CHAPTER 31

Abel’s Theorem, part II

As mentioned at the beginning of the previous chapter, on any Riemann surface $M$, we get a perfect pairing on homology

\[ \langle \ , \ \rangle : H_1(M, \mathbb{Z}) \times H_1(M, \mathbb{Z}) \to \mathbb{Z} \]

by intersecting 1-cycles. With respect to a symplectic basis \{\gamma_j\}_{j=1}^{2g} as described there, this pairing has \((2g \times 2g)\) matrix

\[
Q = \begin{pmatrix}
0 & I_g \\
-I_g & 0
\end{pmatrix}.
\]

We can use (31.0.1) to produce an isomorphism of dual spaces

\[ H_1(M, \mathbb{C}) = H_1(M, \mathbb{Z}) \otimes \mathbb{C} \cong \text{Hom}(H_1(M, \mathbb{Z}), \mathbb{C}) = H^1(M, \mathbb{C}) \]

which is a special case of Poincaré duality.

Recalling the isomorphisms

\[ \Omega^1(M) \oplus \overline{\Omega^1(M)} \cong H^1_{dR}(M, \mathbb{C}) \cong H^1(M, \mathbb{C}), \]

there is also a pairing (the “cup-product”)

\[ H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \to \mathbb{C} \]

induced on the level of 1-forms by

\[(\omega, \eta) \mapsto \int_M \omega \wedge \eta.\]

Notice that since two holomorphic forms wedge to zero, this pairing restricts to zero on $\Omega^1(M) \times \Omega^1(M)$ (and $\Omega^1(M) \times \overline{\Omega^1(M)}$).

Yet another pairing (the “cap-product”)

\[ H_1(M, \mathbb{Z}) \times H^1(M, \mathbb{C}) \to \mathbb{C} \]
is induced by
$$(\gamma, \omega) \mapsto \int_{\gamma} \omega.$$  

The restriction of this pairing to $H_1(M, \mathbb{Z}) \times \Omega^1(M)$ is captured by the period matrix of Chapter 21. An important fact is that, under (31.0.2), both of these integration-induced products are nothing but complex-linear extensions of (31.0.1).

Assuming this compatibility, we can quickly derive the Riemann bilinear relations as follows. If for any closed 1-form $\varphi \in \Omega^1(M) \oplus \overline{\Omega^1(M)}$, we write
$$\pi_j(\varphi) := \int_{\gamma_j} \varphi,$$
then (31.0.2) identifies
\begin{equation}
[\varphi] = \sum_{j=1}^{g} (\pi_j(\varphi)[\gamma_{j+g}] - \pi_{j+g}(\varphi)[\gamma_j])
\end{equation}
in $H^1(M, \mathbb{C})$, i.e. as functionals on homology. One has for $\omega, \varphi \in \Omega^1(M)$
\begin{equation}
0 = \int_M \omega \wedge \varphi = -\sum_{j=1}^{g} (\pi_j(\varphi)\pi_{j+g}(\varphi) - \pi_{j+g}(\varphi)\pi_j(\omega))
\end{equation}
by writing $\int_M \omega \wedge \varphi = \langle [\omega], [\varphi] \rangle$ and expanding both classes as in (31.0.3). Similar reasoning together with the local computation
$$idz \wedge dz = i(dx + idy) \wedge (dx - idy) = i(-2dx \wedge dy) = 2dx \wedge dy,$$
leads to
\begin{equation}
0 < i \int_M \omega \wedge \overline{\omega} = -i \sum_{i=1}^{g} \left( \overline{\pi_i(\omega)} \pi_{i+g}(\omega) - \overline{\pi_{i+g}(\omega)} \pi_i(\omega) \right).
\end{equation}

This is all meant as motivation, though it can be made completely rigorous. We’ll start the first section with a more concrete, classical proof of (31.0.4)-(31.0.5), without the compatibility assumptions on the three bilinear pairings.
31.1. Derivation of the Riemann Bilinear Relations

