CHAPTER 6

More on projective algebraic varieties

We warm up with two examples we can get our hands on immediately: linear varieties and quadric hypersurfaces. Then we turn to what it means for an algebraic variety to be singular resp. smooth at a point, and in the latter case introduce its tangent space at that point (which is a linear variety). This leads to a geometric definition of *dimension* for algebraic varieties. We conclude with a short introduction to plane curve singularities (glossed over in Chapter 2).

6.1. Linear subvarieties of $\mathbb{P}^n$

We start by generalizing the "projectivities" of Chapter 4. Recall that the projective general linear group is defined as the quotient of invertible matrices by the scalar action:

$$PGL(n+1,\mathbb{C}) := \frac{GL(n+1,\mathbb{C})}{\left\langle \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{pmatrix} \middle| \, \alpha \in \mathbb{C}^* \right\rangle}.$$

This group acts on projective space by the rule

$$PGL(n+1,\mathbb{C}) \times \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

$$(M, [\underline{Z}]) \longmapsto [M.\underline{Z}] =: T(M)[\underline{Z}].$$

That is, for each $M \in PGL(n+1,\mathbb{C})$, $T(M)$ gives an automorphism of $\mathbb{P}^n$ as a complex manifold. In fact, the projectivities $T(M)$ give *all* automorphisms (generalizing an exercise from Chapter 4), but we won't prove this here.

A system of k linear equations

$$\left\{ \begin{array}{rcl} \ell_{10}\xi_0 + \cdots + \ell_{1n}\xi_n & = & 0 \\ & \vdots & \\ \ell_{k0}\xi_0 + \cdots + \ell_{kn}\xi_n & = & 0 \end{array} \right\} \tag{6.1.1}$$

defines a linear subspace $V \subseteq \mathbb{P}^n$. Recalling that the rank of a matrix is its number of linearly independent row (or equivalently, column) vectors, the matrix

$$\begin{pmatrix} \ell_{10} & \cdots & \ell_{1n} \\ \vdots & \ddots & \vdots \\ \ell_{k0} & \cdots & \ell_{kn} \end{pmatrix} =: L$$

has $\text{rank}(L) =: r \leq k$. Defining

$$\text{codim}(V) := r \quad (\text{equivalently, } \dim(V) = n - r),$$

we have the

6.1.2. PROPOSITION. *(i) All projective linear subvarieties of the same (co)dimension are projectively equivalent.*

(ii) A linear subvariety of $\mathbb{P}^n$ of codimension r is isomorphic to $\mathbb{P}^{n-r}$ as a complex manifold.

PROOF. Given L and V as above, note that if the equations (6.1.1) are not independent (i.e. $k > r$), then without changing V or r, we can eliminate equations (reducing k) until they are (and $k = r$, i.e. L has maximal rank). Assume this has been done, so that reordering Z_i's if necessary,

$$\det \begin{pmatrix} \ell_{10} & \cdots & \ell_{1,k-1} \\ \vdots & \ddots & \vdots \\ \ell_{k0} & \cdots & \ell_{k,k-1} \end{pmatrix} \neq 0.$$

Let M be the $(n+1) \times (n+1)$ matrix whose first k rows are given by L (a $k \times (n+1)$ matrix) and last $n-k+1$ rows by $(0\ ,\ \mathbb{I}_{n-k+1})$, where 0 denotes a $(n-k+1) \times k$ matrix of zeroes and $\mathbb{I}_m$ always means an $m \times m$ identity matrix.

Consider the automorphism $T(M)$ of $\mathbb{P}^n$. By definition of V, $[\underline{\xi}] \in V$ if and only if (matrix multiplication by) L kills $\underline{\xi}$. So one should view $T(M)$ as taking V to the subspace $V_0 = \{\xi_0 = \cdots = \xi_{k-1} = 0\}$, which proves (i) since V was arbitrary. This also proves (ii) since V_0 is evidently a $\mathbb{P}^{n-k}$ (with homogeneous coordinates $[\xi_k : \cdots : \xi_n]$). $\square$

A linear subvariety of codimension 1 is called a *hyperplane*.

