CHAPTER 8

The connectedness of algebraic curves

The main theorem of this chapter will be that the smooth part[1] $C\backslash\mathrm{sing}(C)$ of an *irreducible* algebraic curve $C \subset \mathbb{P}^2$ is path-connected (and then, of course, so is C). For example, in Exercise 5 of Chapter 3, you showed that the complement of the ODP $\hat{p} = [1 : 0 : 0]$ in the singular cubic curve $\{Y^2Z - X^2Z + X^3 = 0\}$, viewed as a complex 1-manifold, is isomorphic to $\mathbb{C}^*$ — which is certainly connected.

Just so that there is no confusion, we should say what the situation is for reducible curves right away and why the result does not generalize. For plane projective algebraic curves with more than one irreducible component, say $C = \cup C_i$, the components C_i must intersect (this will be one consequence of Bezout's theorem later), making C connected. But the complement of the singularities in C will not be connected, as these will include all of the intersection points.

We begin by introducing a new, somewhat technically involved, tool for dealing with singularities, intersections, and projections of curves.

8.1. Resultants and discriminants

Let $\mathbb{D}$ be a unique factorization domain (UFD), where we recall that this is a commutative domain in which each element has a unique factorization into irreducibles, up to reordering and multiplication by units. In a UFD, amongst other things, the notion of a greatest common divisor[2] has meaning. By the Gauss lemma, $\mathbb{D}[y]$ is

[1]We will show that the set $\mathrm{sing}(C)$ of singular points is always finite

[2]Recall that these are well-defined up to units (invertible elements); for example in $\mathbb{C}[x]$ or $\mathbb{C}[x, y]$ the units are $\mathbb{C}^*$, hence the notion of "monic gcd" (which is completely well-defined).

also a UFD. In practice we will always take $\mathbb{D}$ to be $\mathbb{C}$ or $\mathbb{C}[x]$. (Note that $\mathbb{C}[x]$ is a PID, but $\mathbb{C}[x,y]$ is not.)

Consider $f(y) = a_0y^m + a_1y^{m-1} + \cdots + a_m$, $g(y) = b_0y^n + b_1y^{n-1} + \cdots + b_n$ elements of $\mathbb{D}[y]$ with $a_0, b_0 \neq 0$.

8.1.1. DEFINITION. The *resultant*[3] *of f and g*, written $\mathcal{R}(f,g)$, is the element of $\mathbb{D}$ given by the determinant of the $(n+m)\times(n+m)$ *Sylvester matrix*[4]

$$M_{(f,g)} := \left(\begin{array}{cccccccc} a_0 & a_1 & \cdots & \cdots & a_m & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & 0 \\ 0 & \cdots & 0 & a_0 & \cdots & \cdots & a_{m-1} & a_m \\ \hline b_0 & b_1 & \cdots & \cdots & b_n & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & b_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & 0 \\ 0 & \cdots & 0 & b_0 & \cdots & \cdots & b_{n-1} & b_n \end{array}\right).$$

Now writing K for the field of fractions of $\mathbb{D}$, we have the

8.1.2. PROPOSITION. $\mathcal{R}(f,g) = 0 \iff gcd_{K[y]}(f,g) \neq 1.$[5]

PROOF. The gcd (say, h) is nontrivial if and only if

$$Fg = Gf \tag{8.1.3}$$

for some $F = A_0y^{m-1} + \cdots + A_{m-1}$ and $G = B_0y^{n-1} + \cdots + B_{n-1}$ in $\mathbb{D}[y]$. Indeed, if $h \neq 1$ then put $F = f/h$ and $G = g/h$. Conversely, since $\deg F < \deg f$ and $\deg G < \deg g$, and both sides of (8.1.3) factor into the same irreducibles, f and g have a common factor of degree > 0.

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[3]also called "eliminant", since y is eliminated

[4]the line in the matrix is just an organizational device — it has no meaning

[5]two further equivalent conditions: (i) $\deg_y(\mathrm{gcd}_{\mathbb{D}[y]}(f,g)) > 0$; and, noting that $K[y]$ is a PID, so that the ideal $(f,g)_{K[y]} = (\mathrm{gcd}_{K[y]}(f,g))$, (ii) $(f,g)_{K[y]} \neq (1)_{K[y]}$.

