CHAPTER 5

Complex manifolds and algebraic varieties

In Chapters 2-3 we introduced Riemann surfaces and plane algebraic curves, and stated the Normalization Theorem which produces a strong relation between them. Here we will introduce the arbitrary-dimensional generalizations of these objects. While it is true that an algebraic variety of dimension n has a desingularization[1] which is a complex n-manifold, the converse is false: already there are non-algebraic complex 2-manifolds.

However, it does turn out that any global analytic object (functions or differential forms, for example) on a projective algebraic variety viewed as a complex manifold, is algebraic. This is Serre's "GAGA" (global analytic = global algebraic) principle. For example, global meromorphic functions in this context turn out to be nothing but restrictions to the algebraic variety of rational functions on the ambient projective space (elements of $\mathbb{C}(z_1, \ldots, z_n)$).

5.1. Complex n-manifolds

These are the generalization of complex 1-manifolds (or of Riemann surfaces, if we assume compactness) to higher dimension. Once again, we begin with a (Hausdorff, second-countable) topological space X with open cover $\{U_\alpha\}$ and write $U_{\alpha\beta} = U_\alpha \cap U_\beta$. This is made into a *complex n-manifold* by the additional data of an analytic atlas on X: that is, a collection of holomorphic coordinates[2]

$$\underline{z_\alpha} : U_\alpha \xrightarrow{\simeq} V_\alpha \subseteq \mathbb{C}^n,$$

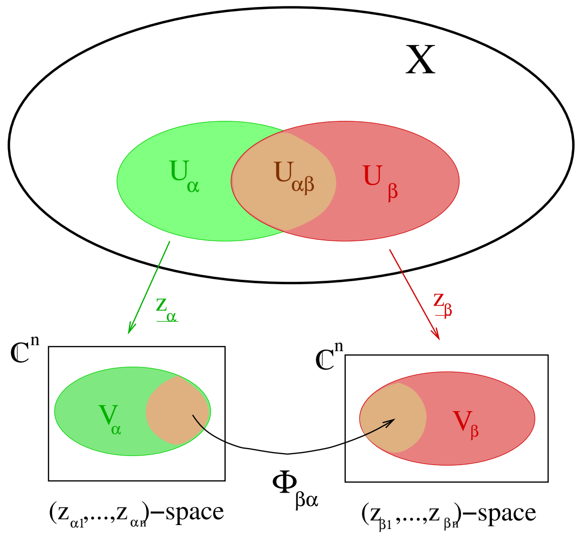
[1]"Normalization" is no longer the correct term: it refers to a weaker process which can still produce a singular object.

[2]As usual, an underline means a vector or tuple of some kind; in this case, $\underline{z_\alpha} = (z_{\alpha 1}, \ldots, z_{\alpha n})$.

or homeomorphisms between U_α and an open set in $\mathbb{C}^n$, such that the transition functions

$$\Phi_{\beta\alpha} := \underline{z_\beta} \circ \underline{z_\alpha}^{-1} : V_\alpha^\beta \to V_\beta^\alpha$$

are biholomorphic.



Here $V_\alpha^\beta := \underline{z_\alpha}(U_{\alpha\beta})$ and $V_\beta^\alpha := \underline{z_\beta}(U_{\alpha\beta})$ are open subsets of $\mathbb{C}^n$, and we need to explain what biholomorphic means. First, a function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if it looks locally (about each point) like $f(\underline{z}) = \sum_{I=(i_1,\ldots,i_n)} a_I \underline{z}^I$ where $\underline{z}^I$ means $z_1^{i_1} z_2^{i_2} \cdots z_n^{i_n}$ and $a_I \in \mathbb{C}$ are constants. A holomorphic map $\mathbb{C}^n \to \mathbb{C}^m$ is just m of these: $(z_1, \ldots z_n) = \underline{z} \mapsto (f_1(\underline{z}), \ldots, f_m(\underline{z}))$. (Since these definitions are local, they immediately have meaning when $\mathbb{C}^n$ etc. are replaced by open sets.) Finally, "biholomorphic" simply indicates a bijective map ($\Phi_{\beta\alpha}$ is bijective by construction) which is holomorphic in each direction.