We start by cutting $M$ open to get the “fundamental domain”, a simply-connected closed region $\tilde{\mathcal{G}}$

with boundary $\partial\tilde{\mathcal{G}}$. (Only a piece of it is shown in the picture.) Let $p_0$ in the interior of $\tilde{\mathcal{G}}$ be fixed. Given $\omega \in \Omega^1(M)$,

$$u(p) := \int_{p_0}^p \omega$$

then yields a well-defined (single valued) holomorphic$^1$ function on $\tilde{\mathcal{G}}$. If we take a second holomorphic form $\varphi \in \Omega^1(M)$, then

$$d(u\varphi) = \omega \wedge \varphi = 0.$$ 

That is, $u\varphi$ is a closed holomorphic form on $\tilde{\mathcal{G}}$ with the consequence that

$$0 = \int_{\tilde{\mathcal{G}}} d(u\varphi) = \int_{\partial\tilde{\mathcal{G}}} u\varphi$$

by Stokes’s theorem. Now, the picture above tells us that $\partial\tilde{\mathcal{G}}$ is the composition of paths

$$\gamma_{2g}^{-1} \gamma_g^{-1} \gamma_2 \gamma_g \cdots \gamma_{g+2}^{-1} \gamma_{g+2} \gamma_2 \gamma_{g+1}^{-1} \gamma_{g+1} \gamma_1,$$

written right to left (with inverse meaning the reverse direction). So the last integral becomes

$$= \sum_{j=1}^g \{ \int_{\gamma_j} (u(p) - u(p')) \varphi + \int_{\gamma_{j+g}} (u(r) - u(r')) \varphi \}$$

$^1$To be holomorphic on a closed set means that the function extends to a holomorphic function on a slightly larger open set (which, in this case, would live on the universal cover of $M$).
and, noting that \( \int_{p'}^q \omega = \int_{p'}^q \omega = - \int_{q}^p \omega \), this
\[
\sum_{j=1}^{g} \left( - \int_{\gamma_{g+j}} \omega \int_{\gamma_j} \varphi + \int_{\gamma_{g+j}} \varphi \int_{\gamma_j} \omega \right).
\]

This calculation of \( \int_{\partial A} u \varphi \) is evidently also valid with \( \varphi \) replaced by a more general (antiholomorphic, meromorphic) 1-form. In particular, replacing it by \( i\bar{\omega} \) yields
\[
0 < i \int_{\partial A} \omega \wedge \bar{\omega} = \int_{\partial A} u(i\bar{\omega}) = i \sum_{j=1}^{g} \left( - \int_{\gamma_{j+g}} \omega \int_{\gamma_j} \bar{\omega} + \int_{\gamma_{j+g}} \bar{\omega} \int_{\gamma_j} \omega \right).
\]

So we have recovered (31.0.4)-(31.0.5).

To reformulate this in matrix terms for any symplectic basis \( \{\gamma_j\}_{j=1}^{2g} \) of \( H_1(M, \mathbb{Z}) \) and any basis \( \{\omega_i\}_{i=1}^{g} \) of \( \Omega^1(M) \), notice that the \( (k, l) \)th entry of
\[
\Pi \cdot Q \cdot \mathring{t} \Pi = \begin{pmatrix}
\uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow \\
\pi_1 & \cdots & \pi_{2g}
\end{pmatrix}
\begin{pmatrix}
0 & \mathbb{I}_{g} \\
-\mathbb{I}_{g} & 0
\end{pmatrix}
\begin{pmatrix}
\leftarrow \pi_1 \rightarrow \\
\leftarrow \pi_{2g} \rightarrow
\end{pmatrix}
\]

is
\[
\sum_{j=1}^{g} \left( \pi_{j}(\omega_k) \pi_{g+j}(\omega_{\ell}) - \pi_{j}(\omega_{\ell}) \pi_{g+j}(\omega_k) \right),
\]
which is zero by (31.0.4); so
\[
(31.1.1) \quad \Pi \cdot Q \cdot \mathring{t} \Pi = 0.
\]

Similarly,
\[
(31.1.2) \quad \sqrt{-1} \Pi \cdot Q \cdot \mathring{t} \Pi > 0
\]
in the sense that \( \mathring{x}(\sqrt{-1} \Pi \cdot Q \cdot \mathring{t} \Pi) \mathring{x} \in \mathbb{R}_{>0} \) for any \( \mathring{x} \in \mathbb{C}^g \). In particular, the diagonal entries of (31.1.2) are positive real.