In turn, (8.1.3) is equivalent to

$$\begin{aligned} a_0B_0 &= b_0A_0 \\ a_1B_0 + a_0B_1 &= b_1A_0 + b_0A_1 \\ &\vdots \\ a_mB_{n-1} &= b_nA_{m-1} \end{aligned} \tag{8.1.4}$$

being satisfied for some $\{A_i\}_{i=0}^{m-1}, \{B_j\}_{j=0}^{n-1} \subset \mathbb{D}$. To get from (8.1.3) to (8.1.4), just take coefficients of y^{m+n-1}, y^{m+n-2}, $\ldots$, 1.

Now notice that (8.1.4) can be rephrased in matrix multiplication terms: there exist $\{A_i\}, \{B_j\}$ such that

$$^tM_{(f,g)} \cdot \begin{pmatrix} B_0 \\ \vdots \\ B_{n-1} \\ -A_0 \\ \vdots \\ -A_{m-1} \end{pmatrix} = \underline{0}.$$

In other words, we have shown $h \neq 1$ is the same as $\ker(^tM_{(f,g)}) \neq \{0\}$, i.e. $\det(M_{(f,g)}) = 0$. $\square$

8.1.5. DEFINITION. $\mathcal{D}(f) := \mathcal{R}(f, f')$ is the *discriminant* of f. Here f' denotes the formal derivative $\frac{\partial f}{\partial y}$.

8.1.6. EXAMPLE. If $f \in \mathbb{C}[y]$, then $\mathcal{D}(f) \in \mathbb{C}$ is a number, and the criterion

$$\mathcal{D}(f) \text{ vanishes} \iff f \text{ has a multiple root} \tag{8.1.7}$$

follows immediately from Prop. 8.1.2. For the affine curve

$$z^2 = 4y^3 + ay + b$$

to be singular, we need two of the roots of the right-hand side to coincide. That is, by (8.1.7), we need

$$0 = \mathcal{R}(4y^3 + ay + b, 12y^2 + a) = \begin{vmatrix} 4 & 0 & a & b & \\ & 4 & 0 & a & b \\ 12 & 0 & a & & \\ & 12 & 0 & a & \\ & & 12 & 0 & a \end{vmatrix}$$

which after a bit of row-reduction

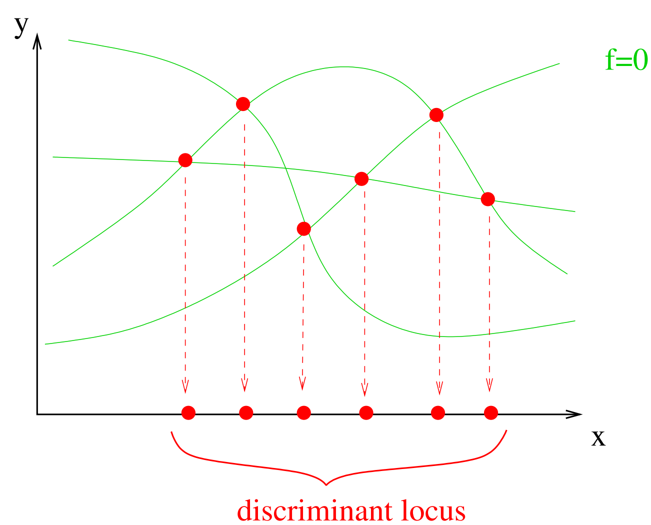
$$= \begin{vmatrix} 4 & 0 & a & b & 0 \\ 0 & 4 & 0 & a & b \\ 0 & 0 & -2a & -3b & 0 \\ 0 & 0 & 0 & -2a & -3b \\ 0 & 0 & 12 & 0 & a \end{vmatrix} = 16(4a^3 + 12 \cdot 9b^2) = 64(a^3 + 27b^2).$$

This recovers the result from Exercise 2 of Chapter 6.