6.2. Quadric hypersurfaces

Recall that a (projective) hypersurface is a subvariety $X \subset \mathbb{P}^n$ cut out by a single homogeneous equation $F(\underline{Z}) = 0$. We are interested in the case where $F \in S^2_{n+1}$ (degree 2), so that X is a quadric. The polynomial can be written

$$F(\underline{Z}) = {}^t\underline{Z}\mathcal{B}\underline{Z} = \begin{pmatrix} Z_0 & \cdots & Z_n \end{pmatrix} \begin{pmatrix} b_{00} & \cdots & b_{0n} \\ \vdots & \ddots & \vdots \\ b_{n0} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} Z_0 \\ \vdots \\ Z_n \end{pmatrix}$$

with $\mathcal{B}$ symmetric. Under a linear change of projective coordinates

$$\begin{pmatrix} Z_0 \\ \vdots \\ Z_n \end{pmatrix} = M \begin{pmatrix} Y_0 \\ \vdots \\ Y_n \end{pmatrix} \qquad (M \in GL(n+1, \mathbb{C})),$$

we find

$$F(\underline{Z}) = \begin{pmatrix} Y_0 & \cdots & Y_n \end{pmatrix} {}^tM\mathcal{B}M \begin{pmatrix} Y_0 \\ \vdots \\ Y_n \end{pmatrix} =: G(\underline{Y}),$$

where (as in Chapter 4) ${}^tM\mathcal{B}M$ is said to be *cogredient* to $\mathcal{B}$.

6.2.1. LEMMA. [SYLVESTER'S THEOREM /$\mathbb{C}$] *Any given symmetric complex $(n+1) \times (n+1)$ matrix $\mathcal{B}$ is cogredient to exactly one of the matrices*

$$M_k = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix},$$

where k is the number of 1*'s.*

6.2.2. COROLLARY. *A given quadric hypersurface in $\mathbb{P}^n$ is projectively equivalent to (or transformable by a linear change of coordinates into) exactly one of the quadrics*

$$\mathcal{Q}_k = \left\{ \sum_{j=0}^{k-1} Y_j^2 = 0 \right\} \qquad (k = 1, \ldots, n+1).$$

Note that $\mathcal{Q}_1 = \{Y_0^2 = 0\}$ is a double hyperplane, $\mathcal{Q}_2 = \{Y_0^2 + Y_1^2 = 0\}$ is a union of two hyperplanes (each $\cong \mathbb{P}^{n-1}$), and $\mathcal{Q}_k$ for $k \geq 3$ is irreducible (equation does not factor). You'll investigate these a tiny bit further in one of the exercises.

6.3. Singularities, tangent planes, and dimension

We'll need the Euler formula from §2.1, so let's prove it first:

6.3.1. LEMMA. [EULER'S FORMULA]

$$F \in S_{n+1}^d \implies \sum_{i=0}^n Z_i \frac{\partial F}{\partial Z_i} = d.F.$$

PROOF. It suffices to check this on monomials $(F =) Z_0^{d_0} \cdots Z_n^{d_n}$, $\sum d_i = d$. We have $\sum_i Z_i \frac{\partial}{\partial Z_i}(Z_0^{d_0} \cdots Z_n^{d_n}) = \sum_i Z_i \frac{d_i}{Z_i}(Z_0^{d_0} \cdots Z_n^{d_n}) = (\sum_i d_i) Z_0^{d_0} \cdots Z_n^{d_n} = d Z_0^{d_0} \cdots Z_n^{d_n}$. □

Now, the definition of smoothness for hypersurfaces is similar to what we have learned for curves; the general case of varieties cut out by more than one equation is trickier. So we'll start, then, with an affine hypersurface

$$V = V(f) \subset \mathbb{C}^n,$$

and a point $p \in V$.

6.3.2. DEFINITION. (i) V is *smooth at* p $\iff$ $\frac{\partial f}{\partial z_j}(p) \neq 0$ for some $j \in \{1, \ldots, n\}$. Otherwise, p is a *singular point* (or *singularity*) of V.

(ii) If V is smooth at all of its points, V is *smooth*. Otherwise, V is *singular*.

(iii) If V is smooth at p, define the *tangent plane*

$$T_p V := \left\{ (z_1(p) + \alpha_1, \ldots, z_n(p) + \alpha_n) \mid \sum_{i=1}^n \alpha_i \frac{\partial f}{\partial z_i}(p) = 0 \right\} \subset \mathbb{C}^n.$$

(Here the $\alpha_i \in \mathbb{C}$.)

