CHAPTER 7

Smooth varieties as complex manifolds

This Chapter starts the long slog toward a proof of part (A) of the Normalization Theorem 3.2.1. After introducing a bit of the theory of several complex variables, we'll use the holomorphic implicit function theorem to put a complex manifold structure on any smooth irreducible (affine or projective) algebraic variety:

7.0.1. THEOREM. *A smooth irreducible algebraic curve $C \subset \mathbb{P}^n$ "is" a Riemann surface. (More precisely, there exists a Riemann surface M and an injective morphism of complex manifolds $\sigma : M \hookrightarrow \mathbb{P}^2$ with C as its image.)*

This is, of course, the "smooth" case of Thm. 3.2.1(A). As for going the other way, from Riemann surfaces to algebraic curves, here is a statement which is different in character from 3.2.1(B):

7.0.2. THEOREM. *A Riemann surface M with $n+1$ linearly independent meromorphic functions $f_0, \ldots, f_n \in \mathcal{K}(M)$, yields an algebraic curve in $\mathbb{P}^n$ not contained in any proper linear subvariety.*

We won't prove this in full — just the existence of a morphism $M \to \mathbb{P}^n$ of complex manifolds which is *nondegenerate*, i.e. whose image is not contained in any $\mathbb{P}^{n-1}$. Proving that the image is described by algebraic equations (hence yields an algebraic curve) is harder.

7.1. Background from several complex variables

Let $\mathcal{O}_n$ (or $\mathbb{C}\{z_1, \ldots, z_n\}$) denote the *ring of convergent power series* $\sum_I a_I \underline{z}^I$ *in* $z_1, \ldots, z_n$, or equivalently, holomorphic functions defined on *some* neighborhood of $\underline{0} \in \mathbb{C}^n$ (cf. §5.1).

7.1.1. PROPOSITION. [W. OSGOOD, 1900] *Let f be a function on an open neighborhood of $\underline{0} \in \mathbb{C}^n$ which is holomorphic in each z_i as the other $\{z_j\}_{j \neq i}$ are held fixed; that is, $\frac{\partial f}{\partial \bar{z}_i} = 0$ $(\forall i)$. Then f is in fact a holomorphic function (and so gives an element of $\mathcal{O}_n$).*

PROOF. We will only give the proof for $n = 2$. Since f is holomorphic in z_2, we have

$$f(z_1, z_2) = \frac{1}{2\pi\sqrt{-1}} \oint \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2;$$

using the holomorphicity in z_1, this

$$= \frac{1}{(2\pi\sqrt{-1})^2} \oint \oint \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2$$
$$= \frac{1}{(2\pi\sqrt{-1})^2} \oint \oint \frac{f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2}{\zeta_1 \zeta_2 \left(1 - \frac{z_1}{\zeta_1}\right)\left(1 - \frac{z_2}{\zeta_2}\right)}.$$

Now using the power-series expansion

$$\frac{1}{1 - \frac{z_i}{\zeta_i}} = \sum_{k \geq 0} \left(\frac{z_i}{\zeta_i}\right)^k,$$

whose uniform convergence allows us to swap integration and summation, we find

$$f(z_1, z_2) = \sum_{k_1, k_2 \geq 0} \left(\frac{1}{(2\pi\sqrt{-1})^2} \oint \oint \frac{f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2}{\zeta_1^{k_1+1} \zeta_2^{k_2+1}} \right) z_1^{k_1} z_2^{k_2}.$$

□

In order to put a complex manifold structure on a smooth variety, we will need a way to parametrize zero-loci of holomorphic functions. This is given by the *holomorphic implicit function theorem* which here I will just state and prove in the two variable case.

7.1.2. PROPOSITION. *Let $f \in \mathcal{O}_2$ with $f(0,0) = 0$, $\frac{\partial f}{\partial z_1}(0,0) \neq 0$. Then there exists $w \in \mathcal{O}_1$ such that in a neighborhood of $(0,0)$ in $\mathbb{C}^2$,*

$$f(z_1, z_2) = 0 \iff z_1 = w(z_2).$$

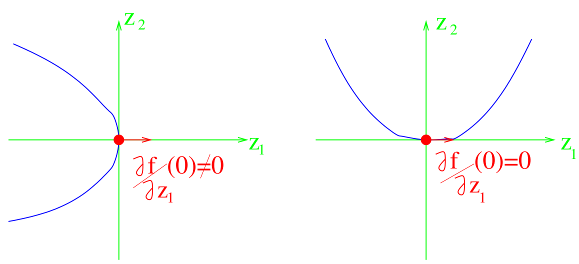
The upshot of this is that z_2 gives a local holomorphic coordinate on $\{f(z_1, z_2) = 0\}$.