To generalize the morphisms of Riemann surfaces introduced in §3.1, we can define a *morphism $F : X \to Y$ of complex manifolds*. Here X and Y need *not* be of the same dimension; for X we keep the above notation and for Y write $\underline{\mathfrak{z}_\gamma} : \mathcal{U}_\gamma \xrightarrow{\simeq} \mathcal{V}_\gamma \subseteq \mathbb{C}^m$. A morphism F is then a collection of continuous functions $F_\alpha : U_\alpha \to Y$ (agreeing on the $U_{\alpha\beta}$) such that each composition

$$\underline{\mathfrak{z}_\gamma} \circ F_\alpha \circ \underline{z_\alpha}^{-1} : \underline{z_\alpha}\left(F_\alpha^{-1}(\mathcal{U}_\gamma \cap F_\alpha(U_\alpha))\right) \to \underline{\mathfrak{z}_\gamma}\left(\mathcal{U}_\gamma \cap F_\alpha(U_\alpha)\right)$$

yields a holomorphic map (from a subset of $V_\alpha \subseteq \mathbb{C}^n$ to a subset of $\mathcal{V}_\gamma \subseteq \mathbb{C}^m$). If $n = m = 1$ then this reproduces Definition 3.1.8.[3] Moreover, compositions of morphisms are morphisms.

Basic examples of complex manifolds include (besides Riemann surfaces when $n = 1$) Cartesian products of Riemann surfaces, complex n-tori

$$\mathbb{C}^n / \mathbb{Z}\left\langle \underline{\lambda_1}, \cdots, \underline{\lambda_{2n}} \right\rangle$$

(where $\underline{\lambda_1}, \ldots, \underline{\lambda_{2n}}$ are linearly independent over $\mathbb{R}$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$), and projective n-space[4]

$$\mathbb{P}^n := \frac{\mathbb{C}^{n+1} \backslash \{\underline{0}\}}{\langle (\xi_0, \ldots, \xi_n) \sim (\gamma\xi_0, \ldots, \gamma\xi_n) \ \ \forall \gamma \in \mathbb{C}^* \rangle}.$$

Demonstrating that $\mathbb{P}^n$ is a complex n-manifold (as we do in the next section) immediately gives meaning to a "morphism of complex manifolds from a Riemann surface to $\mathbb{P}^n$." This notion is equivalent to (but more intrinsic than) Definition 3.1.11, as we shall see.

5.2. $\mathbb{P}^n$ as a complex manifold

$\mathbb{P}^n$ is covered by the open sets $U_i := \{\xi_i \neq 0\}$, with local coordinates

$$\underline{z_i} = (z_{i1}, \ldots, z_{in}) := \left(\frac{\xi_0}{\xi_i}, \ldots, \frac{\widehat{\xi_i}}{\xi_i}, \ldots, \frac{\xi_n}{\xi_i}\right) : \ U_i \xrightarrow{\cong} \mathbb{C}^n.$$

(Here "i" replaces "α", $V_i = \mathbb{C}^n$, and $\widehat{(\cdot)}$ means to omit that term.) We need to check that the transition functions

$$\Phi_{ji} : \ V_i^j \to V_j^i$$

[3] It might be a good idea to glance back at the picture there (for intuition purposes).