So one can think of T_pV as a copy of $\mathbb{C}^{n-1}$ with origin at p and coordinates $\{\alpha_i\}$. It's also worth noting the formal correspondence between "tangent vectors" (points in T_pV) and differential operators "at p", namely $\sum_i \alpha_i \frac{\partial}{\partial z_i}$. This is not misleading at all, and in fact the intrinsic construction of tangent planes (for complex or more generally differentiable manifolds) uses local differential operators.

As for singularities, i.e. points where $f(p) = f_{z_1}(p) = \cdots = f_{z_n}(p) = 0$, we saw some examples for curves in Chapter 2. Here is one more: $y^2 = x^3 - x^2$ is singular at $(0,0)$ since both partials of $y^2 - x^3 + x^2$ vanish there. On the other hand, $y^2 = x^3 - x$ is smooth because this equation together with $0 = \frac{\partial}{\partial x}(y^2 - x^3 + x) = -3x^2 + 1$ and $0 = \frac{\partial}{\partial y}(y^2 - x^3 + x) = 2y$ admit no common solution. This is easy to see: the points $(\frac{1}{\sqrt{3}}, 0)$ and $(\frac{-1}{\sqrt{3}}, 0)$ where both partials vanish, do not lie on the curve.

Next, consider a projective hypersurface

$$V = \bar{V}(F) \subset \mathbb{P}^n,$$

where F is homogeneous and $P \in V$.

6.3.3. Definition. (i) V is *smooth at* P $\iff$ $\frac{\partial F}{\partial Z_j}(P) \neq 0$ for some $j \in \{0, \ldots, n\}$. Otherwise, P is a *singular point* (or *singularity*) of V.

(ii) If V is smooth at all of its points, V is *smooth*. Otherwise, V is *singular*.

(iii) If V is smooth at P, define the *tangent plane* ($\cong \mathbb{P}^{n-1}$)

$$T_PV := \left\{ [Z_0(P) + \alpha_0 : \ldots : Z_n(P) + \alpha_n] \;\middle|\; \textstyle\sum_{i=0}^n \alpha_i \frac{\partial F}{\partial Z_i}(P) = 0 \right\} \subset \mathbb{P}^n.$$

Now *a priori*, the definition of a singular point is one at which $F(P) = F_{Z_0}(P) = \cdots = F_{Z_n}(P) = 0$; but by the Euler formula,

$$\textstyle\sum_i Z_i(q) \frac{\partial F}{\partial Z_i}(P) = \deg(F) \cdot F(P) \tag{6.3.4}$$

and so it suffices to check $F_{Z_0}(P) = \cdots = F_{Z_n}(P) = 0$. In fact, (6.3.4) also implies (in the projective case only!) the simplification

$$T_PV = \left\{ [\alpha_0 : \cdots : \alpha_n] \;\middle|\; \textstyle\sum_i \alpha_i \frac{\partial F}{\partial Z_i}(P) = 0 \right\} \tag{6.3.5}$$

Note that (6.3.5) is really just the solution set of ${}^t\underline{\partial F}(P) \cdot \underline{\alpha} = 0$, as in Chapter 4 (but now in $\mathbb{P}^n$ rather than $\mathbb{P}^2$).

As you might expect, the notions of tangent plane in affine and projective cases "agree", in the sense that – at a point on an affine hypersurface – the tangent plane of the projective completion is the completion of the tangent plane:

6.3.6. PROPOSITION. *$T_p V(F(1, z_1, \ldots, z_n)) = T_{[1:p]} \bar{V}(F) \cap \mathbb{C}^n$, where $(P =)[1 : p]$ means $[1 : z_1(p) : \cdots : z_n(p)]$.*