PROOF. We will assume the C^∞ implicit function theorem, and just check that the w it yields is holomorphic:

$$0 = \frac{\partial}{\partial \overline{z_2}} f(w(z_2), z_2)$$
$$= \frac{\partial f}{\partial \overline{z_2}}(w(z_2), z_2) + \frac{\partial f}{\partial z_1}(w(z_2), z_2) \cdot \frac{\partial w}{\partial \overline{z_2}} + \frac{\partial f}{\partial \overline{z_1}}(w(z_2), z_2) \cdot \frac{\partial \overline{w}}{\partial \overline{z_2}}.$$

Now since $f \in \mathcal{O}_2$, $\frac{\partial f}{\partial \overline{z_1}} = \frac{\partial f}{\partial \overline{z_2}} = 0$; moreover, by assumption $\frac{\partial f}{\partial z_1} \neq 0$ locally. So we find that $\frac{\partial w}{\partial \overline{z_2}} = 0$, so that $w \in \mathcal{O}_1$. □

Here is a visual explanation of why the nonvanishing condition on $\partial f / \partial z_1$ matters:



In the left-hand picture, you can write z_1 as a function of z_2 (as desired); on the right-hand side, you cannot.

7.2. Smooth normalization

The more general statement which implies Theorem 7.0.1 is:

7.2.1. THEOREM. *Given*

- *a closed connected subset Y of a compact complex n-manifold X;*
- *a system of open neighborhoods $\{W_\alpha \subset X\}$ covering Y (with local holomorphic coordinates $\underline{z_\alpha} = (z_{\alpha 1}, \ldots, z_{\alpha n})$);*
- *holomorphic functions $f_{\alpha 1}, \ldots, f_{\alpha \ell} \in \mathcal{O}(W_\alpha)$ (for each α) such that*[1] *$Y \cap W_\alpha = V(\{f_{\alpha j}\}_{j=1,\ldots,\ell}) \cap W_\alpha$; and (also for each α)*
- *rank* $\left(\{\partial f_{\alpha j} / \partial z_{\alpha k}\}_{\substack{j=1,\ldots,\ell \\ k=1,\ldots,n}} \right) = \ell$ [*the "Jacobian condition"*].

Then Y is a compact complex $(n-\ell)$-manifold.

[1] this condition makes Y into an "analytic subvariety" of X; here $V(f_1, \ldots, f_\ell)$ means the vanishing locus $f_1 = \cdots = f_\ell = 0$, just as in the algebraic setting.

We say that Y is a codimension-ℓ complex submanifold of X. In fact, Theorem 7.2.1 immediately gives:

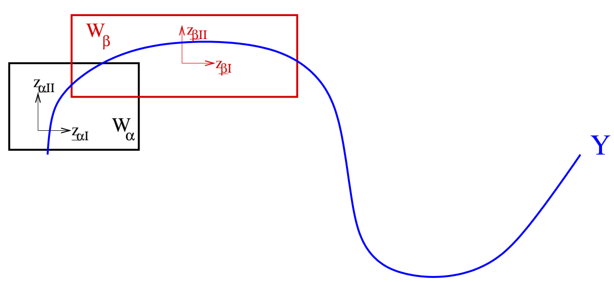
7.2.2. COROLLARY. *Any smooth irreducible projective algebraic variety $Y \subset \mathbb{P}^n$ of dimension d is a compact complex d-manifold.*

PROOF. Put $X = \mathbb{P}^n$ and $\ell = n - d$. That Y is smooth of dimension d (Defn. 6.3.8) implies the Jacobian condition required in Thm. 7.2.1. $\square$

PROOF OF THEOREM 7.2.1. Refining the covering if necessary, we can arrange to have

$$\det\left(\left\{\frac{\partial f_{\alpha j}}{\partial z_{\alpha k}}\right\}_{1\leq j,k\leq \ell}\right) \neq 0. \tag{7.2.3}$$

Write "$\underline{z_\alpha}_I$" for $(z_{\alpha 1}, \ldots, z_{\alpha \ell})$ and "$\underline{z_\alpha}_{II}$" for $(z_{\alpha,\ell+1}, \ldots, z_{\alpha n})$, so that $\underline{z_\alpha} = \left(\underline{z_\alpha}_I, \underline{z_\alpha}_{II}\right)$. A schematic picture:



By the condition (7.2.3), and the general holomorphic implicit function theorem (see the Exercises), we have holomorphic functions $\{w_\alpha\}$ (mapping from open subsets of $\mathbb{C}^{n-\ell}$ to $\mathbb{C}^\ell$) such that

$$Y \cap W_\alpha = \left\{\underline{z_\alpha}_I = w_\alpha\left(\underline{z_\alpha}_{II}\right)\right\}$$

for each α. Hence, the $\left\{\underline{z_\alpha}_{II}\right\}$ give local coordinates on the $\{Y \cap W_\alpha\}$, which constitute an open cover of Y.