\footnote{Recall from Chapter 21 that \( \pi_j \) is the complex g-vector with \( j \)th entry \( \pi_j(\omega_i) \)}
31.1. DERIVATION OF THE RIEMANN BILINEAR RELATIONS

31.1.3. REMARK. Consider any two symplectic integral bases \( \Gamma = \{ \gamma_i \} \) and \( \Gamma' = \{ \gamma'_i \} \) (thought of as row-vectors), so that
\[
\Gamma' = \Gamma A
\]
for some \( A \in \text{SL}_{2g}(\mathbb{Z}) \). Applying the basis \( \{ \omega_i \} \) (viewed as a column-vector of 1-forms) on the left yields
\[
\Pi' = \Pi A.
\]
Furthermore, since both bases are symplectic we have \( Q = \,^t \Gamma \cdot \Gamma \) and
\[
Q = \,^t \Gamma' \cdot \Gamma' = \,^t A \Gamma A = \,^t AQA;
\]
that is, \( A \) belongs to the symplectic group \( \text{Sp}_{2g}(\mathbb{Z}) \). It is for this reason that (31.1.1)-(31.1.2) are compatible with change of symplectic basis: e.g., assuming (31.1.1), we have
\[
\Pi' \,^t Q' \Pi' = \Pi A \,^t Q' A' \Pi = \Pi Q' \Pi = 0.
\]

Now thinking of the \( g \times 2g \) period matrix as two \( g \times g \) blocks, viz.
\[
(31.1.4) \quad \Pi = \begin{pmatrix} A & B \\ \end{pmatrix},
\]
we have
\[
\Pi Q' \Pi = \begin{pmatrix} A & B \\ \end{pmatrix} \begin{pmatrix} I_g \\ -I_g \end{pmatrix} \begin{pmatrix} \,^t A \\ \,^t B \end{pmatrix} = A \cdot \,^t B - B \cdot \,^t A
\]
and
\[
\Pi Q' \Pi = A \cdot \,^t B - B \cdot \,^t A.
\]
In these terms, (31.1.1) reads
\[
(31.1.5) \quad A \cdot \,^t B = B \cdot \,^t A
\]
while (31.1.2) becomes
\[
(31.1.6) \quad \sqrt{-1} \,^t v (A \,^t B - B \,^t A) \overline{\overline{\sigma}} > 0 \quad (\forall v \in \mathbb{C}^g).
\]
If \( \,^t A \) has nonzero kernel, then there exists \( v \in \mathbb{C}^g \) satisfying \( \,^t A v = 0 \), hence \( \,^t v A = 0 \) and \( \,^t A \overline{\sigma} = 0 \), contradicting (31.1.6). It follows that
A is invertible, and so we have proved the statement on C-linear independence asserted at the end of §30.2.

Applying $A^{-1}$ to the left of $\Pi$ amounts to a change of the basis $\{\omega_i\}$ for $\Omega^1(M)$, viz.$^3$

$$A^{-1}\Pi = A^{-1} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} \begin{pmatrix} \gamma_1 & \cdots & \gamma_{2g} \\ \vdots \\ \omega' \end{pmatrix} = \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_g \end{pmatrix} \begin{pmatrix} \gamma_1 & \cdots & \gamma_{2g} \\ \vdots \\ \omega' \end{pmatrix}. $$

If we apply it to (31.1.4), then we get

$$\Pi' := A^{-1}\Pi = \begin{pmatrix} I_g & A^{-1}B \end{pmatrix}. $$

We can therefore always assume that $\{\omega_i\}$ is chosen so that

$$\Pi = \begin{pmatrix} I_g & Z \end{pmatrix},$$

again as claimed in §30.2. The bilinear relations (31.1.5)-(31.1.6) simplify to

(31.1.7) \[
\begin{aligned}
Z &= tZ \\
\sqrt{-1}(Z - Z) &> 0
\end{aligned}
\]

which in particular tell us that the imaginary part $Im(Z)$ is a positive-definite, real symmetric matrix.

