ is well defined there and we can define $\Phi_{\beta\alpha}$ by

$$\begin{cases} z_1 = \frac{w_1 + \sqrt{w_1^2 - 4w_2}}{2} \\ z_2 = \frac{w_2 - \sqrt{w_1^2 - 4w_2}}{2} \end{cases} .$$

More generally, in a neighborhood of

$$\{\underbrace{q_1, \ldots, q_1}_{k_1 \text{ times}}; \ \ldots \ ; \underbrace{q_\ell, \ldots, q_\ell}_{k_\ell \text{ times}}\} \in Sym^d M$$

[5]We could in fact take these as global coordinates, but this situation won't generalize to M.

(where $\sum_{j=1}^{\ell} k_j = d$), the local coordinate system is given in terms of holomorphic coordinates z_j on M near each q_j, by

$$\{\underbrace{p_1, \ldots, p_{k_1}}_{\text{all near } q_1}; \; \ldots \; ; \underbrace{p_{d-k_\ell+1}, \ldots, p_d}_{\text{all near } q_\ell}\} \longmapsto$$

$$\left(\sigma_1\left(z_1(p_1), \ldots, z_1(p_{k_1})\right), \ldots, \sigma_{k_1}\left(z_1(p_1), \ldots, z_1(p_{k_1})\right) \; ; \; \ldots \; ; \right.$$
$$\left. \sigma_1\left(z_\ell(p_{d-k_\ell+1}), \ldots, z_\ell(p_d)\right), \ldots, \sigma_{k_\ell}\left(z_\ell(p_{d-k_\ell+1}), \ldots, z_\ell(p_d)\right)\right).$$

Inelegant, but it gets the job done.

Now let D be any divisor of degree d on M, and consider the mapping

$$\alpha_D : \mathbb{P}(\mathfrak{L}(D)) \to Sym^d M$$

which sends (for $f \in \mathfrak{L}(D)$)

$$[f] \mapsto (f) + D.$$

Here $(f) + D \geq 0$ by definition, and $\deg((f) + D) = \deg D = d$; so $(f) + D$ is indeed of the form $[p_1] + \cdots + [p_d]$. The map sends the projective equivalence class $[f]$, i.e. "f up to a constant multiple", to $\{p_1, \ldots, p_d\}$.

31.3.1. LEMMA. α_D *is* (a) *injective and* (b) *holomorphic.*

31.3.2. DEFINITION. The *linear system*[6] $|D|$ consists of all effective divisors on M rationally equivalent to D. The Lemma evidently realizes $|D| = \text{image}(\alpha_D)$ as a subvariety of $Sym^d M$ isomorphic to $\mathbb{P}^{\ell(D)-1}$.

PROOF OF LEMMA. (a) $(f) + D = (g) + D \implies (f) = (g) \implies (f/g) = 0 \implies f/g$ constant $\implies [f] = [g]$.

(b) To show α_D holomorphic in a neighborhood of $[f_0]$, augment f_0 to a basis $\{f_0, f_1, \ldots, f_{\ell(D)}\} \subset \mathfrak{L}(D)$ and write $f_{\underline{\mu}} := f_0 + \sum_{j=1}^{\ell(D)} \mu_j f_j$ so that $\{\mu_j\}_{j=1}^{\ell(D)}$ are the local holomorphic coordinates (on some small $U \subset \mathfrak{L}(D)$). Let $p \in |D| \cup |(f_0)|$, with open neighborhood $\mathcal{N}_p \subset M$ and local coordinate z ($\text{ord}_p(z) = 1$). Set $k := \text{ord}_p(f_0) + \text{ord}_p(D)$,

[6]The notation $|D|$ is unfortunately standard for both the linear system and the support of D, two completely different concepts!