[4] In §§5.1-5.2, $[\xi_0 : \cdots : \xi_n]$ is used for coordinates on $\mathbb{P}^n$ instead of $[Z_0 : \cdots : Z_n]$ (since the back-and-forth between z_i and Z_j otherwise becomes unreadable).

are holomorphic. Now, Φ_{ji} tells us how to write the $\underline{z_j} = (z_{j1}, \ldots, z_{jn})$ as functions of the $\underline{z_i} = (z_{i1}, \ldots, z_{in})$ in such a way that

$$\left(\frac{\zeta_0}{\zeta_i}, \ldots, \frac{\widehat{\zeta_i}}{\zeta_i}, \ldots, \frac{\zeta_j}{\zeta_i}, \ldots, \frac{\zeta_n}{\zeta_i}\right) \text{ is sent to } \left(\frac{\zeta_0}{\zeta_j}, \ldots, \frac{\zeta_i}{\zeta_j}, \ldots, \frac{\widehat{\zeta_j}}{\zeta_j}, \ldots, \frac{\zeta_n}{\zeta_j}\right)$$

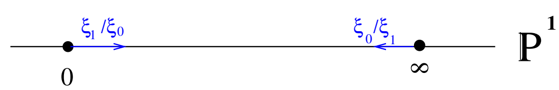
(where for convenience we assume $j > i$). Moreover, $V_i^j \subset \mathbb{C}^n$ is simply the subset where $z_{ij} \neq 0$. So the correct transition function is

$$\Phi_{ji}(z_{i1}, \ldots, z_{in}) = (z_{j1}(z_{i1}, \ldots, z_{in}), \ldots, z_{jn}(z_{i1}, \ldots, z_{in}))$$

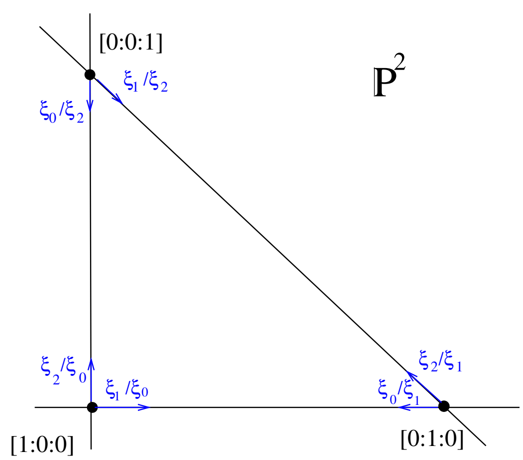
where

$$z_{jk}(z_{i1}, \ldots, z_{in}) = \begin{cases} z_{ik}/z_{ij}, & \text{for } k \leq i,\ k > j \\ z_{i,k-1}/z_{ij}, & \text{for } i+1 < k \leq j \\ 1/z_{ij,} & \text{for } k = i+1 \end{cases} . \tag{5.2.1}$$

For $\mathbb{P}^1$, $\underline{z_i}$ reduces to z_i ($i = 0, 1$). More precisely, $z_0 = \frac{\zeta_1}{\zeta_0}$ and $z_1 = \frac{\zeta_0}{\zeta_1}$ are the two local coordinates, while (5.2.1) becomes $z_1(z_0) = \frac{1}{z_0}$, so that we recover Example 2.2.4. Here is a "schematic picture" of the local coordinates on $\mathbb{P}^1$:



For $\mathbb{P}^2$, we have $\underline{z_0} = (z_{01}, z_{02}) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}\right)$, $\underline{z_1} = (z_{11}, z_{12}) = \left(\frac{\zeta_0}{\zeta_1}, \frac{\zeta_2}{\zeta_1}\right)$, and $\underline{z_2} = (z_{21}, z_{22}) = \left(\frac{\zeta_0}{\zeta_2}, \frac{\zeta_1}{\zeta_2}\right)$, with e.g. $\Phi_{20}\,(z_{01}, z_{02}) = \left(\frac{1}{z_{02}}, \frac{z_{01}}{z_{02}}\right)$. Again, the local coordinates can be visualized as follows:

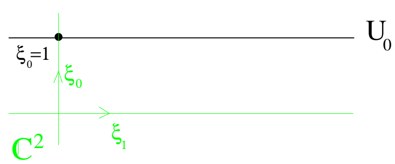


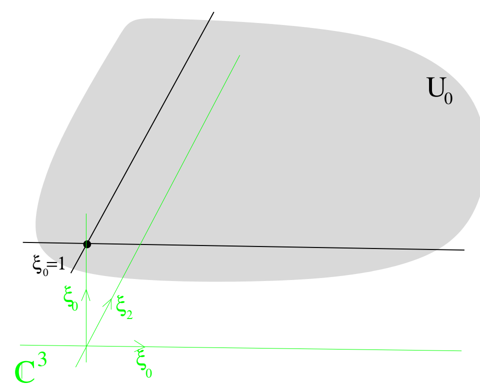
So, for instance, the coordinates $\underline{z_1} = \left(\frac{\xi_0}{\xi_1}, \frac{\xi_2}{\xi_1}\right)$ are defined on the complement U_1 of the vertical line, and both vanish at $[0:1:0]$.

5.2.2. REMARK. Whenever you have a local holomorphic coordinate (system) like $\underline{z_i} : U_i \to V_i \subseteq \mathbb{C}^n$, the inverse mapping $\varphi_i = \underline{z_i}^{-1} : V_i \xrightarrow{\cong} U_i \subset X$ (or just $V_i \hookrightarrow X$) is called a local analytic chart. In case $X = \mathbb{P}^n$, $\varphi_i : \mathbb{C}^n \hookrightarrow \mathbb{P}^n$ is given by

$$\varphi_i(z_{i1}, \ldots, z_{in}) = [z_{i1} : \cdots : z_{ii} : 1 : z_{i,i+1} : \cdots : z_{in}],$$

and one can visualize this as a map from $\mathbb{C}^n \hookrightarrow (\mathbb{C}^{n+1} \backslash \{\underline{0}\}) \twoheadrightarrow \mathbb{P}^n$. Here are pictures of the image of φ_0 for $\mathbb{P}^1$ and $\mathbb{P}^2$:





The next statement says that the notion of "holomorphic map" from a Riemann surface to projective space (Defn. 3.1.11) is just a special case of "morphism of complex manifolds". It is enough to consider (as we do) the situation where the image is not contained in a coordinate hyperplane, since these are just smaller-dimensional projective spaces (included into $\mathbb{P}^n$ by morphisms).

5.2.3. PROPOSITION. *Let M be a complex 1-manifold, and consider a continuous mapping $F : M \to \mathbb{P}^n$ with $F(M)$ not contained in any $\{\xi_i = 0\}$. The following statements are then equivalent:*

(i) F is a morphism of complex manifolds;

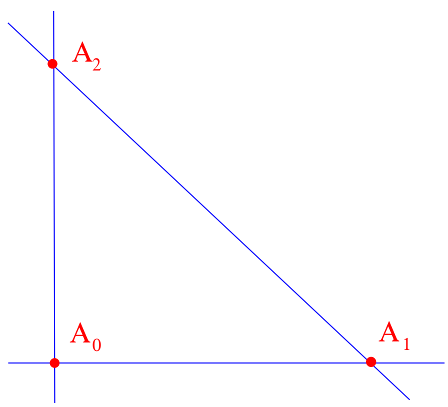
(ii) each composition $[\xi_i \circ F : \xi_j \circ F]$ is a morphism of complex manifolds to $\mathbb{P}^1$ (on the open subset of M where it is well-defined);

(iii) each $\frac{\xi_i}{\xi_j} \circ F$ gives a meromorphic function on M.