Consider the transition functions for X

$$\Phi_{\alpha\beta} : \underline{z_\beta}(W_{\alpha\beta}) \overset{\cong}{\to} \underline{z_\alpha}(W_{\alpha\beta})$$

$$\left(\underline{z_\beta}_I, \underline{z_\beta}_{II}\right) \mapsto \left(\phi_I\left(\underline{z_\beta}_I, \underline{z_\beta}_{II}\right), \phi_{II}\left(\underline{z_\beta}_I, \underline{z_\beta}_{II}\right)\right) =: \left(\underline{z_\alpha}_I, \underline{z_\alpha}_{II}\right)$$

corresponding to change of coordinates on $W_{\alpha\beta}$. Clearly the functions describing change of coordinates on $Y \cap W_{\alpha\beta}$ are then

$$\Phi^Y_{\alpha\beta} : \underline{z_\beta}_{II} \mapsto \phi_{II}\left(w_\beta\left(\underline{z_\beta}_{II}\right), \underline{z_\beta}_{II}\right) =: \underline{z_\alpha}_{II}.$$

This is 1-to-1 because $\Phi_{\alpha\beta}$ is, and holomorphic because ϕ_{II} and w_β are. So we have the data of an analytic atlas on Y. $\square$

7.3. Nondegenerate morphisms

The statement related to Theorem 7.0.2 which we shall prove is:

7.3.1. PROPOSITION. *Given a Riemann surface M, the following data are equivalent:*

(a) $n+1$ linearly independent meromorphic functions $f_i \in \mathcal{K}(M)$;

(b) a nondegenerate holomorphic map (morphism of complex manifolds) $\sigma : M \to \mathbb{P}^n$.

We will need the notion of a meromorphic function on a complex manifold of any dimension.

7.3.2. DEFINITION. A meromorphic function $\mathcal{F} \in \mathcal{K}(X)$ (on a complex manifold X) is a collection $\{(U_\alpha, g_\alpha, h_\alpha)\}$ such that

- $\{U_\alpha\}$ is an open cover of X;
- $g_\alpha, h_\alpha \in \mathcal{O}(U_\alpha)$ (they are holomorphic functions); and
- $g_\alpha h_\beta = g_\beta h_\alpha$ on $U_{\alpha\beta}$.

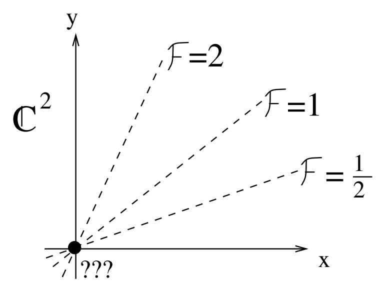
We write "$\mathcal{F} = \frac{g_\alpha}{h_\alpha}$" on U_α.[2]

7.3.3. REMARK. For $\dim(X) = 1$, this coincides with the earlier Definition 3.1.1 (via g/h); by Prop. 3.1.10 meromorphic functions on Riemann surfaces yield morphisms $X \to \mathbb{P}^1$. But this does *not* generalize: if $\dim(X) > 1$, a meromorphic function on X need not even yield a well-defined mapping $X \to \mathbb{P}^1$.

7.3.4. EXAMPLE. Consider $X = \mathbb{C}^2$ with complex coordinates x, y. Then $\mathcal{F} := x/y$ (one $U_\alpha = X$; $g = x$, $h = y$) defines a meromorphic

[2]The third condition says that $\frac{g_\alpha}{h_\alpha} = \frac{g_\beta}{h_\beta}$ on overlaps — at least, where the quotients are defined! (see Remark 7.3.3)

function, which is not well-defined (as a mapping to $\mathbb{P}^1$) at $(0,0)$.