and $\mathcal{W}_{f_0,p} := Sym^k \mathcal{N}_p$ with coordinates $\sigma_1(z_1, \ldots, z_k), \ldots, \sigma_k(z_1, \ldots, z_k)$. We must show that the composition

$$\begin{array}{ccccccc} U & \longrightarrow & \mathcal{O}(\mathcal{N}_p) & \longrightarrow & \mathcal{W}_{f_0,p} & \hookrightarrow & \mathbb{C}^k \\ \underline{\mu} & \mapsto & f_{\underline{\mu}} z^{\mathrm{ord}_p D} & \mapsto & \begin{array}{c} \left(f_{\underline{\mu}} z^{\mathrm{ord}_p D}\right)\Big|_{\mathcal{N}_p} \\ \| \\ p_1(\underline{\mu}) + \cdots + p_k(\underline{\mu}) \end{array} & \mapsto & \begin{pmatrix} \sigma_1\left(z(p_1(\underline{\mu})), \ldots, z(p_k(\underline{\mu}))\right) \\ \vdots \\ \sigma_k\left(z(p_1(\underline{\mu})), \ldots, z(p_k(\underline{\mu}))\right) \end{pmatrix} \end{array}$$

is holomorphic, which in turn boils down to the statement that each σ_ℓ is holomorphic in each μ_j. For $k = 1$, this is the holomorphic implicit function theorem; for $k > 1$, it is this together with Rouché and the Riemann extension theorem in a manner familiar from previous chapters. □

31.3.3. DEFINITION. An effective degree d divisor D (viewed as an element of $Sym^d M$) is called *general* $\iff D = [p_1] + \cdots + [p_d]$ with the $\{p_j\}$ distinct points of M.

Now look at the "Abel-Jacobi" mapping

$$\begin{aligned} u^d \colon Sym^d M &\longrightarrow J(M) \\ [p_1] + \cdots + [p_d] &\longmapsto AJ\left(\sum_{j=1}^{d} [p_j] - d[q]\right), \end{aligned}$$

where $q \in M$ is fixed. This is shown to be holomorphic by using the fundamental theorem of calculus at general D, then applying the Osgood and Riemann extension theorems. (Boundedness is clear by taking a local lifting of the image of u^d to $\mathbb{C}^g$.)

The next result does not require D to be general.

31.3.4. LEMMA. *The fiber of u^d over $u^d(D)$ is $|D|$ $(\cong \mathbb{P}^{\ell(D)-1})$.*

PROOF. (For simplicity write u for u^d.)

$\underline{u^{-1}(u(D)) \subset |D|}$: $u(E) = u(D) \implies AJ(E - D) = 0 \overset{\text{Abel}}{\implies} E - D$ is the divisor of some $f \in \mathcal{K}(M)^* \implies (f) + D = E \geq 0$ (since $E \in Sym^d M$) $\implies f \in \mathfrak{L}(D) \implies E = \alpha_D(f) \in \text{image}(\alpha_D) = |D|$.

$\underline{u^{-1}(u(D)) \supset |D|}$: Given $E \in |D|$, there exists $f \in \mathfrak{L}(D)$ such that $E = (f) + D \implies E - D = (f) \overset{\text{rat}}{\equiv} 0 \implies 0 = AJ(E-D) \implies u(E) = u(D) \implies E \in u^{-1}(u(D))$. $\square$

If $D = [p_1] + \cdots + [p_d]$ is general, then writing z_j for local coordinates about each p_j,

$$(du^d)_D : \; T_D\left(Sym^d M\right) \longrightarrow T_{u(D)}(J(M))$$

is computed by the matrix

$$\left.\begin{pmatrix} \frac{\partial}{\partial z_1}\sum_{i=1}^d \int_q^{z_i}\omega_1 & \cdots & \frac{\partial}{\partial z_1}\sum_{i=1}^d \int_q^{z_i}\omega_g \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_d}\sum_{i=1}^d \int_q^{z_i}\omega_1 & \cdots & \frac{\partial}{\partial z_d}\sum_{i=1}^d \int_q^{z_i}\omega_g \end{pmatrix}\right|_{\{p_1,\ldots,p_d\}}.$$