PROOF. Given $q = (z_1(q), \ldots, z_n(q)) \in \mathbb{C}^n$. Writing $f(\underline{z}) = F(1, \underline{z})$ and $Q = [1 : q]$, we want to show

$$q \in T_p V(f) \iff Q \in T_P \bar{V}(F). \tag{6.3.7}$$

The left-hand (affine) condition is, writing $z_i(q) = z_i(p) + \alpha_i$ in Definition 6.3.2(iii),

$$\sum_{i=1}^{n} (z_i(q) - z_i(p)) \frac{\partial f}{\partial z_i}(p) = 0.$$

This is really

$$\sum_{i=1}^{n} (Z_i(Q) - Z_i(P)) \frac{\partial F}{\partial Z_i}(P) = 0,$$

which by Euler becomes

$$\sum_{i=1}^{n} Z_i(Q) \cdot \frac{\partial F}{\partial Z_i}(P) \; - \; \deg(F) \cdot F(P) \; + \; 1 \cdot \frac{\partial F}{\partial Z_0}(P) = 0.$$

Since $F(P) = 0$, we get

$$1 \cdot \frac{\partial F}{\partial Z_0}(P) + \sum_{i=1}^{n} Z_i(Q) \cdot \frac{\partial F}{\partial Z_i}(P) = 0,$$

which is exactly the right-hand (projective) condition of (6.3.7). □

Now let's have a look at singularities and smoothness in the general projective case. The definition is complicated, but after this chapter we won't use it much. Let

$$V = \bar{V}(F_1, \ldots, F_k) \subseteq \mathbb{P}^n,$$

and p be a point on V.

6.3.8. DEFINITION. (i) V is *smooth at* p if and only if there exists a neighborhood $W \subset \mathbb{P}^n$ of p and sub-index set $\{i_1, \dots, i_c\} \subseteq \{1, \dots, k\}$ such that[1]

(a) $V \cap W = \bar{V}(F_{i_1}, \dots, F_{i_c}) \cap W$, and

(b) $\operatorname{rank} \begin{pmatrix} \frac{\partial F_{i_1}}{\partial Z_0}(p) & \cdots & \frac{\partial F_{i_1}}{\partial Z_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_{i_c}}{\partial Z_0}(p) & \cdots & \frac{\partial F_{i_c}}{\partial Z_n}(p) \end{pmatrix} = c.$

We say V has *codimension* c (or *dimension* $n - c$) at p.

(ii) If V is smooth at each point $p \in V$, then V is *smooth* (otherwise, V is *singular*).

(iii) If V has the same (co)dimension at each smooth point $p \in V$, then V is *equidimensional*. If moreover that codimension is c, we just say V is a variety of codimension c (dimension $n - c$).[2]

(iv) The *tangent plane* $T_pV \subset \mathbb{P}^n$ to V at a smooth point p is the solution set of $L.\underline{p} = 0$, where

$$L = \begin{pmatrix} \frac{\partial F_1}{\partial Z_0}(p) & \cdots & \frac{\partial F_1}{\partial Z_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_k}{\partial Z_0}(p) & \cdots & \frac{\partial F_k}{\partial Z_n}(p) \end{pmatrix}.$$

In Definition 6.3.8(i), condition (a) says that locally about p, once you set $F_{i_1}(\underline{Z}) = \cdots = F_{i_c}(\underline{Z}) = 0$, the remaining equations are redundant; and roughly speaking, the condition (b) on rank says that no more (none of the F_{i_ℓ}) are redundant. In the terminology of §6.1, T_pV is a linear subvariety, and it follows from condition 6.3.8(i)(b) that its codimension is c. That is, we have really just defined the (co)dimension of a variety V at a smooth point p, to be the (co)dimension of T_pV — something we already knew how to define.

Finally, the "neighborhood" W in the definition is an analytic open set containing p (such as a "ball"), but the definition would also

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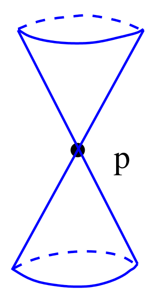
[1]The matrices in this definition assume a particular representative $(Z_0(p), \dots, Z_n(p))$ in $\mathbb{C}^{n+1}$ of the projective coordinate $[Z_0(p) : \cdots : Z_n(p)]$. It doesn't matter which one you take, as long as you are consistent.

