CHAPTER 4

Lines, conics, and duality

To complete our introduction to algebro-geometric concepts on the level of curves, in this chapter we'll study projective transformations, tangent lines, and dual curves. Our convention will be to write elements of $\mathbb{C}^3$ as column vectors

$$\underline{Z} = \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}$$

(with respect to the "standard basis" $\underline{\mathbf{e}}$), and $[\underline{Z}] = [Z_0 : Z_1 : Z_2]$ for the corresponding point in $\mathbb{P}^2$.

4.1. The classification of complex conics

The story begins in even lower degree, with *lines* — i.e. degree 1 algebraic curves. These are subsets of $\mathbb{P}^2$ of the form

$$L_{\underline{\lambda}} = \{ {}^t\underline{\lambda} \cdot \underline{Z} = 0 \}, \tag{4.1.1}$$

where $\underline{\lambda}$ is a nonzero vector in $\mathbb{C}^3$. Note that for $\alpha \in \mathbb{C}^*$, $L_{\alpha\underline{\lambda}} = L_{\underline{\lambda}}$.

By stereographic projection (cf. §3.3), lines and smooth conics are isomorphically parametrized by $\mathbb{P}^1$ in the sense of the Normalization Theorem. (For lines, the projection is done through a point not on the line; for conics, one chooses any point on the conic.) However, not all conics are smooth, and so we will need to classify conics up to projective equivalence.[1] The two key non-smooth examples to keep in mind are the *pair of lines*

$$\{XY = 0\} = \{X = 0\} \cup \{Y = 0\}$$

[1]There is a somewhat subtle point here. For smooth curves in general, projective equivalence is finer (equates fewer curves) than isomorphism as Riemann surfaces. However, you have to consider curves of degree at least 5 to see this discrepancy. As far as conics are concerned, we like projective equivalence simply because it gives a uniform and algebraic treatment of singular and smooth curves.

and the *double line*

$$\{X^2 = 0\}.$$

The first has two irreducible components (and is hence *reducible*), while the second has one component of "multiplicity two" (and is said to be *non-reduced*).[2]

To define projective equivalence, we introduce the projective general linear group

$$\mathrm{PGL}(n,\mathbb{C}) := \frac{\mathrm{GL}(n,\mathbb{C})}{\langle \alpha \cdot \mathrm{id.} \mid \alpha \in \mathbb{C}^* \rangle}.$$

(We have $A \equiv B \iff B = \alpha A$ for some $\alpha \in \mathbb{C}^*$.) Consider the action of $PGL(3,\mathbb{C})$ on $\mathbb{P}^2$ by

$$T\left(\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}\right)[Z_0 : Z_1 : Z_2] =$$

$$\left[\begin{array}{c} a_{00}Z_0 + a_{01}Z_1 \\ +a_{02}Z_2 \end{array} : \begin{array}{c} a_{10}Z_0 + a_{11}Z_1 \\ +a_{12}Z_2 \end{array} : \begin{array}{c} a_{20}Z_0 + a_{21}Z_1 \\ +a_{22}Z_2 \end{array}\right]$$

or in more compact notation

$$T(A)[\underline{Z}] = [A \cdot \underline{Z}].$$

(We are, consistently with the notation mentioned at the beginning of the chapter, letting the matrix A act on $\underline{Z}$ viewed as a column vector.[3]) This action is well-defined:

- it sends no nonzero $\underline{Z}$ to $\underline{0}$ (recall $[\underline{0}]$ is not a point in $\mathbb{P}^2$);
- if $\underline{Z} = \alpha\underline{Y}$, then $T(A)[\underline{Z}] = [A\underline{Z}] = [A \cdot \alpha\underline{Y}] = [\alpha A\underline{Y}] = [A\underline{Y}] = T(A)[\underline{Y}]$;
- if $A = \alpha B$, then $T(A)[\underline{Z}] = [\alpha B\underline{Z}] = [B\underline{Z}] = T(B)[\underline{Z}]$.

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[2]Roughly speaking, "reduced" means "all of its irreducible components are of multiplicity one". So while $\{XY = 0\}$ is reduced, something like $\{XY^3 = 0\}$ is not. Obviously this is going a bit beyond the notion of an algebraic curve as a solution set, since it incorporates multiplicity. To really formalize what such an object is, we would have to work with scheme theory or algebraic cycles, which I do not want to do. So unless otherwise stated, in this course an "algebraic curve" is assumed to be reduced.

[3]The dot indicates matrix multiplication. This will often be omitted.

4.1.2. DEFINITION. The transformations $T(A) : \mathbb{P}^2 \to \mathbb{P}^2$, $A \in PGL(3, \mathbb{C})$, are the projective linear transformations (or *projectivities*) of $\mathbb{P}^2$.