If we write locally (about each p_j) $\omega_i \overset{\text{loc}}{=} f_i(z_j)dz_j$, this

$$= \begin{pmatrix} f_1(p_1) & \cdots & f_g(p_1) \\ \vdots & \ddots & \vdots \\ f_1(p_d) & \cdots & f_g(p_d) \end{pmatrix} = \begin{pmatrix} \leftarrow & \widetilde{\varphi_K(p_1)} & \rightarrow \\ & \vdots & \\ \leftarrow & \widetilde{\varphi_K(p_d)} & \rightarrow \end{pmatrix},$$

where φ_K is the canonical map and $\widetilde{\varphi_K(p_j)} \in \mathbb{C}^g$ is a "lift" of $\varphi_K(p_j) \in \mathbb{P}^{g-1}$. (For $d = 1$ this is just Proposition 30.1.2(b).) From this we see that

$$\text{rank}\left((du^d)_D\right) = \dim\left(\text{span}(\varphi_K(p_1), \ldots, \varphi_K(p_d))\right) + 1,$$

where "span" means the projective linear span in $\mathbb{P}^{g-1}$. Taking $d = g$, we now have the following claim:

31.3.5. LEMMA. rank $((du^g)_D) = g$ *for a generic*[7] *choice of* $D = [p_1] + \cdots + [p_g] \in Sym^g M$, *i.e. for* D *in some Zariski open subset of* $Sym^d M$.

[7]"General" may not be quite enough — D may have to avoid a larger number of subvarieties of $Sym^g M$ then just the ones where two or more p_j's coincide.

PROOF. Pick $p_1, \ldots, p_g$ distinct with span $\left(\varphi_K(p_1), \ldots, \varphi_K(p_g)\right) =$ all of $\mathbb{P}^{g-1}$. This is possible since the canonical map is always non-degenerate by Theorem 28.3.3(a). Consequently rank $((du^g)_D) = g$, and this holds more generally for D in an algebraic open set. This is because its failure is equivalent to $\det(du^g) = 0$, which is an algebraic condition which will hold on some codimension-one subvariety. $\square$

31.3.6. THEOREM. [JACOBI INVERSION] *u^g is surjective and generically injective.*

PROOF. By Lemma 31.3.5, du^g is generically an isomorphism of tangent spaces. So u^g takes an open ball about a general point $D \in Sym^d M$ to an open ball. But u^g is continuous and $Sym^g M$ compact, so image(u^g) is both a closed analytic subvariety of $J(M)$ and contains an open ball, and is therefore all of $J(M)$ (which is connected).

Since at a generic D, du^g is (in particular) injective, we see that any such D is an isolated point of $(u^g)^{-1}\{u^g(D)\}$. But the latter is a projective space by Lemma 31.3.4, and so the only way D is isolated is if $(u^g)^{-1}\{u^g(D)\}$ is isomorphic to $\mathbb{P}^0$, i.e. is just D itself. $\square$

Finally, to address (30.0.2) head-on, surjectivity of AJ follows from the diagram

(31.3.7)
$$\begin{array}{ccc} \mathrm{Div}^0(M) & \xrightarrow{AJ} & J(M) \\ & \nwarrow^{D \mapsto D - g[q]} & \uparrow u^g \\ & & Sym^g M. \end{array}$$

So we conclude that AJ induces an isomorphism $\mathrm{Pic}^0(M) \cong J(M)$ of abelian groups, giving a sort of group law on (linear systems of) g-tuples of points of M.

31.4. A final remark on moduli

For any Riemann surface M (of genus ≥ 1) with given symplectic basis of $H_1(M, \mathbb{Z})$, we know that there is a unique choice of basis for $\Omega^1(M)$ making the period matrix Π of the form $\begin{pmatrix} \mathbb{I}_g & Z \end{pmatrix}$. Moreover, we know by (31.1.7) that Z is symmetric with positive definite imaginary part, i.e. belongs to the g^{th} Siegel upper half space

$$\mathfrak{H}^g := \{Z \in M_g(\mathbb{C}) \mid Z = {}^tZ,\ Im(Z) > 0\}.$$

Note that $\mathfrak{H}^1$ is just $\mathfrak{H}$, the familiar upper half plane.