8.1.8. EXAMPLE. If $f \in \mathbb{C}[x, y]$, then $\mathcal{D}(f) \in \mathbb{C}[x]$ is a polynomial and from Prop. 8.1.2 we have:

$$\mathcal{D}(f) \text{ vanishes at } x_0 \iff f(x_0, y) \text{ has a multiple root in } y. \tag{8.1.9}$$

The collection of x_0's where this happens, that is, the set of roots of $\mathcal{D}(f)$, is called the *discriminant locus* for the projection of the affine curve $\{f(x, y) = 0\}$ onto the x-line:



8.1.10. PROPOSITION. *An irreducible (reduced) algebraic curve* $\{F = 0\} \subset \mathbb{P}^2$ *has (if any) finitely many singularities.*

PROOF. The affine polynomial $f(x, y) = F(1, x, y)$ has multiple roots in y for x in the discriminant locus $\Delta = \{(\mathcal{D}(f))(x) = 0\} \subseteq \mathbb{C}$. We may assume f has positive degree in y, since otherwise $V(f)$ is just a vertical line.

Since f is irreducible in $\mathbb{C}[x,y]$ of positive degree in y, the identical vanishing of $\mathcal{D}(f)$ would imply that $V(f)$, hence $\bar{V}(F)$, was nonreduced. So $\mathcal{D}(f)$ is a nontrivial polynomial, and Δ is finite:

$$\#\{x \in \mathbb{C} \mid \exists y \text{ such that } f(x,y) = f_y(x,y) = 0\} < \infty \tag{8.1.11}$$

It is easy to argue directly[6] that were $V(f)$ to contain a vertical line $\{x = \alpha\}$, then $(x - \alpha)$ would divide f (contradicting irreducibility). So by (8.1.11) and Prop. 2.1.15, in fact

$$\#\{p \in \mathbb{C}^2 \mid f(p) = f_y(p) = 0\} < \infty.$$

The set in brackets includes all singularities of $V(f)$. The only possible *additional* singularities of $\bar{V}(F)$ are the (finitely many) points where it meets the line at ∞. □

8.2. Monodromy and connectedness

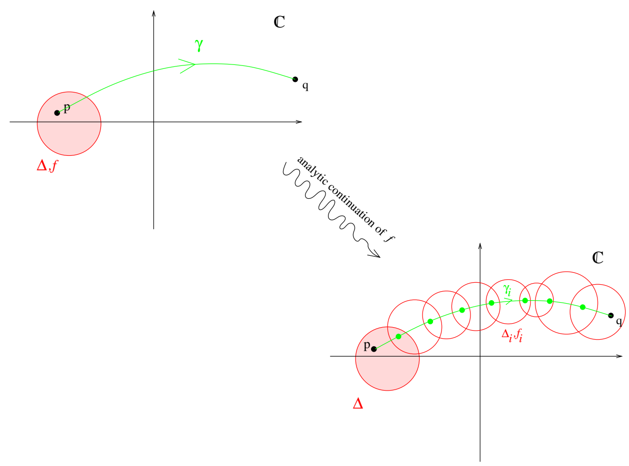
Let $\Omega \subseteq \mathbb{C}$ be a *region*, that is, an open connected subset. Let $\Delta \subset \Omega$ be a small disk about a point $p \in \Omega$ on which one is given a holomorphic function, $f \in \mathcal{O}(\Delta)$. We are interested in the question of when f extends to a holomorphic function on all of Ω. To see why this doesn't always happen, take $\Omega = \mathbb{C}$ and Δ a small disk about $z = 1$: then $f = \frac{1}{z}$ only extends to a holomorphic function on $\mathbb{C}^*$. Even worse, $f = \log(z)$ becomes "multivalued" on $\mathbb{C}^*$ and so (as a holomorphic function) only extends to $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$.