PROOF. First, write $U_i := \{\xi_i \neq 0\} \subset \mathbb{P}^n$ as above. For each $\{i,j\}$ (where $i \neq j$), the projections $\pi_{ij} : U_i \cup U_j \twoheadrightarrow \mathbb{P}^1$ defined by $[\xi_0 : \cdots : \xi_n] \mapsto [\xi_i : \xi_j]$ are morphisms of complex manifolds. So if $M \xrightarrow{F} \mathbb{P}^n$ is one, then $\pi_{ij} \circ F = [\xi_i \circ F : \xi_j \circ F]$ is one too, showing $(i) \implies (ii)$. Next, $(ii) \implies (iii)$ is Remark 3.1.12. Finally, if all the $\frac{\xi_j}{\xi_i} \circ F$ give meromorphic functions on all of M, then in particular $\underline{z_i} \circ F = \left(\frac{\xi_0}{\xi_i} \circ F, \ldots, \widehat{\frac{\xi_i}{\xi_i} \circ F}, \ldots, \frac{\xi_n}{\xi_i} \circ F \right)$ is holomorphic on $F^{-1}(U_i)$ (or a suitable covering of it by coordinate neighborhoods). These give the local holomorphic representations of F required for a morphism, proving $(iii) \implies (i)$. $\square$

We will refine Proposition 5.2.3 in Chapter 7 below.

Whilst we are dwelling on the subject of projective space, I would like to mention (just for $\mathbb{P}^2$) a trick for drawing the *real* solution sets of homogeneous equations on the page: *barycentric coordinates*. First draw 3 points $A^{(0)}$, $A^{(1)}$, $A^{(2)}$ on a piece of paper:

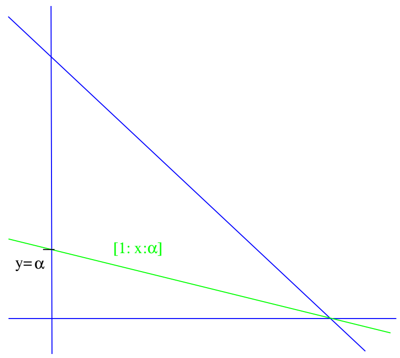


Think of these as vectors $\underline{A^{(i)}} \in \mathbb{R}^2$; it doesn't matter where the origin is. Now, plot $[\xi_0 : \xi_1 : \xi_2]$ as

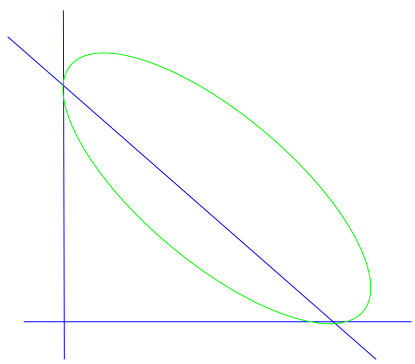
$$\sum_{i=0}^{2} \left(\frac{\xi_i}{\sum_{j=0}^{2} \xi_j} \right) \underline{A^{(i)}}. \tag{5.2.4}$$

To "draw" an algebraic curve, simply find all the solutions $[\xi_0 : \xi_1 : \xi_2]$ with $\xi_i \in \mathbb{R}$, and use (5.2.4) to plot them.

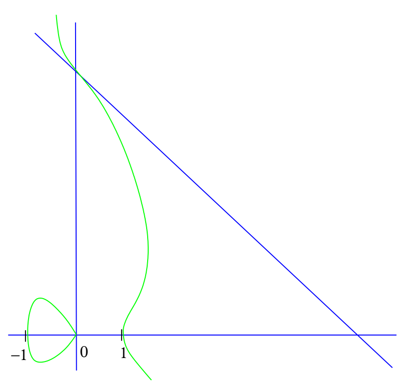
5.2.5. EXAMPLE. (i) The line $y = \alpha$ (assume $\alpha \in \mathbb{R}$) projectively completes to $\xi_2 = \alpha\xi_0$. Plotting the points $[1 : x : \alpha]$ in this way gives



(ii) The conic $xy = 1$ completes to $\xi_1\xi_2 = (\xi_0)^2$, and its real barycentric plot is



(iii) The cubic curve $y^2 = x(x+1)(x-1)$ becomes $(\xi_2)^2\xi_0 = \xi_1(\xi_1 + \xi_0)(\xi_1 - \xi_0)$ with picture



In fact, this is the precise meaning of the "schematic" real one-dimensional pictures of complex algebraic curves which we have been drawing and will continue to draw — we are plotting the real solutions in barycentric coordinates.

