Lecture 10: Fractional linear transformations

I. Group structure

In this course we'll meet the automorphism groups (of 1-to-1 analytic self-maps)

\[ \text{Aut}(\mathbb{C}) = \{ z \mapsto \alpha z + \beta \mid \alpha, \beta \in \mathbb{C}, \alpha \neq 0 \} \]

\[ \text{Aut}(D_1) = \{ z \mapsto e^{i\theta} \frac{z-a}{1-z} \mid \theta \in \mathbb{R}, a \in D_1 \} \]

\[ \text{Aut}(\mathbb{H}) = \{ z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{R}, ad-bc = 1 \} \]

as well as the isomorphism

\[ h \xrightarrow{\equiv} D_1 \]

\[ z \mapsto \frac{z-i}{z+i} \]

All are fractional linear transformations (FLT)

\[ f_M(z) := \frac{az+b}{cz+d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{C}). \]

\[ a, b, c, d \in \mathbb{C} \]

\[ ad-bc \neq 0 \]
If we consider the FLT's as a group under composition of functions, then

\[ M \rightarrow f_M(z) \]

define a surjective group homomorphism

\[ \text{GL}_2(\mathbb{C}) \rightarrow \text{FLT} : \]

- If \( N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \),

\[ (f_M \circ f_N)(z) = \frac{a \left( \frac{A z + B}{C z + D} \right) + b}{c \left( \frac{A z + B}{C z + D} \right) + d} = \ldots \]

\[ = \frac{(aA + bC)z + (cB + dD)}{(cA + dC)z + (cB + dD)} = f_{M \cdot N}(z) . \]

- \( f_M(z) = z \) (identity) \( \iff \ M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) (acting on \( \mathbb{C}^\times \)):

\[ \Rightarrow \frac{dz + 0}{0z + 1} = z \]

\[ \Rightarrow \text{if } \frac{a z + b}{a z + d} = z (\forall z \in \mathbb{C}), \text{ then } 0 = ca^2 + (d-a)b + (b^2) \]

\[ \Rightarrow c = (d-a) = b = 0 \]

\[ \Rightarrow M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \]
So in fact our homomorphism factors through
\[
PGL_2(\mathbb{C}) \cong \frac{GL_2(\mathbb{C})}{\{x \mathbb{C}\} : x \in \mathbb{C}^*}
\]
\[
\cong \frac{SL_2(\mathbb{C})}{\{e^{2\pi i} \} \cdot \{e^{2\pi i a}\}}
\]

But what are the FLT's really transformation of?

We can view \( f_n \) as a function from
\[
\mathbb{C} \setminus \{-\frac{a}{c}\} \cong \mathbb{C} \setminus \left\{ \frac{a}{c}\right\}
\]

which may be extended to
\[
\mathbb{C} \cong \mathbb{C}
\]

by setting
\[
-\frac{a}{c} \mapsto \infty
\]
\[
\infty \mapsto \frac{a}{c}
\]

Remark: In fact, if we define \( \text{Aut}(\mathbb{C}) \) by

- \( f \in \text{Aut}(\mathbb{C}) \) means (i) \( f : \mathbb{C} \setminus \{ \frac{a}{c} \} \to \mathbb{C} \setminus \{ \frac{a}{c} \} \)
- (ii) \( f \) is a small neighborhood
- (iii) \( f(\omega) : U(\omega) \to U(\omega) \)
Then

- \( \exists f^{-1} \) with similar properties (for any such \( f \))

and

- \( \text{FLT} \cong \text{Aut}(\hat{\mathbb{C}}) \)

Heuristic sketch of (\( \star \)): The \( \text{FLT} \cong \text{Aut}(\hat{\mathbb{C}}) \) is easy. Now let \( f \) be an arbitrary \( \hat{\mathbb{C}} \)-analytic automorphism of \( \hat{\mathbb{C}} \): automorphism \( \Rightarrow \) (no essential singularities) \( \Rightarrow \) no common limit points for zeroes & poles.

By our previous results, we can therefore have no limit points of zeroes or poles of \( f \) (otherwise \( f \) or \( \frac{1}{f} \) is identically zero). Since \( \hat{\mathbb{C}} \) is compact, there are therefore only finitely many zeroes and poles; multiplying by a rational function gets rid of these, leaving us with an analytic map \( \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \). Since (again) \( \hat{\mathbb{C}} \) is compact, this is bounded, hence by Liouville's constant. We conclude that \( f \) was rational (= \( P/Q \), \( P \& Q \) polynomials). But then, removing any common factors, the mapping degree of \( f \) is the maximum of \( \deg(P) \& \deg(Q) \). This must be 1 for \( f \) to be \( 1-1 \).