The Jacobian $J(M)$ is the quotient of $\mathbb{C}^g$ by the lattice Λ_M given by integral linear combinations of the columns of Π. More generally, let Z be any $g \times g$ complex matrix such that $\begin{pmatrix} \mathbb{I}_g & Z \end{pmatrix}$ has $\mathbb{R}$-linearly independent column vectors. Writing Λ_Z for their $\mathbb{Z}$-span, we define a complex torus by $A_Z := \mathbb{C}^g / \Lambda_Z$; any complex g-torus is isomorphic to one of this form. A major result is the

31.4.1. THEOREM. [RIEMANN EMBEDDING THEOREM] *A_Z is an abelian variety* (*i.e., has a holomorphic embedding in projective space*) *if and only if $\pm Z$ belongs to $\mathfrak{H}^g$.*

(Of course, any τ $\mathbb{R}$-linearly independent from 1 is in the upper or lower half plane, so every complex 1-torus is algebraic; already for $g = 2$ this is false!) You can find (effectively) two proofs in [Griffiths and Harris], one using generalized theta functions and the other using Kodaira's embedding theorem.

For us, the implications of this theorem are:

(a) Jacobians of Riemann surfaces of genus g are abelian varieties of dimension g; and
(b) abelian varieties of dimension g have $\frac{g(g+1)}{2}$ moduli.

Since Riemann surfaces of genus $g \geq 2$ have $3g - 3$ moduli, the following is believable:

31.4.2. PROPOSITION. *For $g < 4$, all abelian g-folds are Jacobians* (*or products of Jacobians*)*; for $g \geq 4$, "most" of them are not.*

For $g \geq 4$, then, we have the problem of characterizing the "Jacobian locus" in the moduli space $\mathfrak{H}^g / Sp_{2g}(\mathbb{Z})$, which is the (very difficult) *Schottky problem*. There are recent results describing this locus in terms of the vanishing of theta functions.

Exercises

(1) What is the smallest value of d for which it is clear that u_d has all fibers of the same dimension (and what are they)?

(2) Let $D, E \geq 0$ be effective divisors. Show that $\dim |D| + \dim |E| \leq \dim |D + E|$. [Hint: show that addition of divisors gives a map $|D| \times |E| \to |D + E|$ with finite fibers.]

(3) A divisor D on M is *special* if $i(D), \ell(D) > 0$. (a) Show that for D special, $\ell(D) \leq \frac{1}{2}\deg(D) + 1$. [Hint: First apply (2) to D and $K - D$ to bound $\ell(D) + i(D)$, then use Riemann-Roch.] (b) For $D \in Sym^g M$, show that D is special $\iff$ the fiber $u_g^{-1}(u_g(D))(= |D|)$ is not just the point D. (c) Conclude that the "special fibers" of u_g have dimension in $[1, \frac{g}{2}]$.

(4) Let M be hyperelliptic of genus g. Use Exercise (3) to completely describe the fibers of u_g (a) for $g = 2$ and (b) for $g = 3$. [Hint for (b): you will need to show that there is no map $f \colon M \to \mathbb{P}^1$ of degree 3. If there was, and $D := (f)_\infty = [p] + [q] + [r]$, use the degree-2 map $x \colon M \to \mathbb{P}^1$ to construct g with $(g)_\infty = x^{-1}(x(p))$. What functions are in $\mathfrak{L}(D')$? Apply Exercise (3)(a) to $D' := D + (g)_\infty$ to reach a contradiction.]

(5) The automorphism group G of a Riemann surface M is infinite if $g \leq 1$ (why?). By a theorem of Hurwitz,[8] it is finite (of order $\leq 84(g-1)$) if $g > 1$. Here you will just prove an earlier theorem of Schwarz that (for $g > 1$) there are no *continuous families* of automorphisms: suppose otherwise, and let σ_t be a family with $\sigma_0 = \mathrm{id}_M$. For t small, σ_t preserves the homology classes of the $\{\gamma_i\}$ (but $\sigma_t \neq \sigma_{t'}$ for $t \neq t'$). Show that $\sigma_t^* \omega_j = \omega_j$ for $1 \leq j \leq g$, consider $f := \frac{\omega_1}{\omega_2}$, and reach a contradiction.

[8] If you want a really hard computation, assume $|G| < \infty$ and apply the Riemann-Hurwitz formula to the quotient $X \twoheadrightarrow X/G$ to get this bound. Another bound is given by $2W!$, where W is the number of Weierstrass points of M (i.e. those points p for which $g[p]$ is special).