Lecture 12: Constructing

Conformal equivalencies (CE)

When we do the Riemann mapping theorem later on, we'll get more sophisticated methods to do this, simply by considering the map along the boundary. For now, the tools are (for constructing or studying maps)

• FlTs. These give CE's from \( \hat{\mathbb{C}} \to \hat{\mathbb{C}} \).

It's also easy to see what these do since they take \( \mathbb{C} \to \mathbb{C} \).

• Powers. \( z \mapsto z^a = e^{a \log z} \) then

\[
S(\varphi_1, \varphi_2) := \{ z \in \mathbb{C}^* \mid \varphi_1 < \arg z < \varphi_2 \}
\]

\( 0 < \varphi_2 - \varphi_1 \leq 2\pi \) to \( S(\alpha \varphi_1, \alpha \varphi_2) \),

which makes sense expressed this way (and is -1) if \( 0 < \alpha (\varphi_2 - \varphi_1) \leq 2\pi \). \([e^{\varphi} = e^{\log(e^{i\varphi})} = e^{i\varphi}]\)
• level curves. We know that
  \[ F(x=x_0) \] and \[ F(y=y_0) \] are orthogonal (in the image)
  \[ F'(u=u_0) \] and \[ F'(v=v_0) \] are orthogonal (in the domain)
(or use polar coordinates).

• \( \text{exp} \) of \( \log \). \( z \mapsto e^z \) maps strips to sectors.
  \[ \{ x+i y : (x, y) \in (-1, 1) \times (0, \pi) \} \]

Examples

1. \( z \mapsto z^3 \) gives CE at "I, II, III" with the slit disk.
primary at \( v = v_0 \) is given by

\[
v_0 = \text{Im} \left( (x+iy)^3 \right) = 3y \sqrt{x^2 - y^2} = y(3x^2 - y^2).
\]

If \( v_0 = 0 \), this is \( y/x = 0, \sqrt{3}, -\sqrt{3} \).

Rotating the left-hand figure by \( 30^\circ \) gives the preimage of the vertical lines. (The curves in the rotated figure are 1 to those in this one.)

By drawing a (say) vertical line on top of the figure showing the level curves, you can “see” that \( x = x_0 > 0 \) gets sent to

\[\hat{c} \cap [-1,-1] \rightarrow D_1\]
Solve for inverse:

\[
\begin{align*}
\mathbf{B}^{-1} &= \frac{1}{2} \left( \mathbf{w} + \mathbf{w}^T \right) \\
\mathbf{B} &= \mathbf{w} \mathbf{w}^T - 1
\end{align*}
\]
Consider the set of (confocal) ellipses in the $z$-plane with foci $\pm 1$:

$$|z + 1| + |z - 1| = C \ (> 2)$$

Squaring:

$$2|z|^2 + 2|z^2 - 1| = C^2 - 2$$

C.E. $\frac{1}{2} |4z^2 - 4|$.

$$\frac{1}{2} (w + \frac{1}{w})(\bar{w} + \frac{1}{\bar{w}}) + \frac{1}{2} |w^2 + \bar{w}^2 - 2| = C^2 - 2$$

$$|w - \frac{1}{w}|^2 = (w - \frac{1}{w})(\bar{w} - \frac{1}{\bar{w}})$$

$$|w|^2 + \frac{1}{|w|^2} = C^2 - 2$$

$$|w|^4 + (2 - C^2)|w|^2 + 1 = 0$$

$$|w|^2 = \frac{(C^2 - 2) \pm \sqrt{C^4 - 4c^2}}{2} = \frac{C^2 - 2 \pm C \sqrt{c^2 - 4}}{2}$$

Choose $\sqrt{c^2 - 4}$ root so that $|w| < 1$.

$$|w| = \sqrt{\frac{c^2 - 2}{c^2}(1 - \sqrt{1 - \frac{4}{c^2}})} - 1$$

$$= R(C) \left( \approx \frac{1}{C^2} \text{ for } C \text{ large} \right)$$

For $C$ small $\to 2^+$.