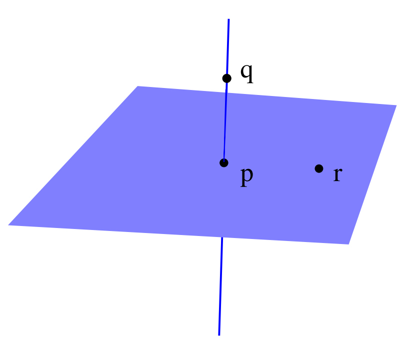
[2]Note that the dimension of V at a *singular* point is not defined, so the definition of a m-dimensional variety *must* be "one that is of dimension m at all *smooth* points".

work if we only permitted "algebraic" open sets defined by complements of (other) subvarieties,[3] known as *Zariski open sets*. In general, if you want to view an algebraic variety as a complex analytic space (or manifold, if it is smooth), then you must use analytic open sets; on the other hand, the Zariski open sets introduce a different topology on V or $\mathbb{P}^n$, which is coarser (and no longer Hausdorff) but has the advantage of being algebraic. We need both. In brief, when we study varieties analytically, we use the analytic topology; when we want to make heavy use of the correspondence between varieties and ideals in commutative rings, we use the Zariski topology.

6.3.9. EXAMPLE. (i) Let V be the affine variety $\{z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}^3$. The partial derivatives $\frac{\partial}{\partial z_i}(\sum_j z_j^2) = 2z_i$ all vanish at $p = (0,0,0)$ and so V is singular there:



(ii) Now for a nasty one. Let $V \subset \mathbb{P}^3$ be defined by $\left\{ \begin{array}{c} Z_1 Z_3 = 0 \\ Z_2 Z_3 = 0 \end{array} \right\}$, and take $p = [1:0:0:0]$, $q = [1:0:0:1]$, $r = [1:1:0:0]$:



[3]This is exactly how Example 6.3.9(iii) is started below.

Locally about r, $Z_1 \neq 0$ and so having set $Z_1Z_3 = 0$ (i.e. $Z_3 = 0$), the second equation $Z_2Z_3 = 0$ is redundant. So the relevant matrix from 6.3.5(i)(b) is $\left.\begin{pmatrix} \frac{\partial}{\partial Z_0}(Z_1Z_3) & \frac{\partial}{\partial Z_1}(Z_1Z_3) & \frac{\partial}{\partial Z_2}(Z_1Z_3) & \frac{\partial}{\partial Z_3}(Z_1Z_3) \end{pmatrix}\right|_r =$ $\begin{pmatrix} 0 & Z_3 & 0 & Z_1 \end{pmatrix}|_r = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$, which has rank 1, proving that V has codimension 1 (dimension 2) at r. Locally about q, $Z_3 \neq 0$ and so the equations are effectively $Z_1 = 0$ and $Z_2 = 0$, neither of which is redundant. The matrix in 6.3.5(i)(b) is now $\left.\begin{pmatrix} 0 & Z_3 & 0 & Z_1 \\ 0 & 0 & Z_3 & Z_2 \end{pmatrix}\right|_q =$ $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, which does have rank 2, confirming that V has codimension 2 at q. So V is not equidimensional. Finally, at p neither equation is redundant but the matrix 6.3.8(i)(b) is $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, meaning V is singular at p.

(iii) Finally, consider the variety $C \subset \mathbb{P}^3$ defined by the three equations

$$\left\{ \begin{array}{lr} Z_0Z_3 - Z_1Z_2 = 0 & (\mathbf{I}) \\ Z_1^2 - Z_0Z_2 = 0 & (\mathbf{II}) \\ Z_2^2 - Z_1Z_3 = 0 & (\mathbf{III}) \end{array} \right\},$$

and the covering $U_i := \{Z_i \neq 0\}$ of $\mathbb{P}^3$. In U_0, we can divide by Z_0, so that $(\mathbf{II})$ becomes $Z_2 = \frac{Z_1^2}{Z_0}$. (You may already recognize C as the *twisted cubic curve* from the Chapter 5 exercises.) Together with $(\mathbf{I})$, this gives $Z_2^2 = Z_2Z_2 = Z_1Z_2\frac{Z_1}{Z_0} = Z_0Z_3\frac{Z_1}{Z_0} = Z_1Z_3$. Consquently, $(\mathbf{III})$ is redundant on U_0. Now, for any point in $U_0 \cap C$, you can check that the matrix in 6.3.5(i)(b) has rank 2, showing that C is smooth of dimension 1 at all of those points. To finish, and show that C is a 1-dimensional smooth variety, carry out a similar analysis in each of U_1, U_2, and U_3 (exercise).