5.3. Affine and projective algebraic varieties

We are going to approach this from a slightly more algebraic angle than, "take the common solution of a bunch of polynomial equations". Start with the commutative ring $S_n := \mathbb{C}[z_1, \ldots, z_n]$ of polynomials in n variables.

Let $J \subset S_n$ be an ideal. The *affine variety* associated to J is

$$V(J) := \{\underline{z} = (z_1, \ldots, z_n) \mid f(\underline{z}) = 0 \ \ \forall f \in J\} \subseteq \mathbb{C}^n,$$

which is the vanishing locus of all polynomials in J. By a result in algebra known as *Hilbert's basis theorem*, any ideal in S_n is finitely generated, that is, of the form $(f_1, \ldots, f_k)$; consequently $V(J)$ is simply of the form $f_1(\underline{z}) = \cdots = f_k(\underline{z}) = 0$. However, working in terms of ideals does have a payoff, in the form of the famous "theorem on zeroes" or *Nullstellensatz*:

5.3.1. THEOREM. [D. HILBERT, 1893] *If $g \in S_n$ vanishes identically on $V(J)$, then for some $m \in \mathbb{N}$, g^m belongs to J.*

If $J = (f)$, then this just says "if g vanishes (in $\mathbb{C}^n$) wherever f does, then f divides some power of g."

Next we consider the projective case, writing $S_{n+1} = \mathbb{C}[Z_0, \ldots, Z_n]$. Its underlying additive group can be viewed as the direct sum $\oplus_d S^d_{n+1}$, where S^d_{n+1} denotes homogeneous polynomials of degree d in $n+1$ variables. Hence, any polynomial G can be written uniquely as a finite sum of homogeneous terms $\sum_d G_d$.

5.3.2. DEFINITION. An ideal $I \subset S_{n+1}$ is *homogeneous* if and only if the condition

$$G \in I \quad \Longrightarrow \quad G_d \in I \ \ (\forall d)$$

is satisfied.

Given a homogeneous ideal $I \subset S_{n+1}$, the associated *projective variety* is defined by[5]

$$\bar{V}(I) := \{[\underline{Z}] = [Z_0 : \cdots : Z_n] \mid F(\underline{Z}) = 0 \ \forall F \in I\} \subseteq \mathbb{P}^n.$$

A version of the Nullstellensatz suited to this case, which is an immediate consequence of Theorem 5.3.1, is:

5.3.3. COROLLARY. *Given a homogeneous polynomial g vanishing on all of $\bar{V}(I)$, some power of g belongs to I.*

5.3.4. REMARK. If $F_1, \ldots, F_k$ are homogeneous polynomials (of various degrees), then
(i) $I := (F_1, \ldots, F_k)$ is a homogeneous ideal (exercise); and
(ii) $\bar{V}(I) = \{F_1(\underline{Z}) = \cdots = F_k(\underline{Z}) = 0\}$.

As in the case of curves, we want to be able to go between the affine and projective settings. To "restrict" a projective variety to the affine world, start with the surjective ring (or algebra) homomorphism

$$S_{n+1} \twoheadrightarrow S_n$$

induced by

$$F(Z_0, Z_1, \ldots, Z_n) \mapsto F(1, z_1, \ldots, z_n).$$

If we write I° for the image of a homogeneous ideal I under this map, then

$$V(I^\circ) = \bar{V}(I) \cap \mathbb{C}^n.$$

To go the other way, recall the space $\mathcal{P}_n^d = \oplus_{j=1}^d S_n^j$ of polynomials of degree at most d. We have a homomorphism of abelian groups (or vector spaces)

$$\mathcal{P}_n^d \xrightarrow{\theta_d} S_{n+1}^d$$

called the *homogenization* map, which is defined by

$$f(\underline{z}) \longmapsto (Z_0)^d f(\underline{Z}/Z_0).$$

[5] Technically, one should keep track of multiplicities of irreducible components, rather than just defining $\bar{V}(I)$ as a set. For the most part we will suppress this detail.