### 31.2. Proof of Abel’s Theorem

With the holomorphic basis as normalized above, we can now quickly establish (30.2.4) and hence (30.0.1). Write $D = \sum n_i[P_i]$ (with $\sum n_i = 0$) and let $\varphi := \tilde{\eta_D}$ be as in §30.2, so that

(31.2.1) \[
\text{Res}_{P_i}(\varphi) = \frac{n_i}{2\pi\sqrt{-1}} \quad (\forall i)
\]

and

$$\int_{\gamma_j} \varphi = 0 \quad (j = 1, \ldots, g).$$

$^3$Here the “product” of $\omega_i$ and $\gamma_j$ is just the integral $\int_{\gamma_j} \omega_i$.
For each $k = 1, \ldots, g$ set
\[ u_k(p) := \int_{p_0}^p \omega_k \]
on $\mathfrak{F}$, and let $\Gamma$ be a 1-chain (sum of paths) on $\mathfrak{F}$ with $\partial \Gamma = D$. Then noting $D = \sum_j n_j([P_j] - [p_0])$, we have
\[ \int_{\Gamma} \omega_k = \sum_i n_i u_k(P_i) = 2\pi \sqrt{-1} \sum_{p \in |D|} \text{Res}_p(u_k \varphi) \]
which by the Residue Theorem
\[ = \int_{\partial \mathfrak{F}} u_k \varphi = \sum_j (\pi_j(\omega_k) \pi_{g+j}(\varphi) - \pi_{g+j}(\omega_k) \pi_j(\varphi)) = \pi_{g+k}(\varphi). \]

If $AJ(D) = 0$ then there are integers $m_j (j = 1, \ldots, 2g)$ such that for every $k$
\[ \int_{\Gamma} \omega_k = \sum_{j=1}^{2g} m_j \int_{\gamma_j} \omega_k. \]
Using $\int_{\gamma_j} \omega_k = \delta_{jk}$ and $Z = ^t Z$ (from (31.1.7)), this
\[ = m_k + \sum_{j=1}^{g} m_{j+g} \pi_{j+g}(\omega_k) = m_k + \sum_{j=1}^{g} m_{j+g} \pi_{k+g}(\omega_j). \]
Now
\[ \hat{\varphi} := \varphi - \sum_{j=1}^{g} m_{j+g} \omega_j \]
is still an element of $\mathcal{I}(- \sum_{p \in |D|} [p])$ satisfying (31.2.1). Moreover, for $k \in \{1, \ldots, g\}$
\[ \pi_{k+g}(\hat{\varphi}) = \pi_{k+g}(\varphi) - \sum_{j=1}^{g} m_{j+g} \pi_{k+g}(\omega_j) \]
\[ = \int_{\Gamma} \omega_k - \sum_{j=1}^{g} m_{j+g} \pi_{k+g}(\omega_j) = m_k \in \mathbb{Z} \]
and
\[ \pi_k(\hat{\phi}) = \pi_k(\varphi) - \sum_{j=1}^{g} m_{j+g} \pi_k(\omega_j) = -m_{k+g} \in \mathbb{Z}. \]

By Lemma 30.2.2, \( \exp (2\pi \sqrt{-1} \int \hat{\phi}) \) now gives a meromorphic function with \( (f) = D. \)

### 31.3. Proof of Jacobi Inversion

To show that \( AJ \) is surjective, we will study the image of a certain class of (degree zero) divisor on \( M \), namely those of the form
\[ [p_1] + \cdots + [p_d] - d[q] \]
given some fixed point \( q \in M \) and natural number \( d \). Such divisors are obviously in 1-to-1 correspondence with unordered \( d \)-tuples of points on \( M \), in other words with elements of the \( d \)th symmetric power
\[ Sym^d M := \underbrace{M \times \cdots \times M}_{\text{d copies}} \sim (p_{\sigma(1)}, \ldots, p_{\sigma(d)}) \]
for all \( \sigma \in \mathfrak{S}_d \).

(These elements are written either \( p_1 + \cdots + p_d \) or \( \{p_1, \ldots, p_d\} \).) In order to be able to use complex analytic techniques we need to put the structure of a \( d \)-dimensional complex manifold on this.\(^4\)

To get a feel for how this works, take \( d = 2 \) and consider \( \mathbb{C} \) instead of a (compact) Riemann surface. The symmetric square \( Sym^2 \mathbb{C} \) is the quotient of \( \mathbb{C} \times \mathbb{C} \) by the involution \( (z_1, z_2) \mapsto (z_2, z_1) \). What causes difficulty is the locus consisting of its fixed points, i.e. the

\(^4\)Had we started with \( M \) itself of dimension > 1, its symmetric powers would be singular complex analytic spaces, hence not manifolds. So what happens next is special for \( \dim(M) = 1 \).
diagonal line. Take two small open sets in $\text{Sym}^2 M$, one which intersects the diagonal and one which does not:

Clearly $(z_1, z_2)$ give local holomorphic coordinates on $U_\beta$. On $U_\alpha$, they are not well-defined, but their elementary symmetric polynomials $\sigma_1(z_1, z_2) = z_1 + z_2$ and $\sigma_2(z_1, z_2) = z_1 z_2$ are. Moreover, these functions generate all polynomials in $z_1, z_2$ which are invariant under the involution and hence well-defined on $U_\alpha \subset \text{Sym}^2 \mathbb{C}$. Taking $(w_1, w_2) := (z_1 + z_2, z_1 z_2)$ as the holomorphic coordinates there,\(^5\) the transition function $\Phi_{\alpha \beta}$ is then just $(\sigma_1, \sigma_2)$. To see that this is invertible on $U_{\alpha \beta}$, notice that in $U_\alpha$ the diagonal is defined by $w_1^2 = 4w_2$ (since $z_1 = z_2 \iff (z_1 + z_2)^2 = 4z_1 z_2$). Since $U_{\alpha \beta}$ avoids this locus (and is simply connected), $\sqrt{w_1^2 - 4w_2}$ is well defined there and we can define $\Phi_{\beta \alpha}$ by

$$
\begin{align*}
    z_1 &= \frac{w_1 + \sqrt{w_1^2 - 4w_2}}{2} \\
    z_2 &= \frac{w_1 - \sqrt{w_1^2 - 4w_2}}{2}.
\end{align*}
$$

More generally, in a neighborhood of

$$
\{q_1, \ldots, q_{k_1}; \ldots; q_{k_l}, \ldots, q_{k_l}\} \in \text{Sym}^d M
$$

\(^5\)We could in fact take these as global coordinates, but this situation won’t generalize to $M$.\)
(where $\sum_{j=1}^{\ell} k_j = d$), the local coordinate system is given in terms of holomorphic coordinates $z_j$ on $M$ near each $q_j$, by

$$\{p_{11}, \ldots, p_{k_1}, \ldots; p_{d-k_\ell+1}, \ldots, p_d\} \mapsto (z_1(p_1), \ldots, z_1(p_{k_1})); \ldots; z_1(p_1), \ldots, z_1(p_{k_1})); \ldots;$$

$$z_\ell(p_{d-k_\ell+1}), \ldots, z_\ell(p_d)), z_\ell(p_{d-k_\ell+1}), \ldots, z_\ell(p_d)) \}. \]

Inelegant, but it gets the job done.

Now let $D$ be any divisor of degree $d$ on $M$, and consider the mapping

$$\alpha_D : \mathbb{P}(\mathcal{L}(D)) \to \text{Sym}^d M$$

which sends (for $f \in \mathcal{L}(D)$)

$$[f] \mapsto (f) + D.$$

Here $(f) + D \geq 0$ by definition, and $\deg((f) + D) = \deg D = d$; so $(f) + D$ is indeed of the form $[p_1] + \cdots + [p_d]$. The map sends the projective equivalence class $[f]$, i.e. “$f$ up to a constant multiple”, to $\{p_1, \ldots, p_d\}$.

31.3.1. **Lemma.** $\alpha_D$ is (a) injective and (b) holomorphic.

31.3.2. **Definition.** The linear system $^6 |D|$ consists of all effective divisors on $M$ rationally equivalent to $D$. The Lemma evidently realizes $|D| = \text{image}(\alpha_D)$ as a subvariety of $\text{Sym}^d M$ isomorphic to $\mathbb{P}^{\ell(D) - 1}$.

**Proof of Lemma.** (a) $(f) + D = (g) + D \implies (f) = (g) \implies (f/g) = 0 \implies f/g \text{ constant} \implies [f] = [g]$.