The notion of "blowing up" in algebraic geometry is motivated (in part) by the desire to remove such indeterminacies. In this example, the idea would be to replace the origin in $X = \mathbb{C}^2$ by the $\mathbb{P}^1$ of lines through the origin, yielding a new space $\tilde{X}$ (mapping down onto X) on which the meromorphic function becomes well-defined as a morphism.

7.3.5. EXAMPLE. The meromorphic functions on $\mathbb{P}^n$ and its smooth subvarieties (viewed as complex manifolds) are the *rational functions* $\mathcal{F} = \frac{P(\underline{Z})}{Q(\underline{Z})}$ for $P, Q \in S^d_{n+1}$. For instance, the affine coordinates $z_i = \frac{Z_i}{Z_0}$ are meromorphic functions (and more generally, $z_{ji} = \frac{Z_i}{Z_j}$ is one).

Here is how to see at least that "rational functions are meromorphic" in the sense of Definition 7.3.2. (That meromorphic functions are rational is more nontrivial.) In $U_j = \{Z_j \neq 0\}$, set

$$g_j(\underline{z_j}) := P(z_{j0}, \ldots, \underset{\substack{j^{\text{th}}\\ \text{entry}}}{1}, \ldots, z_{jn}) = P\left(\underline{Z}/Z_j\right) = \frac{1}{Z_j^d} P(\underline{Z})$$

$$h_j(\underline{z_j}) := Q(z_{j0}, \ldots, \underset{\substack{j^{\text{th}}\\ \text{entry}}}{1}, \ldots, z_{jn}) = \frac{1}{Z_j^d} Q(\underline{Z});$$

then

$$g_j h_i = \frac{1}{Z_j^d} \frac{1}{Z_i^d} P(\underline{Z}) Q(\underline{Z}) = g_i h_j.$$

7.3.6. EXAMPLE. Consider a holomorphic map $f : C \to X$ from a Riemann surface to a complex manifold, and let $\mathcal{F} \in \mathcal{K}(X)$ be given by $\{(g_\alpha, h_\alpha, U_\alpha)\}$. Assume that $f(C)|_{U_\alpha} \cap \{h_\alpha = 0\}$ is a finite point set, and put $W_\alpha := f^{-1}(U_\alpha)$, $G_\alpha := g_\alpha \circ f$, $H_\alpha := h_\alpha \circ f$. Then $f^*\mathcal{F} := \{(G_\alpha, H_\alpha, W_\alpha)\}$ (or rather, G/H) belongs to $\mathcal{K}(C)$.

The last two examples will now be used in the

PROOF OF PROPOSITION 7.3.1. The first issue is how we get from $n+1$ meromorphic functions to a morphism to $\mathbb{P}^n$. The set of points in M which cause a problem is

$$\Delta := \{q \in M \,|\, f_i(q) = 0 \text{ for all } i\} \cup \{q \in M \,|\, f_i(q) = \infty \text{ for some } i\}.$$

Define

$$f : (M \backslash \Delta) \to \mathbb{P}^n$$

by

$$p \longmapsto [f_0(p) : \cdots : f_n(p)].$$

Near $q \in \Delta$ let z be a local holomorphic coordinate with $z(q) = 0$, then write $f_i(z) = z^{\nu_q(f_i)} h_i(z)$ (where h_i are local holomorphic functions not vanishing at q), and put $\nu := \min_{i \in \{0,\ldots,n\}} \{\nu_q(f_i)\}$. For $z \neq 0$,

$$f(z) = \left[z^{-\nu} f_0(z) : \cdots : z^{-\nu} f_n(z)\right];$$

none of the entries in this blows up locally, and at least one does not vanish at $z = 0$ (i.e. at q). Hence, f extends to all of M, and it is evident that this extension is still holomorphic as a map to $\mathbb{P}^1$.

Next, given a morphism $f : M \to \mathbb{P}^n$, we want to product an $(n+1)$-tuple of meromorphic functions. Referring to Examples 7.3.5 (for z_i) and 7.3.6 (for f^*), simply take $f_i := f^* z_i$ and you're done.

Finally, to see that f is degenerate iff the $\{f_i\}$ are linearly dependent, consider the correspondence between nonzero vectors $\underline{v} \in \mathbb{C}^{n+1}$ (up to scale) and hyperplanes in $\mathbb{P}^n$, by taking $\mathbb{P}^{n-1}_{\underline{v}}$ to be the projectification of $(\mathbb{C}^{n+1})^{\perp \underline{v}}$. Degeneracy of f occurs iff $f(M) \subset \mathbb{P}^{n-1}_{\underline{v}}$ for some $\underline{v}$, which is to say $(f_0(p), \ldots, f_n(p)) \perp \underline{v}$ for all $p \in M$. But this just reads $\sum v_i f_i(p) = 0$ $(\forall p)$, which is a nontrivial linear relation. $\square$

We give two examples of nondegenerate projective embeddings of Riemann surfaces (the first is actually a series of examples). For these cases we actually give algebraic equations for the image.