So \( P \& Q \) are constant or linear \( \Rightarrow f \in \text{FLT} \). \( \Box \)
This used "everything", including Casorati-Weierstrass, Fundamental Thm. of Algebra, Liouville, etc.!! Mathematica for what still has to be proved, I guess...

So we have

\[ \frac{\text{SL}_2(\mathbb{C})}{\{\pm 1\}} \cong \text{FLT} \cong \text{Aut}(\hat{\mathbb{C}}). \]

(or $\text{PGL}_2(\mathbb{C})$) (unofficially)

The group structure on FLT clarifies lots of stuff:

- Composition inverses:
  \[ (\begin{array}{cc} a & b \\ c & d \end{array})^{-1} = \left( \begin{array}{cc} d & -b \\ -c & a \end{array} \right) \]  
  (using $ad-bc=1$)

\[ \Rightarrow f^{-1}_{(ab, cd)}(w) = \frac{dw-b}{-cw+a}. \]

Indeed, that FLT is a group (under composition) means that all FLT's are 1-to-1.

- Iwasawa decomposition: for real FLT's (more on these below)

  one has

  \[ \text{SL}_2(\mathbb{R}) \cong N \cdot A \cdot K \]

  any $M = \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} \sqrt{g} & 0 \\ 0 & \sqrt{g}^{-1} \end{array} \right) \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right)$
Why 2x2 matrices? If you think of C as
\[ P' = \mathbb{C}^2 \setminus \{(0,0)\}, \]
\[ \mathbb{C} \xrightarrow{a \in \mathbb{C}^\times} \left[ \begin{array}{c} x_1 + x_2 \\ x_1 \end{array} \right] \]
and write elements \([x_1 : x_2]\) instead of \([x_1 : x_2]\), then
\[
\left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right] = \left[ \begin{array}{c} ax_1 + bx_2 \\ cx_1 + dx_2 \end{array} \right]
\]
\[
\left[ \begin{array}{c} x_1 : x_2 \end{array} \right] \xrightarrow{a \neq 0} \left[ \begin{array}{c} x_1 : x_2 \\ x_1 \end{array} \right] \xrightarrow{\infty} \left[ \begin{array}{c} x_1 : x_2 \\ 1 \end{array} \right] \xrightarrow{\infty} \left[ \begin{array}{c} x_1 : x_2 \\ 1 : 0 \end{array} \right]
\]
So really, FCTs are linear transformations acting on lines in \( \mathbb{C}^2 \) through the origin.
II. Action on circles

Let
\[
C := \text{set of circles & lines on } \mathbb{C}
\]

\[
\hat{C} := \text{set of circles on } \hat{\mathbb{C}}
\]

(viewed as a sphere via stereographic projection)

(See end of Lecture 1)

**Theorem**
(a) \( f \in \text{FLT} \) takes \( C \rightarrow C \)
(b) This action is transitive.

**Examples**

1. Lines are special kinds of circles: ones that contain \( \{ \infty \} \).

\[ C \]

\[ \hat{C} \]
2. \( z \mapsto \frac{z - i}{z + i} = w \) (takes \( z \to \infty \) to \( D_2 \))

In some sense, the last picture ("degenerate Steiner circles") is telling you "what Cartesian coordinates look like at \( \infty \)" (about \( w = 1 \)).

3. \( z \mapsto \frac{1}{z} = \zeta(z) \) (inversion)

Looks like rotating \( C \) 180° about the orange line.
f(z) = \frac{az+b}{cz+d} \text{ with } a, b, c, d \in \mathbb{C} \\
\text{ sends } \mathbb{R} \to \mathbb{R} \text{ (and these are the only } \text{ FLTs doing this); if } ad - bc > 0, \text{ then also sends } \mathbb{H} \to \mathbb{H}. \text{ These are } \text{ automorphisms of } \mathbb{H} \text{ which (in the } \mathbb{H} \text{ picture) don't mix the upper & lower hemispheres.}

\text{The N.A.K decomposition certainly suggests that they preserve circles: translation by } \alpha; \text{ dilation by } \gamma; \text{ only issue is what the } "K" \text{ notions/FLT do.}

\text{Proof of (a): } F(z) = \frac{az+b}{cz+d} \text{, } a, b, c, d \in \mathbb{C}.

c = 0 \Rightarrow F(z) = \frac{a}{d} z + \frac{b}{d} = \mathcal{L}_{b/d} \circ \mathcal{M}_{a/d} \text{.} \\
c \neq 0 \Rightarrow F(z) = \frac{cz+a}{cz+d} + \frac{b-d}{cz+d} \text{.}
\[ \frac{a}{c} \frac{c^2 + d}{c^2 + d} + \frac{b/c - \frac{d^2}{c^2}}{2 + d/c} \]

\[ = \tau_{a/c} \circ \mu_{bc-d^2/c^2} \circ J \circ \tau_{d/c} . \]

Now, \( \tau \) preserves \( C \); how about \( J \)?