To give a condition which will ensure the existence of a well-defined holomorphic extension, we need the concept of *analytic continuation*. Define a *path* $\gamma \subset \Omega$ from p to q to be the image of a continuous function $\mathsf{P} : [0,1] \to \Omega$ with $\mathsf{P}(0) = p$ and $\mathsf{P}(1) = q$. (Here we are allowed to pick $q = p$.) An analytic continuation of f along γ consists of

- a partition of γ into segments $\{\gamma_i\}_{i=0}^N$,
- a covering of γ by disks $\Delta_i \supset \gamma_i$ (with $\Delta_0 = \Delta$), and

[6] or you can wait for Study's lemma in the next Chapter

- functions $f_i \in \mathcal{O}(\Delta_i)$ (with $f_0 = f$) satisfying $f_i \equiv f_{i+1}$ on $\Delta_i \cap \Delta_{i+1}$:



If we continue f along two different paths from p to q and compare the "results", i.e. the last function $f_N \in \mathcal{O}(\Delta_N)$ (in the neighborhood of q) in each case, these need not agree. In the above example of $f = \log(z)$ on a disk about $p = \{z = 1\}$, we can analytically continue f along any path in $\mathbb{C}^*$. However, if we take $q = p$ so that the path is closed, then we do not have $f_N(p) = f_{(0)}(p)$: they differ by $2\pi\sqrt{-1}$ times the winding number of the path about $z = 0$, hence the "multivaluedness" referred to above. This problem only occurs, however, for non-simply-connected regions:

8.2.1. PROPOSITION. [RIEMANN MONODROMY PRINCIPLE] *Given a region $\Omega \subseteq \mathbb{C}$ which is simply connected, i.e. $\pi_1(\Omega) = \{0\}$. Let $\Delta \subset \Omega$ be a small disk, and assume that $f \in \mathcal{O}(\Delta)$ can be analytically continued along any path $\gamma \subset \Omega$ starting at $p \in \Delta$. Then there exists $\tilde{f} \in \mathcal{O}(\Omega)$ extending f.*

We will frequently use this together with the

8.2.2. PROPOSITION. [HEREDITY PRINCIPLE] *For $F(x, y) \in \mathcal{O}(\mathbb{C}^2)$ and $f \in \mathcal{O}(\Delta)$ satisfying*

$$F(x, f(x)) = 0, \tag{8.2.3}$$

the analytic continuation of f along any path γ will also satisfy (8.2.3).

PROOF. Since F and each f_i in the analytic continuation are holomorphic, so is each $F(x, f_i(x))$ (on Δ_i). But $F(x, f(x)) \equiv 0$ on $\Delta = \Delta_0$ by assumption, and since $f = f_0 \equiv f_1$ on $\Delta_0 \cap \Delta_1$, we have $F(x, f_1(x)) \equiv 0$ on $\Delta_0 \cap \Delta_1$ and therefore (by basic complex analysis) on all of Δ_1. Simply iterate this argument for $i = 1, \ldots, N$. $\square$

Now given an affine algebraic curve $C = \{f_0(x_0, y) = 0\}$ with f_0 of degree n, it is convenient to write C as the vanishing locus of a monic polynomial in y over $\mathbb{C}[x]$:

$$f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) = 0. \tag{8.2.4}$$

This is acheived by performing a change of variable $x_0 = x + \lambda y$ and writing $f(x,y) := f_0(x_0, y) = f_0(x + \lambda y, y)$, which has coefficient of y^n depending polynomially on λ; choose λ so that this coefficient is 1. (The main point is that in $f_0(x_0, y)$, the y^n term may be zero, and we want to remedy that.)

Having put the equation of C in this form, we write

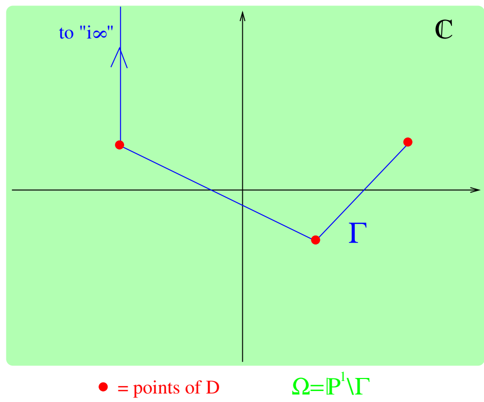
$$\begin{array}{rccc} \pi : & C & \longrightarrow & \mathbb{C} \\ & (x,y) & \longmapsto & x \end{array}$$