4.1.3. REMARK. The analogue of projectivities on $\mathbb{P}^1$ are simply the fractional linear transformations:

$$T\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right)[Z_0 : Z_1] = [aZ_0 + bZ_1 : cZ_0 + dZ_1].$$

So writing "z" for the point $[1 : z]$,

$$T\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right)(z) = \frac{c + dz}{a + bz}.$$

You probably know from complex analysis that such transformations preserve the cross-ratio of 4 points. Furthermore, they are the only automorphisms of $\mathbb{P}^1$ (invertible morphisms from $\mathbb{P}^1 \to \mathbb{P}^1$) as a complex 1-manifold.

How do projectivities affect algebraic curves? For a curve $C = \{F(\underline{Z}) = 0\}$ of degree d, points in $T(A)C$ are of the form $T(A)\underline{\mathfrak{z}}$ for $\underline{\mathfrak{z}} \in C$. These are precisely the solutions of the equation

$$\{F\left(T(A^{-1})(\cdot)\right) = 0\} \ (= T(A)C) \tag{4.1.4}$$

since then

$$F(T(A^{-1})T(A)\underline{\mathfrak{z}}) = F(T(A^{-1}A)\underline{\mathfrak{z}}) = F(\underline{\mathfrak{z}}) = 0.$$

Since (4.1.4) just substitutes linear forms[4] for $Z_0, \ldots, Z_n$ in the equation for C, we find:

4.1.5. PROPOSITION. *The images of (smooth resp. singular) algebraic curves of degree d under projectivities, are again (smooth resp. singular) algebraic curves of degree d.*

So lines are carried to lines, conics to conics, and so on. In general, if $T(A)C = C'$ for some A, then the curves C, C' are said to be *projectively equivalent*.

[4] i.e. homogeneous polynomials of degree 1 in $Z_0, \ldots, Z_n$

PROOF. To see why smoothness is preserved, write

$$\tilde{F}(\underline{\xi}) = F(T(A^{-1})\underline{\xi})$$

(where we have in mind $[\underline{\xi}] = T(A)[\underline{Z}]$); and suppose (for a contradiction) that for some $\underline{\mathfrak{z}} \in C$ we have $\frac{\partial F}{\partial Z_0}(\underline{\mathfrak{z}}) \neq 0$ but $\frac{\partial \tilde{F}}{\partial \xi_i}(T(A)\underline{\mathfrak{z}}) = 0$ $(\forall i)$.

If $\tilde{F} = F \circ T(A^{-1})$, then $F = \tilde{F} \circ T(A)$, and by the (multivariable) chain rule

$$\frac{\partial F}{\partial Z_0} = \sum_i \frac{\partial T(A)_i}{\partial Z_0}\frac{\partial \tilde{F}}{\partial \xi_i} = \sum_i a_{i0}\frac{\partial \tilde{F}}{\partial \xi_i},$$

so that

$$0 \neq \frac{\partial F}{\partial Z_0}(\underline{\mathfrak{z}}) = \sum_i a_{i0}\frac{\partial \tilde{F}}{\partial \xi_i}(T(A)\underline{\mathfrak{z}}) = 0,$$

a contradiction. □

Next we want to get formulas for the effect of projectivities on lines and conics. For a line $L_{\underline{\lambda}} = \{{}^t\underline{\lambda} \cdot \underline{Z} = 0\}$, (4.1.4) gives $0 = {}^t\underline{\lambda}A^{-1}\underline{Z} = {}^t({}^tA^{-1}\underline{\lambda})\underline{Z}$, so that

$$T(A)L_{\underline{\lambda}} = L_{({}^tA^{-1}\underline{\lambda})}. \tag{4.1.6}$$

Since $GL(3,\mathbb{C})$ acts transitively on $\mathbb{C}^3 \setminus \{\underline{0}\}$, this implies the (relatively trivial)

4.1.7. PROPOSITION. *All lines in $\mathbb{P}^2$ are projectively equivalent.*

Let $Q = \{0 = aZ_0^2 + bZ_1^2 + cZ_2^2 + dZ_0Z_1 + eZ_0Z_2 + fZ_1Z_2\}$ be an arbitrary conic. We can rewrite its equation

$$0 = \begin{pmatrix} Z_0 & Z_1 & Z_2 \end{pmatrix} \begin{pmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix} =: {}^t\underline{Z}\mathcal{B}\underline{Z}$$

in terms of a (unique) *symmetric*[5] matrix $\mathcal{B}$, and accordingly rename it $Q_{\mathcal{B}}$. (The expression ${}^t\underline{Z}\mathcal{B}\underline{Z}$ is called a symmetric bilinear form.) For the action of a projectivity, (4.1.4) substitutes in $A^{-1}\underline{Z}$ for $\underline{Z}$, yielding

$$0 = {}^t(A^{-1}\underline{Z})\mathcal{B}A^{-1}\underline{Z} = {}^t\underline{Z}({}^tA^{-1}\mathcal{B}A^{-1})\underline{Z}$$