6.3.10. Remark. For an affine variety defined by a prime ideal $I \subset S_n = \mathbb{C}[z_1, \ldots, z_n]$, one can consider the coordinate ring $R = S_n/I$ and its fraction field F (see Chapter 9). The Krull dimension of R (the maximum length of a chain of prime ideals) and transcendence degree of $F/\mathbb{C}$ (which agrees with the former by Noether normalization) are algebraic definitions of dimension that agree with our geometric one, and also work for fields other than $\mathbb{C}$.

Somewhat unsurprisingly, a variety of dimension 1 is called a curve, of dimension 2 a surface, and of dimension $d \geq 3$ a d-fold. So-called "Calabi-Yau threefolds", such as quintic hypersurfaces in $\mathbb{P}^4$, play a central role in mathematical string theory.

6.4. Singularities of plane curves

Consider a curve

$$C = \{F(\underline{Z}) = 0\} \subset \mathbb{P}^2$$

defined by a homogeneous polynomial $F \in S_3^d$ (i.e., of degree 3 in Z_0, Z_1, Z_2). A point $p \in C$ is a singularity if and only if $F_{Z_0}(p) = F_{Z_1}(p) = F_{Z_2}(p) = 0$, and (moving C by a projectivity if necessary) we may assume that $p = [1:0:0]$. To locally analyze C at p, we can pass to affine coordinates $x = \frac{Z_1}{Z_0}$, $y = \frac{Z_2}{Z_0}$ and replace F by

$$f(x,y) = F(1,x,y) = \sum_{m=k}^{d} f_m(x,y),$$

where $f_m \in S_2^m$ for each m, and $f_k \neq 0$. Now $0 = f_x(p) = f_y(p) = f(p)$ $(p = (0,0))$ translates to $0 = f_1 = f_0$, so that $k \geq 2$. We say that p is a *k-tuple point* of C, or a singularity of *order* k.

So far, we have not defined tangent planes at singular points. Indeed, this can really only be done for curves in general. To decide what the tangent lines to C at 0 should be, we think of $[x : y]$ as homogenous coordinates on the $\mathbb{P}^1$ of lines through $(0,0) = p$. The lowest-order homogeneous term f_k of f defines a 0-dimensional variety $\tau_p(C) := \{f_k(x,y) = 0\}$ in this $\mathbb{P}^1$. For each $[x_0 : y_0] \in \tau_p(C)$, one should think of $\frac{y_0}{x_0}$ as the slope of a line tangent to some "local irreducible component"[4] of the curve C at p.

6.4.1. DEFINITION. The tangent lines to a curve C at a singularity p are the lines through p corresponding to points of $\tau_p(C)$.

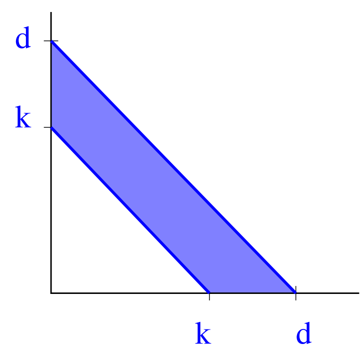
Now, $f_k(x,y) = 0$ has k solutions counted with multplicity. If these are all distinct, i.e. if $\tau_p(C)$ is reduced, then we say p is an

[4] this will be made precise when we do local normalization

ordinary k-tuple point. The most geometric way to think of this is that C has k distinct tangent lines at p.

Any line through a k-tuple point p *other* than one of C's tangent lines there, meets C with multiplicity k at p: if L is given parametrically by $t \mapsto (at, bt)$ ($f_k(a,b) \neq 0$) then the intersection multiplicity is computed as in Chapter 2 by taking the order of $f(at, bt) = t^k f_k(a,b) + \cdots$ at $t = 0$.

6.4.2. REMARK. Given a polynomial $f(x,y) = \sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}} \alpha_{ab} x^a y^b$ it can be useful (for various purposes) to plot the finitely many (a,b) with $\alpha_{ab} \neq 0$. If you do this when $(0,0)$ is a k-tuple point and f has degree d, then these lie in the shaded region



This may be useful for one of the exercises below.