$R(C) \to 1^-$

i.e. they map to circles!
Well, \( z \mapsto z^2 \) would at least open up the \( \frac{1}{4} \)-plane \( R \) sits in, to a \( \frac{1}{2} \)-plane!

But look:

\[
    z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy) = u + iv
\]

\[\Rightarrow x^2 - y^2 = -k \quad \text{(i.e. } y^2 = x^2 + k \text{) goes to } u = -k.\]

Since \( R \) is filled out by the hyperbolae \( y^2 = x^2 + k \) for \( k > 1 \) (by definition) and these are mapped in 1-1 fashion onto lines \( u = -k \) (for \( -k < -1 \)),

all we then have to do is add 1 and multiply by \(-i\): so we get

\[ z \mapsto -i(z^2 + 1). \]
Works b/c FLT takes circle from a & b to
"circle" thru O & a
\( b \rightarrow \frac{b-a}{a-b} \)

\( S(\varphi, \theta + \varphi) \)

take appropriate power,
mult. by some e

You better know what to do here...

\( \frac{1}{z-a} \)

Rotate and translate, copy loop
6. \n\[ D \xrightarrow{\text{Segment}} (\mathbb{C} \setminus \{a\}) \xrightarrow{\text{map sending}} D_1 \xrightarrow{\varphi} h \]

\[ z \mapsto -i \frac{z+1}{z-1} \]

\[ h \setminus \text{Segment} \]

\[ w \mapsto w/b \text{ followed by } z \mapsto \frac{z^3}{b} + 1 \]

7. \n\[ \text{composite map:} \]
\[ z \mapsto \frac{\sqrt{\left(-\frac{z+1}{b} + 1\right)^2 + 1} - i}{\sqrt{\left(-\frac{z+1}{b} - 1\right)^2 + 1} + i} \]

\[ 2 \mapsto \frac{2-a}{2-b} \]

\[ \text{power, etc.} \]

\[ \text{now in situating of } \mathbb{C} \]

\[ \beta \]
\[ z \mapsto z + \frac{1}{z} \quad (= w = f(z)) \]

Proof:

\[ w = x + iy + \frac{x - iy}{(x+iy)(x-iy)} \]

\[ = x \left( 1 + \frac{1}{x^2 + y^2} \right) + iy \left( 1 - \frac{1}{x^2 + y^2} \right) \]

For \( z \in U \Rightarrow w \in h \):

\[ |z| > 1 \quad \Rightarrow \quad \{ x^2 + y^2 > 1 \} \quad \Rightarrow \quad \{ \frac{1}{x^2 + y^2} < 1 \} \]

\[ \text{Im}(z) > 0 \quad \Rightarrow \quad \{ y > 0 \} \]

\[ \Rightarrow \text{Im}(w) = y \cdot \left( 1 - \frac{1}{x^2 + y^2} \right) > 0. \]

Given \( w \in h \), find solution of \( w = z + \frac{1}{z} \) in \( U \):

\[ wz = z^2 + 1 \]

\[ 0 = z^2 - wz + 1 = (z - z_+)(z - z_-) \]

Clearly \( z_+ z_- = 1 \), we can't have \( |z_+| + |z_-| = 1 \),
because then \( z_+ \cdot \overline{z}_+ \Rightarrow (z-z_+)\overline{(z-z_-)} = e^w (z_+ + \overline{z}_+) z + 1 \).

\[ w = 2 \Re(e^{g \varphi}) \]

So let \( |z_+| > 1, |z_-| < 1 \); then

\[ W = z_+ \frac{1}{z_+} = z_+ + z_- \in U \Rightarrow \text{Im}(z_+) > 0, \]

which together with \( |z_+| > 1 \Rightarrow \overline{z_+} \in U \).

Use fact that \( |z_-| < 1 \) so not in \( U \). \[ \square \]