[5] i.e. the transpose ${}^t\mathcal{B}$ equals $\mathcal{B}$

so that

$$T(A)Q_{\mathcal{B}} = Q_{({}^tA^{-1}\mathcal{B}A^{-1})}.$$

Given an invertible complex matrix M, the transformation

$$\mathcal{B} \mapsto {}^tM\mathcal{B}M =: \mathcal{B}'$$

is called a *cogredience*, and $\mathcal{B}$, $\mathcal{B}'$ are *cogredient over* $\mathbb{C}$. All nonzero symmetric matrices are cogredient $/\mathbb{C}$ to one of the form[6]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We conclude:

4.1.8. PROPOSITION. *All conics in* $\mathbb{P}^2$ *are projectively equivalent to one of*

$$\{X^2 = 0\},\ \{X^2 + Y^2 = 0\},\ \text{or } \{X^2 + Y^2 + Z^2 = 0\}.$$

Notice that $X^2 + Y^2 = (X + \sqrt{-1}Y)(X - \sqrt{-1}Y)$ is a pair of lines, and so projectively equivalent to $XY = 0$.

4.1.9. COROLLARY. *(i) All smooth*[7] *conics are projectively equivalent.*
(ii) $Q_{\mathcal{B}}$ *is smooth* $\iff \det \mathcal{B} \neq 0$.

PROOF. (i) Since $\{X^2 + Y^2 + Z^2 = 0\}$ is the only smooth option in Prop. 4.1.8, by Prop. 4.1.5 all smooth conics must be equivalent to this hence to each other.

(ii) Cogredience $\mathcal{B} \mapsto {}^tM\mathcal{B}M$ multiplies determinant by $(\det M)^2$, which is always nonzero (as $M \in GL(3, \mathbb{C})$); so projectivities preserve non-zero-ness of $\det \mathcal{B}$. □

[6] This is Sylvester's theorem. For an easy proof of the *real* version, see §VII.B of my linear algebra notes. (To get rid of the "-1" entries, hence arrive at the simpler *complex* version, just multiply the relevant basis vectors by $\sqrt{-1}$.)
[7] or equivalently, irreducible

4.2. Tangent lines

Let $C = \{F(Z_0, Z_1, Z_2) = 0\}$ be a projective algebraic curve, and suppose

$$\mathbb{R} \supset (-\epsilon, \epsilon) \to C$$
$$t \mapsto \underline{f}(t) = [Z_0(t) : Z_1(t) : Z_2(t)]$$

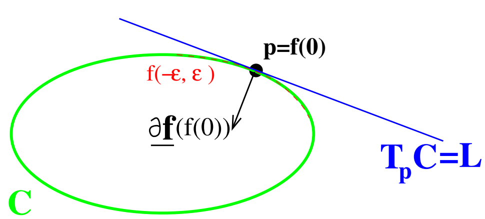
is a differentiable path segment in C. Then,

$$0 = (F \circ f)'(0) = \sum_{i=0}^{2} \frac{\partial F}{\partial Z_i}(f(0)) \cdot \frac{dZ_i}{dt}(0)$$
$$= \begin{pmatrix} \frac{\partial F}{\partial Z_0}(f(0)) & \frac{\partial F}{\partial Z_1}(f(0)) & \frac{\partial F}{\partial Z_2}(f(0)) \end{pmatrix} \begin{pmatrix} Z_0'(0) \\ Z_1'(0) \\ Z_2'(0) \end{pmatrix}$$
$$= {}^t\underline{\partial F}(f(0)) \cdot \underline{f}'(0),$$

and so the line $L_{\underline{\partial F}(f(0))}$ contains all tangent vectors $\underline{f}'(0)$ to all such paths in C through $f(0)$.

There is one catch: if the gradient vector $\underline{\partial F}(f(0)) = \underline{0}$, then it does not define a line at all. So we must ask C to be smooth at $f(0)$ for this computation to work.

4.2.1. DEFINITION. The tangent line T_pC to a curve $C = \{F = 0\} \subset \mathbb{P}^2$ at a smooth point $p = [p_0 : p_1 : p_2] \in C$ is $L_{\underline{\partial F}(p)}$.



The next proposition makes the intuitively obvious statement that "projectivities respect tangent lines":

4.2.2. PROPOSITION. *If L is the tangent line to C at p, then $T(A)L$ is the tangent line to $T(A)C$ at $T(A)p$.*

PROOF. We must show

$$T(A)L_{\underline{\partial F}(p)} = L_{\underline{\partial (F\circ T(A^{-1}))}(T(A)p)}.$$

Writing $\tilde{F} = F \circ T(A)^{-1}$, this is equivalent to

$$T(A)L_{\underline{\partial(\tilde{F}\circ T(A))}(p)} = L_{\underline{\partial\tilde{F}}(T(A)p)}$$

hence to

$$L_{{}^tA^{-1}\underline{\partial(\tilde{F}\circ T(A))}(p)} = L_{\underline{\partial\tilde{F}}(T(A)p)}$$

or

$$\underline{\partial(\tilde{F}\circ T(A))}(p) \equiv {}^tA\underline{\partial\tilde{F}}(T(A)p) \tag{4.2.3}$$

where $\equiv$ means up to multiplication by $\mathbb{C}^*$. As you may wish to check by writing everything out, *equality* of both sides of (4.2.3) is just an expression of the chain rule. □