for the projection of the curve to the x-axis. Writing $D := \{\mathcal{D}(f)(x) = 0\}$ for the *discriminant locus* of this projection, by (8.1.9) we have that for $x \in \mathbb{C} \backslash D$, the fibre $\pi^{-1}(x)$ consists of n distinct points. For some fixed disk $\Delta \subset \mathbb{C} \backslash D$, label these points $\{y_1(x), \ldots, y_n(x)\}$. Notice that $\mathcal{R}(f, \frac{\partial f}{\partial y}) = \mathcal{D}(f)(x) \neq 0$ implies that $\frac{\partial f}{\partial y} \neq 0$ on $\{f = 0\} \cap \pi^{-1}(\Delta)$, so that the holomorphic IFT (Prop. 7.1.2) gives $y_i(x) \in \mathcal{O}(\Delta)$. The point here is that the "roots" of (8.2.4) in y are algebraic — hence multivalued — functions of x, but we can take well-defined holomorphic branches of them over Δ. As we shall see, the multivaluedness will intertwine them outside Δ.

Label the points of $D = \{p_1, \ldots, p_K\}$, and let Γ be the path in $\mathbb{P}^1$ consisting of segments connecting ∞ to p_1, p_1 to p_2, and so on up to p_K. Then the region $\Omega := (\mathbb{P}^1 \backslash \Gamma) \subset \mathbb{C}$ is simply connected. By Propositions 8.2.1-8.2.2, the $\{y_i(x)\}$ extend to functions in $\mathcal{O}(\Omega)$

which still satisfy

$$f(x, y_i(x)) = 0. \tag{8.2.5}$$



Analytically continued *through* Γ in $\mathbb{C}\backslash D$, the y_i continue to satisfy (8.2.5) by the heredity principle, but may *swap*.

8.2.6. EXAMPLE. $f(x,y) = y^3 - x$, $D = \{0\}$, $\Gamma = \mathbb{R}_{\leq 0}$. Passing through Γ cyclically permutes $y_1(x) = \sqrt[3]{x}$, $y_2(x) = e^{\frac{2\pi\sqrt{-1}}{3}}\sqrt[3]{x}$, $y_3(x) = e^{\frac{4\pi\sqrt{-1}}{3}}\sqrt[3]{x}$.

This swapping (or permutation)[7] of the $y_i(x)$ gives rise to an equivalence relation "$\sim$": $y_i(x) \sim y_j(x)$ if one may be analytically continued into the other in $\mathbb{C}\backslash D$. An equivalence class is just all the $\{y_\lambda\}$ which are equivalent to a given y_i in this sense.

8.2.7. PROPOSITION. *For any equivalence class E of $\sim$, formed (reordering if necessary) by $y_1(x), \ldots, y_m(x)$,*

$$\prod_{\lambda=1}^{m}(y - y_\lambda(x)) \tag{8.2.8}$$

belongs to $\mathbb{C}[x,y]$.

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[7]The transformations of an algebraic structure arising from its transport around loops (in this case, loops in $\mathbb{C}$ about points of D) are what is meant by the word *monodromy* in general. So the Riemann monodromy principle is really a statement about the *absence* of monodromy.

Put differently: while the $\{y_\lambda(x)\}_{\lambda=1}^m$ are multivalued algebraic functions on $\mathbb{C}\backslash D$, the elementary symmetric polynomials in them are not multivalued; in fact, they are polynomials![8]

8.2.9. COROLLARY. *C irreducible $\implies C\backslash\pi^{-1}(D)$ is connected ($\implies$ C connected).*

PROOF ASSUMING PROP. 8.2.7. If $f \in \mathbb{C}[x,y]$ doesn't factor, then by the Proposition there can be only one equivalence class: $E = \{1,\ldots,n\}$. So the complete set of "branches" $\{y_i(x)\}$ is acted on transitively by monodromy about D, and one can therefore draw a path on $C\backslash\pi^{-1}(D)$ connecting any two points. $\square$

We now prove Prop. 8.2.7, using some theorems from complex analysis. In particular, recall that *Rouché's theorem* asserts that for two holomorphic functions $f, g \in \mathcal{O}(\mathfrak{R})$ on a simply connected region[9] with $|f| > |g|$ on a simple closed curve $\gamma \subset \mathfrak{R}$, $f+g$ and f have the same number of zeroes (counted with multiplicity) inside γ.