There is more to singularities, it turns out, than their order or even the tangent line configuration reflected by $\tau_p(C)$. A local analytic classification of so-called *simple*[5] singularities of curves has been carried out. For the purposes of this classification, if C, C' are two curves through $p = (0,0)$, their singularities at p are considered equivalent if there are small neighborhoods U, U' of $(0,0)$ in $\mathbb{C}^2$ and a biholomorphism $U \xrightarrow{\cong} U'$ carrying p to p and C to C'. The different classes of simple singularities carry "A-D-E" labels, which reflect a relation to other classifications in mathematics (simple Lie algebras/Dynkin diagrams, rational surface singularities, etc.)

[5]cf. [Barth, Hulek, Peters, and van de Ven, "Compact complex surfaces", Springer, 2004] for the definition. Simple singularities encompass all double (2-tuple) points and some triple (3-tuple) points, and nothing of higher order.

The results are that double points are all equivalent to one of

An: $x^2 + y^{n+1} = 0 \quad (n \geq 1)$,

and (simple) triple points to one of

Dn: $y(x^2 + y^{n-2}) = 0 \quad (n \geq 4)$,
E6: $x^3 + y^4 = 0$,
E7: $x(x^2 + y^3) = 0$,
E8: $x^3 + y^5 = 0$.

The ODP's (ordinary double points: two distinct tangents) are all of type A_1, as all $A_{n\geq 2}$ have only one tangent; amongst the latter, "cusps" are the singularities of type A_2.[6] OTP's (ordinary triple points) are all of type D_4; we note that the tangent lines to $y(x^2 + y^2) = 0$ have slopes $0, i, -i$. All $D_{n\geq 5}$ have two distinct tangents (one with "mutliplicity 2") and the $E_{6,7,8}$ each have one tangent (of "multiplicity 3").

Exercises

(1) (i) Prove that a quadric hypersurface in $\mathbb{P}^n$ defined by a symmetric bilinear form B is smooth if and only if $\det(B) \neq 0$.
(ii) Cor. 6.2.2 associates a number k to each projective quadric hypersurface in $\mathbb{P}^n$. Show that any two are projectively equivalent if and only if they have the same value of k. [This is easy.]
(2) Show that [the closure in $\mathbb{P}^2$ of] $y^2 = 4x^3 + ax + b$ is smooth unless $a^3 + 27b^2 = 0$.
(3) Find the tangent plane to the complex surface $2x^4 + y^4 + z^4 - 4xyz = 0$ (in $\mathbb{C}^3$) at the point $p = (1, 1, 1)$.
(4) Finish the proof in Example 6.3.9(iii) that C is a smooth curve.
(5) What form does a degree k projective algebraic curve (in $\mathbb{P}^2$) take if it has a singularity of order k?
(6) Analyze the singularity of $C = \{(x^2 + y^2)^2 + 3x^2y - y^3 = 0\} \subseteq \mathbb{C}^2$ at the origin. (What is its order, and type?)
(7) For which values of μ are the algebraic curves $F(X, Y, Z) = 0$ in $\mathbb{P}^2$ singular (in (a) and (b) below)? Attempt a sketch of each

[6] Refer back to Chapter 2 for a few pictures (cusps, ODP, OTP).

of the singular curves, saying where the singularities are located and what type they are.

(a) $F(X, Y, Z) = X^3 + Y^3 + Z^3 + \mu(X + Y + Z)^3$,

(b) $F(X, Y, Z) = X^3 + Y^3 + Z^3 + 3\mu XYZ$.

(8) Show that a smooth quadric surface in $\mathbb{P}^3$ contains two infinite families of lines. Starting with the right smooth quadric (to which all others are projectively equivalent), which may not be the one from Cor. 6.2.2, is the key step. [Hint: how would you map $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$? Doing it this way gives you a bonus result...]

(9) Calculate the number of singularities on the hypersurface in $\mathbb{P}^n$ defined by $\sum_{i=0}^n Z_i^{n+1} = (n+1)\prod_{i=0}^n Z_i$. (These are all nodes.) [Hint: set $Z_i \frac{\partial}{\partial Z_i}$ of the equation to zero for each i.]