Now, the tangent line to a *line* (at any point) is the line itself; for conics the story is less trivial. First we write

$$F(\underline{Z}) = {}^t\underline{Z}\mathcal{B}\underline{Z},$$

$\mathcal{B}$ symmetric, and compute the gradient: writing $\underline{e}_0, \underline{e}_1, \underline{e}_2$ for the standard basis vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$,

$$\underline{\partial F}(\underline{Z}) = \begin{pmatrix}\partial F/\partial Z_0\\ \partial F/\partial Z_1\\ \partial F/\partial Z_2\end{pmatrix} = \begin{pmatrix}{}^t\underline{Z}\mathcal{B}\underline{e}_0 + {}^t\underline{e}_0\mathcal{B}\underline{Z}\\ {}^t\underline{Z}\mathcal{B}\underline{e}_1 + {}^t\underline{e}_1\mathcal{B}\underline{Z}\\ {}^t\underline{Z}\mathcal{B}\underline{e}_2 + {}^t\underline{e}_2\mathcal{B}\underline{Z}\end{pmatrix}$$

$$= 2\begin{pmatrix}{}^t\underline{e}_0\mathcal{B}\underline{Z}\\ {}^t\underline{e}_1\mathcal{B}\underline{Z}\\ {}^t\underline{e}_2\mathcal{B}\underline{Z}\end{pmatrix} = 2\mathcal{B}\underline{Z}.$$

(Here ${}^t\underline{Z}\mathcal{B}\underline{e}_i = {}^t({}^t\underline{e}_i{}^t\mathcal{B}\underline{Z}) = {}^t({}^t\underline{e}_i\mathcal{B}\underline{Z}) = {}^t\underline{e}_i\mathcal{B}\underline{Z}$ uses the fact that $\mathcal{B}$ is symmetric and ${}^t\underline{e}_i\mathcal{B}\underline{Z}$ is "1×1", i.e. a scalar.)

4.2.4. PROPOSITION. *The tangent line to $Q_{\mathcal{B}}$ at $[\underline{p}] \in Q_{\mathcal{B}}$ is $L_{\mathcal{B}\underline{p}}$.*[8]

[8] Here we are treating $\underline{p}$ as a column vector (in concert with earlier notation).

4.3. The dual projective plane

Suppose we have a vector space $V/\mathbb{C}$, a basis $\mathbf{e} = \{\underline{e}_i\}$ of V and an invertible linear transformation $\mathcal{T} : V \to V$ with matrix $[\mathcal{T}]_{\mathbf{e}} =: M$. Recall that the dual of V is the vector space

$$\check{V} := Hom(V, \mathbb{C})$$

of linear functionals ($f : V \to \mathbb{C}$); one has the tautological pairing

$$\check{V} \times V \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C} \tag{4.3.1}$$

given by $\langle f, \underline{v} \rangle = f(\underline{v})$. We have a dual basis $\mathbf{e}^* = \{\underline{e}_i^*\}$ ($\left\langle \underline{e}_i^*, \underline{e}_j \right\rangle = \delta_{ij}$), with respect to which one has

$$\langle f, \underline{v} \rangle = \underbrace{{}^t([f]_{\mathbf{e}^*})[\underline{v}]_{\mathbf{e}}}_{\text{matrix multiplication}} .$$

Finally, there is a dual transformation $\check{\mathcal{T}} : \check{V} \to \check{V}$ defined by

$$\left\langle \check{\mathcal{T}} f, \mathcal{T} \underline{v} \right\rangle = \langle f, \underline{v} \rangle$$

with matrix

$$[\check{\mathcal{T}}]_{\mathbf{e}^*} = {}^t M^{-1}.$$

This gives a more conceptual way to look at the story of lines in $\mathbb{P}^2$ above: put $V = \mathbb{C}^3$, $[\mathcal{T}]_{\mathbf{e}} = A$, $f \in \check{V}$, $\underline{\lambda} = [f]_{\mathbf{e}^*}$ and so on. Of course $\mathbb{C}^3 \cong \check{\mathbb{C}}^3$ as vector spaces, but we want to keep them conceptually separate.