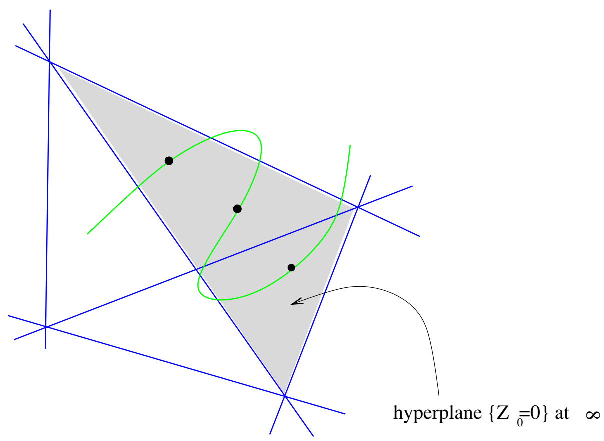
Now, take $J \subset S_n$ an ideal. To homogenize J, simply set

$$\bar{J} := (\{\theta_d(f) \mid f \in J,\ \deg(f) = d\}); \tag{5.3.5}$$

we then evidently have

$$\bar{V}(\bar{J}) \cap \mathbb{C}^n = V(J).$$

So if $n = 3$, then $\bar{V}(\bar{J})$ is adding stuff in the "$\mathbb{P}^2$ at infinity" $\{Z_0 = 0\}$ to complete your affine variety to a projective one, as suggested by the picture:



in which the black points get added in the process of completing the curve.

5.3.6. EXAMPLE. A key example of affine or projective varieties are the *hypersurfaces* cut out of $\mathbb{C}^n$ or $\mathbb{P}^n$ by a single equation. Let $F \in S^d_{n+1}$ and[6]

$$X = V(F) = \{F(\underline{Z}) = 0\} \subseteq \mathbb{P}^n$$

the corresponding projective hypersurface of degree d. (Like algebraic curves, these are called linear, quadric, cubic, quartic, quintic, etc. according as $d = 1, 2, 3, 4, 5, \ldots$.) We will define dimension rigorously below, but X is $(n-1)$-dimensional (and thus of *codimension* one).

[6]Here we really mean $V((F))$, the variety of the ideal (F), but we shorten this to $V(F)$.

Now since S_n is a unique factorization domain, we can factor

$$F = \prod_{i=1}^{k} F_i^{m_i},$$

uniquely (up to order), where each F_i is prime (irreducible). We can then write unambiguously[7]

$$X = \sum_{i=1}^{k} m_i X_i,$$

and say X is *reduced* when all $m_i = 1$, and *irreducible* when $k = 1$.

5.3.7. REMARK. Given generators $\{f_i\}_{i=1}^k$ (f_i of degree d_i) for an ideal $J \subset S_n$, you may wonder whether $\bar{J} = \left(\theta_{d_1}(f_1), \ldots, \theta_{d_k}(f_k)\right)$. This is true when J is a principal ideal, making the projective closure of a hypersurface an easy matter. However, it is not true in general: already the *twisted cubic curve* in $\mathbb{P}^3$ provides a counterexample (see the Exercises).

Here is one more bit of terminology: a variety inside another variety is called a *subvariety*. So for instance one may refer to a projective variety $X \subset \mathbb{P}^3$ as a subvariety of $\mathbb{P}^3$. (See the Exercises for a more interesting example.)

Exercises

(1) Show that each projection $\pi_{ij} : U_i \cup U_j \twoheadrightarrow \mathbb{P}^1$ described in the proof of Prop. 5.2.3 is a morphism of complex manifolds. [This is a quick one.]