7.3.7. EXAMPLE. The so-called *rational canonical curves* are the images of the nondegenerate morphisms

$$f : \mathbb{P}^1 \hookrightarrow \mathbb{P}^n$$

given, for each $n \in \mathbb{N}$, by

$$[Z_0 : Z_1] \mapsto [Z_0^n : Z_0^{n-1}Z_1 : \cdots : Z_1^n].$$

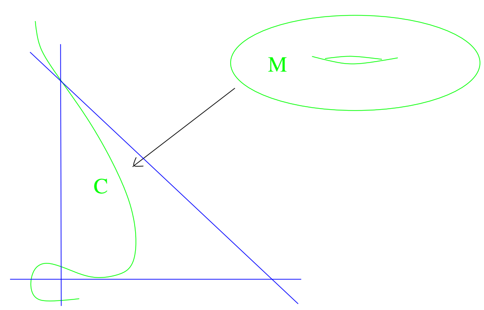
(In affine terms, one can think of this as $z \mapsto [1 : z : \ldots : z^n]$, with $\infty \mapsto [0 : \cdots : 0 : 1]$.)

Let's see what this looks like for the first few values of n:

- for $n = 1$, f sends $[Z_0 : Z_1] \mapsto [Z_0 : Z_1]$ and so is just the identity map.
- for $n = 2$, we have $[Z_0 : Z_1] \mapsto [Z_0^2 : Z_0Z_1 : Z_1^2]$. If we write $[Y_0 : Y_1 : Y_2]$ for the homogeneous coordinates on $\mathbb{P}^2$, then the image is the conic $\{Y_1^2 - Y_0Y_2 = 0\} \subset \mathbb{P}^2$.
- for $n = 3$, $[Z_0 : Z_1] \mapsto [Z_0^3 : Z_0^2Z_1 : Z_0Z_1^2 : Z_1^3](= [Y_0 : Y_1 : Y_2 : Y_3])$ has image $V := \overline{V}(Y_0Y_3 - Y_1Y_2, Y_1^2 - Y_0Y_2, Y_2^2 - Y_1Y_3) \subset \mathbb{P}^3$.

By Exercise 4 from Chapter 6 you know that V is smooth.

7.3.8. EXAMPLE. Let $M = \mathbb{C}/\Lambda$ ($\Lambda \subset \mathbb{C}$ a lattice) be a complex 1-torus. We want to demonstrate that there is a (nondegenerate) morphism from M to $\mathbb{P}^2$ with a cubic curve as image. Note that this will present M as the normalization of such a cubic curve:



Some of the steps will be exercises.

First, there exists a unique meromorphic function $\wp \in \mathcal{K}(\mathbb{C})$ satisfying

- $\wp(u+\lambda) = \wp(u)$ for every $\lambda \in \Lambda$ and $u \in \mathbb{C}$
- $\wp(u) = u^{-2} + h(u)$, where $h \in \mathcal{K}(\mathbb{C})$ is holomorphic in a neighborhood of 0, has all its poles in $\Lambda \backslash \{0\}$, and $h(0) = 0$.

Existence is an exercise. Uniqueness is easy: if $\mathcal{Q}$ were another such function, $\wp - \mathcal{Q} = (\wp - u^{-2}) - (\mathcal{Q} - u^{-2})$ has no pole at 0 *and* is Λ-periodic, hence has no poles in Λ either. But the only possible zeroes were in Λ, and so $\wp - \mathcal{Q}$ is entire. By compactness of a fundamental region for Λ, any Λ-periodic entire function is bounded hence (by Liouville) constant. Since $\wp - \mathcal{Q}$ is zero at 0, this constant is zero and $\wp = \mathcal{Q}$.