The crucial point is to projectivize V *and* $\check{V}$: writing

$$\mathbb{P}^2 = \frac{\mathbb{C}^3 \backslash \{0\}}{\mathbb{C}^*}, \quad \check{\mathbb{P}}^2 = \frac{\check{\mathbb{C}}^3 \backslash \{0\}}{\mathbb{C}^*},$$

we see that lines in $\mathbb{P}^2$ correspond to points $[\underline{\lambda}] \in \check{\mathbb{P}}^2$. In fact, since the notion of duality is defined by (4.3.1), it is symmetric: $\check{\check{V}} = V$, and so points in $\mathbb{P}^2$ correspond to lines in $\check{\mathbb{P}}^2$. This entire correspondence is invariant under projectivities provided one operates simultaneously on $\mathbb{P}^2$ with $T(A)$ and $\check{\mathbb{P}}^2$ with $T({}^t A^{-1})$. A bit more formally, then:

4.3.2. DEFINITION. The dual projective plane $\check{\mathbb{P}}^2$ is the space of lines in $\mathbb{P}^2$.

Now write

$$\underline{p} = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}, \quad \underline{\lambda} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}$$

for column vectors. Though this is maybe a little awkward, here is how I want to standardize notation:

$$p = [\underline{p}] = [p_0 : p_1 : p_2] \in \mathbb{P}^2$$
$$\lambda = [{}^t\underline{\lambda}] = [\lambda_0 : \lambda_1 : \lambda_2] \in \check{\mathbb{P}}^2,$$

in other words, points in $\mathbb{P}^2$ are thought of as *column* vectors and points in $\check{\mathbb{P}}^2$ as row vectors. As above the line $L_{\underline{\lambda}} \subset \mathbb{P}^2$ is defined by the equation

$${}^t\underline{\lambda} \cdot \underline{Z} = 0 \qquad \left(\text{solve for } \underline{Z} = \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix} \right),$$

and we say that its dual $\check{L}_{\underline{\lambda}} = \lambda$. Moreover, p defines a line $L_{\underline{p}} \subset \check{\mathbb{P}}^2$ via

$${}^t\underline{W} \cdot \underline{p} = 0 \qquad \left(\text{solve for } {}^t\underline{W} = \begin{pmatrix} W_0 & W_1 & W_2 \end{pmatrix} \right),$$

and we write $\check{p} = L_{\underline{p}}$.

What about the dual of a configuration of

(a) a point p on a line $L_{(\underline{\lambda})}$ (important in Poncelet)?
(b) a pair of lines L, $\mathcal{L}$ through a point p?
(c) a pair of points p, q on a line L?

For the first one, the equation

$${}^t\underline{\lambda} \cdot \underline{p} = 0$$

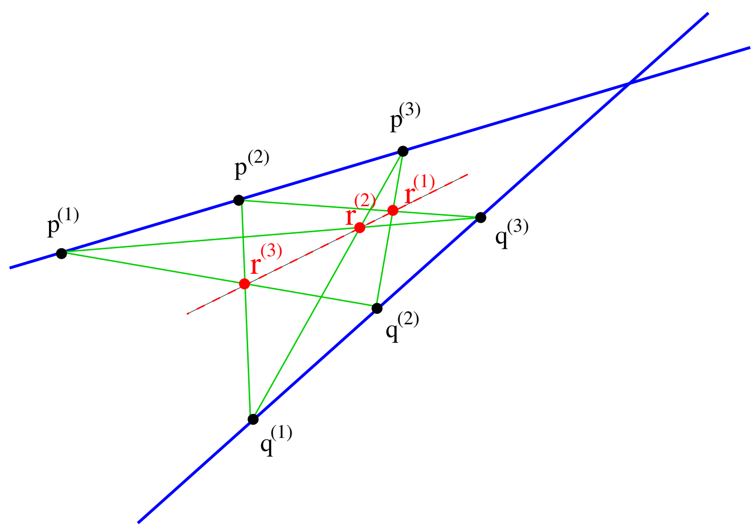
expresses "$p \in L_{\underline{\lambda}}$"; but from the above it also expresses $\check{L}_{\underline{\lambda}} \in \check{p}$ (i.e. $\lambda \in L_{\underline{p}}$). Repeating this reasoning, we have

(ǎ) a line $\check{p}$ through a point $\check{L}$;

($\check{\text{b}}$) a pair of points $\check{L}$, $\check{\mathcal{L}}$ on a line $\check{p}$;
($\check{\text{c}}$) a pair of lines $\check{p}$, $\check{q}$ through a point $\check{L}$.

Here is something more interesting to dualize, which is left as an exercise for you.

4.3.3. THEOREM. [PAPPUS OF ALEXANDRIA, c. 300 AD] *Let* $L, \mathcal{L} \subset \mathbb{P}^2$ *be two distinct lines, and write* $s = L \cap \mathcal{L}$. *On* L *(resp.* $\mathcal{L}$*) take distinct* $p^{(1)}$, $p^{(2)}$, $p^{(3)}$ *(resp.* $q^{(1)}$, $q^{(2)}$, $q^{(3)}$*) different from* s*, and set (for* $k = 1,2,3$*)* $r^{(k)} := \overline{p^{(i)}q^{(j)}} \cap \overline{p^{(j)}q^{(i)}}$ *(where* $\{i,j,k\} = \{1,2,3\}$*). Then* $r^{(1)}$, $r^{(2)}$, $r^{(3)}$ *are collinear.*