(b) To show $\alpha_D$ holomorphic in a neighborhood of $[f_0]$, augment $f_0$ to a basis $\{f_0, f_1, \ldots, f_{\ell(D)}\} \subset \mathcal{L}(D)$ and write $f_\mu := f_0 + \sum_{j=1}^{\ell(D)} \mu_j f_j$ so that $\{f_j\}_{j=1}^{\ell(D)}$ are the local holomorphic coordinates (on some small $U \subset \mathcal{L}(D)$). Let $p \in |D| \cup |(f_0)|$, with open neighborhood $\mathcal{N}_p \subset M$ and local coordinate $z$ (ord$_p(z) = 1$). Set $k := \text{ord}_p(f_0) + \text{ord}_p(D)$,
and $W_{f_0,p} := \text{Sym}^k N_p$ with coordinates $\sigma_1(z_1, \ldots, z_k), \ldots, \sigma_k(z_1, \ldots, z_k)$.
We must show that the composition

$$U \rightarrow \mathcal{O}(N_p) \rightarrow W_{f_0,p} \hookrightarrow \mathbb{C}^k$$

$$\mu \mapsto f_\mu z^{\text{ord}_p D} \mapsto \left. \left( f_\mu z^{\text{ord}_p D} \right) \right|_{N_p} \mapsto \left( \begin{array}{c}
\sigma_1(z(p_1(\mu)), \ldots, z(p_k(\mu))) \\
\vdots \\
\sigma_k(z(p_1(\mu)), \ldots, z(p_k(\mu)))
\end{array} \right)$$

$p_1(\mu) + \cdots + p_k(\mu)$

is holomorphic, which in turn boils down to the statement that each $\sigma_\ell$ is holomorphic in each $\mu_j$. For $k = 1$, this is the holomorphic implicit function theorem; for $k > 1$, it is this together with Rouché and the Riemann extension theorem in a manner familiar from previous chapters.

31.3.3. Definition. An effective degree $d$ divisor $D$ (viewed as an element of $\text{Sym}^d M$) is called general $\iff D = [p_1] + \cdots + [p_d]$ with the $\{p_j\}$ distinct points of $M$.

Now look at the “Abel-Jacobi” mapping

$$u^d : \text{Sym}^d M \rightarrow J(M)$$

$$[p_1] + \cdots + [p_d] \mapsto AJ \left( \sum_{j=1}^d [p_j] - d[q] \right),$$

where $q \in M$ is fixed. This is shown to be holomorphic by using the fundamental theorem of calculus at general $D$, then applying the Osgood and Riemann extension theorems. (Boundedness is clear by taking a local lifting of the image of $u^d$ to $\mathcal{C}^8$.)

The next result does not require $D$ to be general.

31.3.4. Lemma. The fiber of $u^d$ over $u^d(D)$ is $|D| (\cong \mathbb{P}^{\ell(D) - 1})$.

Proof. (For simplicity write $u$ for $u^d$.)

$$u^{-1}(u(D)) \subset |D| : u(E) = u(D) \iff \text{AJ}(E - D) = 0 \overset{\text{Abel}}{\implies} E - D$$

is the divisor of some $f \in \mathcal{K}(M)^* \implies (f) + D = E \geq 0$ (since $E \in \text{Sym}^d M \iff f \in \mathcal{L}(D) \implies E = \alpha_D(f) \in \text{image}(\alpha_D) = |D|$.
\[ u^{-1}(u(D)) \supset |D|: \text{Given } E \in |D|, \text{there exists } f \in L(D) \text{ such that} \]
\[ E = (f) + D \implies E - D = (f) \overset{\text{rat}}{=} 0 \implies 0 = AJ(E - D) \implies u(E) = u(D) \implies E \in u^{-1}(u(D)). \]
\[ \square \]

If \( D = [p_1] + \cdots + [p_d] \) is general, then writing \( z_j \) for local coordinates about each \( p_j \),
\[ (du^d)_D : T_D \left( \text{Sym}^d M \right) \to T_{u(D)}(J(M)) \]

is computed by the matrix
\[
\left( \begin{array}{cccc}
\frac{\partial}{\partial z_1} \sum_{i=1}^d \int_q^{|z_i|} \omega_1 \\
\vdots \\
\frac{\partial}{\partial z_d} \sum_{i=1}^d \int_q^{|z_i|} \omega_d
\end{array} \right)_{\{p_1, \ldots, p_d\}}
\]