In the exercises below, you will also show that $\wp$ is an even function ($\wp(u) = \wp(-u)$) and $(\wp')^2 = 4\wp^3 + a\wp + b$ for some $a, b \in \mathbb{C}$. In each case, you get equality by showing the right-hand side minus the left-hand side has no poles and is zero at some point (as in the uniqueness argument just described). The upshot is that

$$f : \mathbb{C}/\Lambda \to \mathbb{P}^2$$

defined by

$$\begin{aligned} u &\mapsto [1 : \wp(u) : \wp'(u)] \qquad \text{for } u \neq \bar{0} \\ \text{and} \quad \bar{0} &\mapsto [0 : 0 : 1] \end{aligned}$$

parametrizes (or normalizes) $C = \{Z_0Z_2^2 = 4Z_1^3 + aZ_1Z_0^2 + bZ_0^3\}$, a smooth cubic with the affine equation

$$y^2 = 4x^3 + ax + b.$$

What we have said so far only gives that $f(M) \subseteq C$, but viewing the smooth curve C as a complex manifold, and f as a morphism $M \to C$, the open mapping theorem from complex analysis says the image is open; while on the other hand the image of a compact set by a continuous map is compact (hence closed in C). So $f(M)$ is open and closed in C, and thus $f(M) = C$.

Exercises

(1) Show that the rational canonical map $f : \mathbb{P}^1 \to \mathbb{P}^n$ has the following property: the image of any collection of $k\,(\leq n+1)$ distinct points $\{w_1, \ldots, w_k\} \subset \mathbb{P}^1$ is in general position (spans a $\mathbb{P}^{k-1}$ in $\mathbb{P}^n$). [Hint: Vandermonde determinant.] Then, taking $k = 2$, explain why this shows f is injective.

(2) Turning to the case $n = 3$ in Example 7.3.7 (i.e. the twisted cubic), (a) actually *prove* that $V = \mathit{Image}(f)$ and (b) that you cannot throw out any of the three equations defining V.

(3) Show that the map $\mathbb{P}^2 \to \mathbb{P}^5$ given by $[Z_0 : Z_1 : Z_2] \mapsto [(Z_0)^2 : Z_0Z_1 : Z_0Z_2 : (Z_1)^2 : Z_1Z_2 : (Z_2)^2]$ is (a) well-defined and (b) holomorphic (i.e. a "morphism of complex manifolds"), then (c) write (polynomial) equations expressing the image as an algebraic variety. (For (c) you can just write the equations and not prove it.)

(4) Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be two complex numbers which are $\mathbb{R}$-linearly independent, and let

$$\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 = \{n_1\lambda_1 + n_2\lambda_2 |\; n_1, n_2 \in \mathbb{Z}\}$$

be the lattice in $\mathbb{C}$ that they generate.

(a) Show that the series

$$\wp(u) = \frac{1}{u^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left(\frac{1}{(u-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

is absolutely and uniformly convergent on any compact subset of the complex u-plane which does not contain any of the points of Λ. [Hint: any compact subset is contained inside one of the following form: $|u| \leq K \cap |u - \lambda| \geq \epsilon\,(\forall \lambda)$. Break the sum into terms with $|\lambda| \leq 2K$, and $|\lambda| > 2K$, and use (essentially) the Weierstrass M-test.]

(b) Verify the pole condition in Example 7.3.8: that all poles are on Λ, and in a neighborhood U of 0, $\wp(u) = u^{-2} + h(u)$ with h holomorphic and $h(0) = 0$. [Hint: what do you know about an absolutely and uniformly convergent series of analytic functions?]

(c) Show that $\wp$ is a doubly-periodic function; that is, show that

$$\wp(u + \lambda) = \wp(u) \text{ for every } u \in \mathbb{C} \text{ and every } \lambda \in \Lambda.$$

[Hint: From (a), you can calculate the derivative $\wp'(u)$ by differentiating each term of the series defining $\wp(u)$. First prove $\wp'(u + \lambda) = \wp'(u)$, then integrate.]

(5) Now forget the explicit formula for $\wp(u)$ just given, and retain just these facts: that $\wp \in \mathcal{K}(\mathbb{C})$ is Λ-periodic with all poles $\in \Lambda$, and locally of the form $\wp(u) = u^{-2} + h(u)$ with h holomorphic (on some $U \subset \mathbb{C}$ containing a fundamental domain) and $h(0) = 0$. Prove that (a) $\wp(u) = \wp(-u)$ [$\implies$ h even $\implies$ h' odd] and (b) $(\wp'(u))^2 = 4(\wp(u))^3 + a\wp(u) + b$ for some $a, b \in \mathbb{C}$. [See hint given in Example 7.3.8.]

(6) State and prove the general holomorphic IFT used in the proof of Theorem 7.2.1. [Hint: imitate the proof of Prop. 7.1.2, but now making use of Prop. 7.1.1.]