PROOF. The product (8.2.8) is clearly well-defined on $\mathbb{C}\backslash D$, since monodromy about D simply swaps its factors; hence it is in $\mathcal{O}(\mathbb{C}\backslash D)$. Write

$$\prod_{\lambda=1}^m (y - y_\lambda(x)) = \sum_{j=0}^m (-1)^{m-j} e_{m-j}(y_1(x), \ldots y_m(x)) y^j \tag{8.2.10}$$

where $e_{m-j}(y_1(x),\ldots,y_m(x)) =: e_{m-j}(x)$ denotes the elementary symmetric polynomials in the $\{y_\lambda\}$. Again, because these are not changed under monodromy, we have $e_{m-j}(x) \in \mathcal{O}(\mathbb{C}\backslash D)$. Observe that given $\alpha \in D$ with neighborhood $\mathcal{N}_\alpha$ (a small disk about α), the polynomials $a_j(x)$ from (8.2.4) satisfy

$$x \in \mathcal{N}_\alpha \implies |a_j(x)| \leq M \ (\forall j)$$

for some $M \in \mathbb{N}$. Fixing $x_0 \in \mathcal{N}_\alpha \setminus \{\alpha\}$, put $a_j = a_j(x_0)$ and

$$\mathfrak{F}(y) = y^n \ , \qquad \mathfrak{G}(y) = y^n + a_1 y^{n-1} + \cdots + a_n,$$

8 we use λ to index E (i.e. $1,\ldots,m$) and i to index $\{1,\ldots,n\}$

9 the main point is that $\mathfrak{R}$ should contain the "interior" of γ

so that the $\{y_i(x_0)\}$ are the roots of $\mathfrak{G}$. On $\gamma = \{|y| = M+1\} \subset \mathbb{C}$, we have

$$|\mathfrak{G} - \mathfrak{F}| = |a_1 y^{n-1} + \cdots + a_n| \leq M\left((M+1)^{n-1} + \cdots + 1\right)$$
$$= (M+1)^n - 1 < (M+1)^n = |\mathfrak{F}|.$$

By Rouché, $\mathfrak{F}$ and $\mathfrak{G}$ have the same number of zeroes inside γ; since $\mathfrak{F} = y^n$ has n zeroes (at $y = 0$!), we find that

$$|y_j(x_0)| < M+1 \text{ for all } j = 1, \ldots, n \text{ and } x_0 \in \mathcal{N}_\alpha.$$

Consequently the $e_k(x) \in \mathcal{O}(\mathbb{C}\backslash D)$ are bounded on $\mathcal{N}_\alpha \cap (\mathbb{C}\backslash D) = \mathcal{N}_\alpha \backslash \{\alpha\}$, and so by the Riemann removable singularity theorem extend across $\{\alpha\}$. Doing this for each $\alpha \in D$, we conclude that $e_k(x) \in \mathcal{O}(\mathbb{C})$.

So the coefficients of the y^j's in (8.2.10) are entire functions of x. To prove that they are polynomials in x, we shall have to consider their behavior about $x = \infty$. If we work in the local coordinates $\tilde{x} = \frac{1}{x}$, $\tilde{y} = \frac{y}{x}$ about $[0 : 1 : 0]$ in $\mathbb{P}^2$, then the polynomial (8.2.4) defining C becomes[10]

$$\tilde{x}^n f\left(\tfrac{1}{\tilde{x}}, \tfrac{\tilde{y}}{\tilde{x}}\right) = \tilde{y}^n + \left(\tilde{x} a_1(\tfrac{1}{\tilde{x}})\right)\tilde{y}^{n-1} + \cdots + \tilde{x}^n a_n(\tfrac{1}{\tilde{x}}),$$

with roots

$$\tilde{y}_i(\tilde{x}) = \tilde{x} y_i(\tfrac{1}{\tilde{x}}). \tag{8.2.11}$$

Let $\mathcal{N}_\infty \subset \mathbb{P}^1$ be a small neighborhood of $\tilde{x} = 0$ (i.e. $x = \infty$) and $\mathcal{N}_\infty^* := \mathcal{N}_\infty \backslash \{\tilde{x} = 0\}$. By (8.2.11), the monodromy of the $\{\tilde{y}_i\}_{i=1}^n$ about $\tilde{x} = 0$ stabilizes the subset $\{\tilde{y}_\lambda\}_{\lambda=1}^m$, so that the