If we write locally (about each \( p_j \)) \( \omega_i^\text{loc} = f_i(z_j)dz_j \), this
\[
\left( \begin{array}{cccc}
f_1(p_1) & \cdots & f_1(p_1) \\
\vdots & \ddots & \vdots \\
f_d(p_1) & \cdots & f_d(p_1)
\end{array} \right) = \left( \begin{array}{c}
\varphi_K(p_1) \\
\vdots \\
\varphi_K(p_d)
\end{array} \right),
\]

where \( \varphi_K \) is the canonical map and \( \varphi_K(p_j) \in C^s \) is a “lift” of \( \varphi_K(p_j) \in \mathbb{P}^{s-1} \). (For \( d = 1 \) this is just Proposition 30.1.2(b).) From this we see that
\[ \text{rank} \left( (du^d)_D \right) = \text{dim} \left( \text{span}(\varphi_K(p_1), \ldots, \varphi_K(p_d)) \right) + 1, \]
where “span” means the projective linear span in \( \mathbb{P}^{s-1} \). Taking \( d = g \), we now have the following claim:

31.3.5. Lemma. \( \text{rank} \left( (du^g)_D \right) = g \) for a generic\(^7 \) choice of \( D = [p_1] + \cdots + [p_g] \in \text{Sym}^g M \), i.e. for \( D \) in some Zariski open subset of \( \text{Sym}^d M \).

\(^7\)“General” may not be quite enough — \( D \) may have to avoid a larger number of subvarieties of \( \text{Sym}^g M \) then just the ones where two or more \( p_j \)'s coincide.
Proof. Pick \( p_1, \ldots, p_g \) distinct with \( \text{span} \left( \varphi_K(p_1), \ldots, \varphi_K(p_g) \right) = \) all of \( \mathbb{P}^{g-1} \). This is possible since the canonical map is always non-degenerate by Theorem 28.3.3(a). Consequently \( \text{rank} \left( (du^g)_D \right) = g \), and this holds more generally for \( D \) in an algebraic open set. This is because its failure is equivalent to \( \det(du^g) = 0 \), which is an algebraic condition which will hold on some codimension-one subvariety. \( \square \)

31.3.6. Theorem. [Jacobi inversion] \( u^g \) is surjective and generically injective.

Proof. By Lemma 31.3.5, \( du^g \) is generically an isomorphism of tangent spaces. So \( u^g \) takes an open ball about a general point \( D \in \text{Sym}^d M \) to an open ball. But \( u^g \) is continuous and \( \text{Sym}^g M \) compact, so \( \text{image}(u^g) \) is both a closed analytic subvariety of \( J(M) \) and contains an open ball, and is therefore all of \( J(M) \) (which is connected).

Since at a generic \( D \), \( du^g \) is (in particular) injective, we see that any such \( D \) is an isolated point of \( (u^g)^{-1}\{u^g(D)\} \). But the latter is a projective space by Lemma 31.3.4, and so the only way \( D \) is isolated is if \( (u^g)^{-1}\{u^g(D)\} \) is isomorphic to \( \mathbb{P}^0 \), i.e. is just \( D \) itself. \( \square \)

Finally, to address (30.0.2) head-on, surjectivity of \( AJ \) follows from the diagram

\[
\begin{array}{ccc}
\text{Div}^0(M) & \xrightarrow{AJ} & J(M) \\
\downarrow & & \uparrow u^g \\
D \mapsto D-g[q] & \swarrow & \text{Sym}^g M.
\end{array}
\]

So we conclude that \( AJ \) induces an isomorphism \( \text{Pic}^0(M) \cong J(M) \) of abelian groups, giving a sort of group law on (linear systems of) \( g \)-tuples of points of \( M \).

31.4. A final remark on moduli

For any Riemann surface \( M \) (of genus \( \geq 1 \)) with given symplectic basis of \( H_1(M, \mathbb{Z}) \), we know that there is a unique choice of basis
for \( \Omega^1(M) \) making the period matrix \( \Pi \) of the form \( \left( I_g \quad Z \right) \). Moreover, we know by (31.1.7) that \( Z \) is symmetric with positive definite imaginary part, i.e. belongs to the \( g \)-th Siegel upper half space

\[
\mathcal{H}^g := \{ Z \in M_g(C) \mid Z = ^tZ, \text{Im}(Z) > 0 \}.
\]

Note that \( \mathcal{H}^1 \) is just \( \mathcal{H} \), the familiar upper half plane.