(2) Sketch the real solutions of [the projective closure of]

$$y^2 = \prod_{i=1}^{2g+2} (x - a_i)$$

in $\mathbb{P}^2$, if $a_1 < a_2 < \cdots < a_{2g+2}$ are real numbers.

(3) Use $S_{n+1}^d = \oplus_{k \le d} S_n^d$ (as abelian groups) to compute $\sum_{k \le d} \binom{n+k-1}{n-1}$.

(4) (a) Check Remark 5.3.4(i). (b) Show that homogeneous ideals are generated by finitely many homogeneous polynomials.

[7]but completely heuristically, since as defined above X is just a set.

(5) Show that the Fermat cubic surface $X \subset \mathbb{P}^3$ defined by $\sum_{i=0}^3 Z_i^3 = 0$ contains 27 lines. [You can describe the lines parametrically or by equations. Here is one to start you off: the line parametrized (by $[T_0{:}T_1] \in \mathbb{P}^1$) via $[T_0{:}T_1{:}-T_0{:}-T_1]$.]

(6) Prove that $\bar{V}(\bar{J})$ is the smallest subvariety of $\mathbb{P}^n$ whose intersection with $\mathbb{C}^n$ (i.e. $\mathbb{P}^n \setminus \{Z_0 = 0\}$) contains $V(J)$. [Hint: use Theorem 5.3.1.]

(7) The *twisted cubic curve* $C \subset \mathbb{C}^3$ parametrized by $t \mapsto (t, t^2, t^3)$ can be defined by the ideal $J = (f_1, f_2) \subset S_3$ where $f_1 = z_2 - z_1^2$ and $f_2 = z_3 - z_1^3$; that is, $C = V(J)$. Show that the homogeneous ideal $I := (F_1, F_2) \subset S_4$ where $F_1 = \theta_2(f_1) = Z_0Z_2 - Z_1^2$ and $F_2 = \theta_3(f_2) = Z_0^2Z_3 - Z_1^3$ is *not* the same as $\bar{J}$. [Hint: consider $f_3 = z_2^2 - z_1z_3$; first, is it in J?]

(8) Introduce the *grlex* order on monomials $\underline{z}^{\underline{a}}$ by first using total degree, then using lexicographic order to break ties. Let $\mathrm{lt}(f)$ denote the leading (i.e. highest order) term of $f \in S_n$ under grlex order, and $\mathrm{lt}(J)$ the ideal generated by leading terms of elements of J. A *Groebner basis* of J is a generating set $\{f_i\}_{i=1}^k$ with the property that $(\mathrm{lt}(f_1), \ldots, \mathrm{lt}(f_k)) = \mathrm{lt}(J)$; when $\{f_i\}_{i=1}^k$ is a Groebner basis, it is a theorem[8] that $\bar{J} = (\theta_{d_1}(f_1), \ldots, \theta_{d_k}(f_k))$. Assuming this theorem, find a Groebner basis for the ideal J from the previous problem. [Hint: consider, for each $n \in \mathbb{N}$, all monomials $\underline{z}^{\underline{a}}$ with $a_1 + 2a_2 + 3a_3 = n$. Your Groebner basis should have four elements.]

(9) Reduce the basis found in the last problem to three homogeneous polynomials of degree 2 (as generators for $\bar{J}$). [Alternatively, just use the three polynomials appearing in Example 6.3.9(iii).] Use this to describe the intersections of $\bar{V}(\bar{J})$ and $\bar{V}(I)$ with $\{Z_0 = 0\}$ (the "$\mathbb{P}^2$ at ∞" in $\mathbb{P}^3$).

[8] A proof can be found in [Cox-Little-O'Shea, *Ideals, Varieties, and Algorithms*, §8.4]. Algorithms for producing Groebner bases are a part of "computational algebraic geometry".