$$e_k(\tilde{y}_1(\tilde{x}), \ldots, \tilde{y}_m(\tilde{x})) = \tilde{x}^k e_k(\tfrac{1}{\tilde{x}})$$

are well-defined holomorphic functions on $\mathcal{N}_\infty^*$. Since $\deg(a_j(x)) \leq j$, the $\tilde{x}^j a_j(\frac{1}{\tilde{x}})$ are polynomials in $\tilde{x}$ hence bounded on $\mathcal{N}_\infty$. Using Rouché as above, the $e_k(\{\tilde{y}_\lambda(\tilde{x})\}_{\lambda=1}^m)$ are also bounded on $\mathcal{N}_\infty^*$, and thus extend to holomorphic functions on $\mathcal{N}_\infty$.

[10]Here we are essentially taking the projective completion of C and restricting that to $U_1 \subset \mathbb{P}^2$.

In other words, $e_k(x) = e_k(\frac{1}{\tilde{x}})$ has a pole at $x = \infty$ of order at most k. Since $e_k(x)$ was also holomorphic on $\mathbb{C}$, we have $e_k \in \mathcal{K}(\mathbb{P}^1)$. Now $\mathcal{K}(\mathbb{P}^1) \cong \mathbb{C}(\mathbb{P}^1)$ and so $e_k(x) = \frac{P(x)}{Q(x)}$ where P, Q are polynomials; since its only pole is at ∞, Q is a constant. Therefore each $e_k \in \mathbb{C}[x]$, and with (8.2.10) we see that (8.2.8) is a polynomial in $\mathbb{C}[x, y]$. $\square$

Exercises

(1) Are the real points[11] of a smooth algebraic curve $\subset \mathbb{P}^2$ necessarily connected?

(2) For what values of a, b does $x^4 + ax + b$ have a multiple root?

(3) Find the intersection points of the two conics $x^2 + 2y^2 = 3$ and $x^2 + xy + y^2 = 3$ in $\mathbb{C}^2$, starting by taking a resultant.

(4) Consider the family of affine curves $\{C_\lambda\}_{\lambda \in \mathbb{C}}$ defined by $\lambda xy = (x+1)(y+1)(x+y+1)$. Take discriminants twice, first in $\mathbb{C}[\lambda, x]$ (eliminating y) and then in $\mathbb{C}[\lambda]$ (eliminating x), to find the set of (three) values of λ for which C_λ is singular. Why does this work? [Hint: you may wish to use a computer to take the second discriminant.] This is called the *discriminant* locus of the family of curves.

(5) Let C be defined by $y^{24} = x^{12}(x-1)^3(x+1)^3(x-2)^4(x+2)^2$, with covering map $\pi\colon C \to \mathbb{C}$ sending $(x, y) \mapsto x$ as above. Explicitly describe the action of monodromy about $D = \{0, \pm 1, \pm 2\}$ on the "branches" (or "decks", or "sheets") of C over $\mathbb{C}$, as given by the $\{y_i(x)\}$ on $\mathbb{C} \setminus \Gamma$. Conclude that $C \setminus \mathrm{sing}(C)$ is connected. [You can use $\Gamma = [-2, 2]$ here.]

(6) In the previous exercise, the fundamental group $\pi_1(\mathbb{C} \setminus D)$ acts on the set of branches through a cyclic (abelian) group. If we instead take C to be the curve $(1-x)y^6 + (1+x)y^3 + (1-x) = 0$, can you show that π_1 acts through a *non*abelian group? [Start by finding D, which consists of 3 points. Interpret $\mathbb{C}(x)$ as a subfield of $\mathbb{C}(y)$ (by presenting x as a rational function of y) and describe the monodromy action via automorphisms of $\mathbb{C}(y)/\mathbb{C}(x)$.]

[11] i.e. points on the curve which can be written $[X_0 : X_1 : X_2]$, with all $X_i \in \mathbb{R}$. See also Exercise 5 of Chapter 2 and Exercise 2 of Chapter 5.