PROOF. In fact, $p^{(1)}q^{(2)}p^{(3)}q^{(1)}p^{(2)}q^{(3)}$ is a hexagon "inscribed" in the conic $L \cup \mathcal{L}$, and the $\{r^{(i)}\}$ are the intercepts of its opposite edges. After changing by a projectivity (which preserves the "figure"), this conic is $XY = 0$, which is obviously the limit of the smooth conic $XY - \alpha Z^2 = 0$ as $\alpha \to 0$. Since Pascal's theorem implies collinearity of the hexagon edge intercepts for all $\alpha \neq 0$, this remains true at $\alpha = 0$. □

4.4. Dual conics and polar lines

4.4.1. DEFINITION. The dual $\check{C} \subset \check{\mathbb{P}}^2$ of a smooth algebraic curve $C = \{F = 0\} \subset \mathbb{P}^2$ is the set of (dual points of) tangent lines to C. That is, $\check{C} = \{\check{T_p C} \in \check{\mathbb{P}}^2 \mid p \in C\}$.

This is consistent with our definition for lines. For higher degree curves, however, the dual is not one point: consider the duality map

$$\mathcal{D}_C : \mathbb{P}^2 \longrightarrow \check{\mathbb{P}}^2$$

sending[9]

$$p \longmapsto [{}^t\underline{\partial F}(p)].$$

4.4.2. PROPOSITION. *(a)* $\check{C} = \mathcal{D}_C(C)$.
(b) If $\check{C}$ *is smooth at* $\lambda = \check{T_p C}$*, then* $T_\lambda \check{C} = \check{p}$.

PROOF. (a) For $p \in C$, $T_p C = L_{\underline{\partial F}(p)} \implies \check{T_p C} = [{}^t\underline{\partial F}(p)]$. So this is practically a tautology.

(b) Here we jump into a little deep water. It suffices to show that for any path ${}^t\underline{\lambda}(\cdot) : (-\epsilon, \epsilon) \to \check{C}$ through $\check{T_p C} = [{}^t\underline{\partial F}(p)] = \mathcal{D}_C(p)$,

$$\frac{d\,{}^t\underline{\lambda}}{dt}(0) \cdot \underline{p} = 0. \tag{4.4.3}$$

Since $\check{C}$ is the image of C by $\mathcal{D}_C$, ${}^t\underline{\lambda}(t) = (\mathcal{D}_C \circ \underline{q})(t)$ for some $\underline{q}(\cdot) : (-\epsilon, \epsilon) \to C$ through p. So the left-hand side of (4.4.3) becomes

$$\frac{d}{dt}(\mathcal{D}_C \circ \underline{q})(0) \cdot \underline{p} =$$

(4.4.4)

$$\begin{pmatrix} q_0'(0) & q_1'(0) & q_2'(0) \end{pmatrix} \left.\begin{pmatrix} \frac{\partial^2 F}{\partial Z_0^2} & \frac{\partial^2 F}{\partial Z_0 \partial Z_1} & \frac{\partial^2 F}{\partial Z_0 \partial Z_2} \\ \frac{\partial^2 F}{\partial Z_1 \partial Z_0} & \frac{\partial^2 F}{\partial Z_1^2} & \frac{\partial^2 F}{\partial Z_1 \partial Z_2} \\ \frac{\partial^2 F}{\partial Z_2 \partial Z_0} & \frac{\partial^2 F}{\partial Z_2 \partial Z_1} & \frac{\partial^2 F}{\partial Z_2^2} \end{pmatrix}\right|_p \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}.$$

The matrix in the middle is the *Hessian* of F at p and has nonvanishing determinant if $\check{C}$ is nonsingular at $\mathcal{D}_C(p)$. We will return to the Hessian later in this course.

[9] i.e., $[Z_0 : Z_1 : Z_2] \mapsto \left[\frac{\partial F}{\partial Z_0}(Z_0, Z_1, Z_2) : \frac{\partial F}{\partial Z_1}(Z_0, Z_1, Z_2) : \frac{\partial F}{\partial Z_2}(Z_0, Z_1, Z_2)\right]$

Now, since each $\frac{\partial F}{\partial Z_i}$ is homogeneous (of degree $d-1$, if $d = \deg(C)$), the Euler formula ((2.1.18), with $\underline{Z}$ set equal to $\underline{p}$) collapses (4.4.4) to

$$(d-1).\begin{pmatrix} q_0'(0) & q_1'(0) & q_2'(0) \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial Z_0}(p) \\ \frac{\partial F}{\partial Z_1}(p) \\ \frac{\partial F}{\partial Z_2}(p) \end{pmatrix} =$$

$$(d-1).\underline{\partial F}(\underline{q}(0)) \cdot \frac{d\underline{q}}{dt}(0),$$

which is indeed zero by the beginning of §4.2. □

4.4.5. REMARK. One can show that the dual of a smooth algebraic curve of degree d is an algebraic curve of degree $d(d-1)$; moreover, for $d \geq 2$ this dual is *singular*, so the duality map cannot be "reversed" as defined.