(Then \( F \) is a composition of such things; done.)

\[ J(x+iy) = u+iv \Rightarrow J(u+iv) = x+iy \]

\[ \frac{1}{u+iv} = \frac{u}{u^2+v^2} + i \frac{-v}{u^2+v^2} \]

Hence, given an object

\[ G = \{ A(x^2+y^2) + Bx + C_y + D = 0 \} \]

\( G \), we can rewrite the equation in \( u,v \) to get

\[ J(G) = \{ A \left( \frac{u^2+v^2}{u^2+v^2} \right) + B \frac{u}{u^2+v^2} - C \frac{v}{u^2+v^2} + D = 0 \} \]

\[ = \{ A + Bu - Cv + D (u^2+v^2) = 0 \} \]

which is again in \( C \).
To prove (b), we'll need a

**Lemma** Given \( \{z_1, z_2, z_3\} \) distinct \((\in \mathbb{C})\), \( w_1, w_2, w_3 \) distinct, 

\[ \exists \! f_m \in \text{FLT} \text{ sending } z_i \mapsto w_i \ (i=1,2,3). \]

**Proof of Theorem 3:** First, send \( w_3/z_3 \) to \( 0,1,\infty \):

\[
\begin{align*}
  f(z) &= \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}, \\
  g(w) &= \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}, \\
  z_3 &\mapsto f = 0,1,\infty & w_3 &\mapsto g
\end{align*}
\]

\[
(g^{-1} f)(z_i) = w_i \ (\forall i).
\]

So take \( w = g^{-1}(f(z)) \), i.e. \( g(w) = f(z) \).

**Example:** Find \( f_m \) sending \(-1,0,1 \mapsto -1, i, 1\).

Set \( \frac{z+1}{z-1} = \frac{w+1}{w-1} \), i.e. \( \boxed{w = \frac{z+1}{iz+1}} = f_m(z) \Rightarrow M = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \).
Proof: If \( f, g \) both map \( \{2i, 3\} \to \{\omega_i, \omega_3\} \), then \( g^{-1} \) of fixes \( \{2i, 3\} \). Let \( F \) send \( \{2i, 3\} \) to \( \{0, 1, 2\} \). Then \( h = F \circ g^{-1} \circ f \) fixes 0, \( \infty \).

\[
h(z) = \frac{a + b}{c + d} : x \to \infty \Rightarrow c = 0 \quad a \neq 0
\]

\[
\Rightarrow \quad h(z) = A + B : 0 \to 0 \Rightarrow B = 0
\]

\[
\Rightarrow \quad h(z) = A : 1 \to 1 \Rightarrow A = 1
\]

\[
\Rightarrow \quad z = t = \text{id}(z).
\]

So \( \text{id} = F \circ g^{-1} \circ f \).

\[
F \circ \text{id} \circ F = g^{-1} \circ f
\]

\[
\text{id} = g^{-1} \circ f
\]

\[
g = f.
\]

**Corollary:** \( \exists! \) \( g \in C \) through any 3 distinct \( \omega_i \in C \).

Proof: \( \boxed{\text{The lemma provides } f \in \text{Filt}} \)

sending \( 0, 1, 2 \to \omega_1, \omega_2, \omega_3 \).
Since \( \hat{\mathbb{R}} \subset S \), part (a) of the Theorem implies \( f(\hat{\mathbb{R}}) \subset S \).

So take \( C = f(\hat{\mathbb{R}}) \).

(!) If there are two, then applying \( f^{-1} \) gives 2 elements of \( C \) through \( 0, 1, \infty \). But there is no circle in \( C \) through these points, and the only line is \( \hat{\mathbb{R}} \).

(proof of 6)

Given \( C, C' \in S \), take

\( \{z_1, z_2, z_3 \in C \} \) distinct

\( \{w_1, w_2, w_3 \in C' \} \) distinct.

Let \( f \) send

\( z_i \rightarrow w_i \) \( (vi) \).

Then \( f(C) \subset S \) and contains the \( \{w_i\} \).

By the Corollary, \( f(C) = C' \).

Coming out of this discussion are 3 things we'd like to investigate in the next lecture:

- Cross-ratio
- Symmetry & orientation (if \( C \rightarrow C' \), what about fixed points)