The Jacobian \( J(M) \) is the quotient of \( \mathbb{C}^g \) by the lattice \( \Lambda_M \) given by integral linear combinations of the columns of \( \Pi \). More generally, let \( Z \) be any \( g \times g \) complex matrix such that \( \left( I_g \quad Z \right) \) has \( \mathbb{R} \)-linearly independent column vectors. Writing \( \Lambda_Z \) for their \( \mathbb{Z} \)-span, we define a complex torus by \( A_Z := \mathbb{C}^g / \Lambda_Z \); any complex \( g \)-torus is isomorphic to one of this form. A major result is the

**31.4.1. Theorem.** [Riemann Embedding Theorem] \( A_Z \) is an abelian variety (i.e., has a holomorphic embedding in projective space) if and only if \( \pm Z \) belongs to \( \mathcal{H}^g \).

(Of course, any \( \tau \) \( \mathbb{R} \)-linearly independent from 1 is in the upper or lower half plane, so every complex 1-torus is algebraic; already for \( g = 2 \) this is false!) You can find (effectively) two proofs in [Griffiths and Harris], one using generalized theta functions and the other using Kodaira’s embedding theorem.

For us, the implications of this theorem are:

(a) Jacobians of Riemann surfaces of genus \( g \) are abelian varieties of dimension \( g \); and

(b) abelian varieties of dimension \( g \) have \( \frac{g(g+1)}{2} \) moduli.

Since Riemann surfaces of genus \( g \geq 2 \) have \( 3g - 3 \) moduli, the following is believable:

**31.4.2. Proposition.** For \( g < 4 \), all abelian \( g \)-folds are Jacobians (or products of Jacobians); for \( g \geq 4 \), “most” of them are not.

For \( g \geq 4 \), then, we have the problem of characterizing the “Jacobian locus” in the moduli space \( \mathcal{H}^g / Sp_{2g}(\mathbb{Z}) \), which is the (very difficult) Schottky problem. There are recent results describing this locus in terms of the vanishing of theta functions.
Exercises

(1) What is the smallest value of $d$ for which it is clear that $u_d$ has all fibers of the same dimension (and what are they)?

(2) Let $D, E \geq 0$ be effective divisors. Show that $\dim |D| + \dim |E| \leq \dim |D + E|$. [Hint: show that addition of divisors gives a map $|D| \times |E| \to |D + E|$ with finite fibers.]

(3) A divisor $D$ on $M$ is special if $i(D), \ell(D) > 0$. (a) Show that for $D$ special, $\ell(D) \leq \frac{1}{2} \deg(D) + 1$. [Hint: First apply (2) to $D$ and $K - D$ to bound $\ell(D) + i(D)$, then use Riemann-Roch.] (b) For $D \in Sym^gM$, show that $D$ is special $\iff$ the fiber $u_g^{-1}(u_g(D))(=|D|)$ is not just the point $D$. (c) Conclude that the “special fibers” of $u_g$ have dimension in $[1, \frac{g}{2}]$.

(4) Let $M$ be hyperelliptic of genus $g$. Use Exercise (3) to completely describe the fibers of $u_g$ (a) for $g = 2$ and (b) for $g = 3$. [Hint for (b): you will need to show that there is no map $f : M \to \mathbb{P}^1$ of degree 3. If there was, and $D := (f)_\infty = [p] + [q] + [r]$, use the degree-2 map $x : M \to \mathbb{P}^1$ to construct $g$ with $(g)_\infty = x^{-1}(x(p))$. What functions are in $L(D')$? Apply Exercise (3)(a) to $D' := D + (g)_\infty$ to reach a contradiction.]

(5) The automorphism group $G$ of a Riemann surface $M$ is infinite if $g \leq 1$ (why?). By a theorem of Hurwitz,\(^8\) it is finite (of order $\leq 84(g - 1)$) if $g > 1$. Here you will just prove an earlier theorem of Schwarz that (for $g > 1$) there are no continuous families of automorphisms: suppose otherwise, and let $\sigma_t$ be a family with $\sigma_0 = \text{id}_M$. For $t$ small, $\sigma_t$ preserves the homology classes of the $\{\gamma_i\}$ (but $\sigma_t \neq \sigma_{t'}$ for $t \neq t'$). Show that $\sigma_t^* \omega_j = \omega_j$ for $1 \leq j \leq g$, consider $f := \frac{\omega_1}{\omega_2}$, and reach a contradiction.

---

\(^8\)If you want a really hard computation, assume $|G| < \infty$ and apply the Riemann-Hurwitz formula to the quotient $X \to X/G$ to get this bound. Another bound is given by $2W!$, where $W$ is the number of Weierstrass points of $M$ (i.e. those points $p$ for which $g[p]$ is special).