We consider the dual of the conic $Q_{\mathcal{B}}$. By the computation in §4.2 (for $F(\underline{Z}) = {}^t\underline{Z}\mathcal{B}\underline{Z}$), ${}^t\underline{\partial F}(\underline{Z}) = 2\,{}^t\underline{Z}\mathcal{B} \equiv {}^t\underline{Z}\mathcal{B}$ and so

$$\mathcal{D}_{Q_{\mathcal{B}}}(Q_{\mathcal{B}}) = \{[\,{}^t\underline{Z}\mathcal{B} \,|\, [\underline{Z}] \in C]\}$$
$$= \{[\,{}^t\underline{Z}\mathcal{B}] \,|\, {}^t\underline{Z}\mathcal{B}\underline{Z} = 0\}.$$

Making the substitition ${}^t\underline{\lambda} = {}^t\underline{Z}\mathcal{B} \longleftrightarrow \underline{Z} = {}^t\mathcal{B}^{-1}\underline{\lambda}$, this becomes

$$\check{Q}_{\mathcal{B}} = \{[\,{}^t\underline{\lambda}] \in \check{\mathbb{P}}^2 \,|\, {}^t\underline{W}\mathcal{B}^{-1}\mathcal{B}\,{}^t\mathcal{B}^{-1}\underline{W} = 0\}$$
$$= \{\lambda \in \check{\mathbb{P}}^2 \,|\, {}^t\underline{\lambda}\mathcal{B}^{-1}\underline{\lambda} = 0\}$$

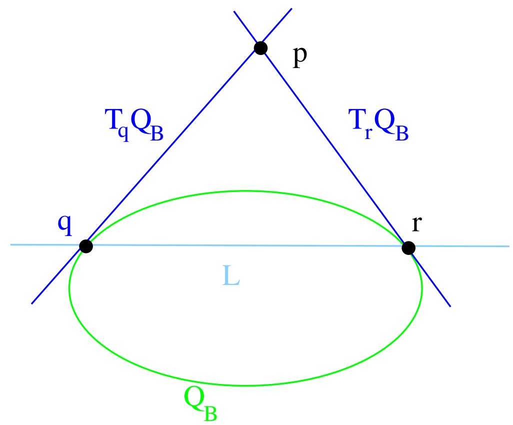
where we have used the fact that $\mathcal{B}$ is symmetric. This gives part (i) of:

4.4.6. PROPOSITION. *(i)* $\check{Q}_{\mathcal{B}} = Q_{\mathcal{B}^{-1}}$, *and* $\check{\check{Q}}_{\mathcal{B}} = Q_{\mathcal{B}}$. *(In particular, the dual of a smooth conic is a* smooth *conic, since* $\det \mathcal{B} \neq 0 \implies \det \mathcal{B}^{-1} \neq 0$.*)*

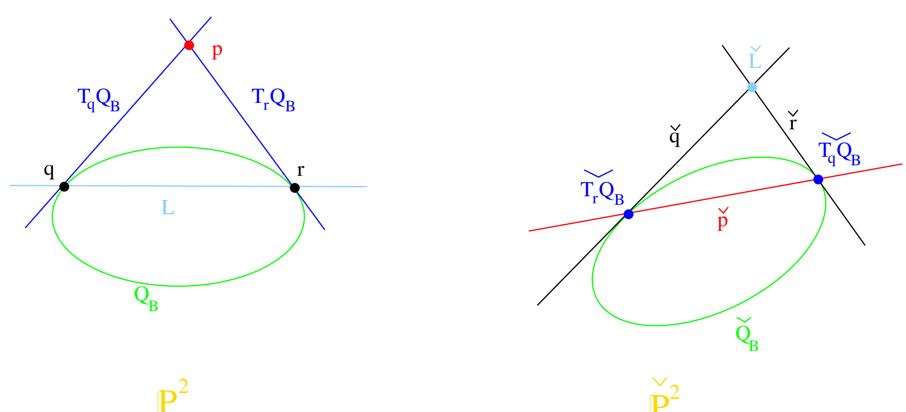
(ii) Given $p \in \mathbb{P}^2 \backslash Q_{\mathcal{B}}$, *there exist exactly two lines through* p *and tangent to* $Q_{\mathcal{B}}$.

PROOF. (ii) By Proposition 4.4.2(b), this is *dual* to the statement: if the line $\check{p} \subset \check{\mathbb{P}}^2$ is not tangent to $\check{Q}_{\mathcal{B}}$, then it meets $\check{Q}_{\mathcal{B}}$ in exactly 2 points. This last statement then follows from Proposition 2.1.15. □

4.4.7. DEFINITION. Let $Q_{\mathcal{B}}$ be a smooth conic and p be a point not on $Q_{\mathcal{B}}$, with $T_q Q_{\mathcal{B}}$ and $T_r Q_{\mathcal{B}}$ the two tangent lines to $Q_{\mathcal{B}}$ containing p. (Here $q, r \in Q_{\mathcal{B}}$.) Then the *polar line* $L_{(p,Q_{\mathcal{B}})} \subset \mathbb{P}^2$ of p with respect to $Q_{\mathcal{B}}$ is the line through q and r.



4.4.8. PROPOSITION. *Let $p \in \mathbb{P}^2 \backslash Q_{\mathcal{B}}$, with polar line $L = L_{(p,Q_{\mathcal{B}})} (\subset \mathbb{P}^2)$. Then the polar line $L_{(\check{L},\check{Q}_{\mathcal{B}})} \subset \check{\mathbb{P}}^2$ (of the dual point $\check{L}$ with respect to the dual conic) is $\check{p}$ (the dual line of p). In a picture, where dual objects are the same color:*



PROOF. This is an immediate consequence of the rules (a,b,c) $\longleftrightarrow$ ($\check{a}, \check{b}, \check{c}$), the definition of the dual curve, and Proposition 4.4.2(b). □

4.4.9. EXAMPLE. $Q = \{-4Z_0^2 + Z_1^2 + Z_2^2 = 0\} \longrightarrow \mathcal{B} = \begin{pmatrix} -4 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Let $p = [1:2:2]$ ($\notin Q$) so that the polar line L is given by[10]

$$0 = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -4 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix} = -4Z + 2X + 2Y,$$

i.e. $L = L_{\underline{\lambda}}$ where $\lambda = [-4:2:2]$.

On the dual $\check{\mathbb{P}}^2$ side: $\check{L} = \lambda$; $\check{Q}$ has matrix $\mathcal{B}^{-1} = \begin{pmatrix} -\frac{1}{4} & & \\ & 1 & \\ & & 1 \end{pmatrix}$ hence equation $0 = -\frac{1}{4}Z_0^2 + Z_1^2 + Z_2^2$; and $\check{p}$ is the line $0 = W_0 + 2W_1 + 2W_2$. On the other hand, the polar line of λ with respect to $\check{Q}$ is

$$0 = \begin{pmatrix} -4 & 2 & 2 \end{pmatrix} \begin{pmatrix} -\frac{1}{4} & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix} = W_0 + 2W_1 + 2W_2,$$

agreeing with $\check{p}$.

It is instructive to think about what happens to Pascal and Poncelet under duality. While the dual of Poncelet is again just Poncelet (but in $\check{\mathbb{P}}^2$), we *do* find that if polygons inscribed in C and circumscribed about D close up after n sides, then so do the polygons (in $\check{\mathbb{P}}^2$) inscribed in $\check{D}$ and circumscribed about $\check{C}$.

The dual of Pascal, on the other hand, does give a different statement:

4.4.10. PROPOSITION. *The three lines through opposite vertices of a hexagon circumscribed about a conic, pass through a single point.*

Proof is basically the same as for Proposition 4.4.8; I'll let you work it out.

Exercises

(1) Given a configuration of 4 points $p, q, r, s \in \mathbb{P}^2$ in "general position", i.e. no three of them collinear, show there exists a unique projectivity sending $p \mapsto [1:0:0]$, $q \mapsto [0:1:0]$, $r \mapsto [0:$

[10]See Exercise 3 below.

$0 : 1]$, $s \mapsto [1 : 1 : 1]$. [Hint: work with vectors $\underline{p}, \underline{q}, \underline{r}, \underline{s} \in \mathbb{C}^3$. You only have to send $\underline{p}$ (resp. $\underline{q}, \underline{r}, \underline{s}$) to a multiple of $\underline{e}_0$ (resp. $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_0 + \underline{e}_1 + \underline{e}_2$).]

(2) (a) Give a direct proof of Pappus's theorem. [Hint: use the last exercise to first simplify the coordinates of several of the points.] (b) State a dual version of Pappus's theorem, and draw a figure. [Note: it would be better to state the dual version in $\mathbb{P}^2$: think first of Pappus in $\check{\mathbb{P}}^2$ and then dualize that.]

(3) Show that the equation of the polar line of p with respect to $Q_{\mathcal{B}}$ has equation ${}^t\underline{p}\mathcal{B}\underline{Z} = 0$. Use this to give another (short) proof of Proposition 4.4.8.

(4) Prove that all automorphisms of $\mathbb{P}^1$ (as a complex manifold) are fractional linear transformations. Deduce that

$$Aut(\mathbb{P}^1) \cong \mathrm{PGL}_2(\mathbb{C}).$$

[Use material from §3.1.]

(5) Given five points, no four of which are collinear, show that there is a *unique* conic through them by the following steps: (a) Show that a projectivity sends the points to [1:0:0], [0:1:0], [0:0:1], [1:a:1], and [1:1:b], with $a, b \in \mathbb{C}$ not both 1. (b) Consider the map from $S_3^2 :=$ {homogeneous polynomials of degree two in Z_0, Z_1, Z_2} to $\mathbb{C}^5$ given by evaluating at these 5 points (or rather, at the five points $(1,0,0), \ldots, (1,1,b) \in \mathbb{C}^3$, so as to get a well-defined map). Write a matrix for this map and show it has maximal rank. (c) As projectivities send conics to conics, conclude the desired uniqueness.