Lecture 16: Some interesting functions

I. The complex logarithm

Recall that \( \mathbb{C} \setminus \mathbb{R} \leq 0 \) is star-shaped and therefore simply-connected, and that on a simply-connected region, every holomorphic function has a global holomorphic primitive. Therefore, on \( \mathbb{C} \setminus \mathbb{R} \leq 0 \), \( \frac{1}{z} \) has a global primitive:

\[
\log(z) := \int_1^z \frac{1}{w} \, dw.
\]

Properties:

- \( \log(z) \in \text{hol} (\mathbb{C} \setminus \mathbb{R} \leq 0) \)
- \( \text{arg}(\log(z)) \in (-\pi, \pi) \)
- \( \log(z) \) is analytic:
  \[
  \log(z) = \int_{z_1}^z \frac{1}{w} \, dw + \int_{z_1}^{z_2} \frac{1}{w} \, dw
  \]
  \[
  \text{[want: const. power series]}
  \]
  \( \) will now expand on this:
Let $w_1$ be such that $e^{w_1} = -z_1$, $\arg(w_1) \in (-\pi, \pi)$.

Put $F[z_1](z) = w_1 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \frac{(z - z_1)^n}{z_1^n}$ on $D(z_1, 2|z_1|$.

Now $F'[z_1](z) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \frac{(z - z_1)^{n-1}}{z_1^n} = \frac{1}{z_1} \sum_{m \geq 0} (-1)^m \left( \frac{z - z_1}{z_1} \right)^m = \frac{1}{z_1} \left( \frac{z}{z_1} - 1 \right) = \frac{1}{z} = \log'(z)$.

$\implies \log(z) - F[z_1](z) = C[z_1]$ (const.)

PTC (since $F$ and $\log$ both have logs.)

$\implies \log$ analytic at $z_1$ (and $z_1$ was arbitrary).

$\implies \log = C[z_1] + F[z_1](z)$

Take $z_1 = 1$, $w_1 > 0 \implies \log(1) = 0 = F[z_1](1)$

$\implies C[z_1] = 0$

$\implies$ on $D(1, 1)$, $e^{\log(z)} = e^{F[z_1](z)} = e^{\frac{(1-z)^n}{n}}$

since both of these are analytic, and they agree on $D(1, 1)$, they agree on all of $\mathbb{C} \setminus \overline{D(1, 1)}$.

This is true, as usual, for $z \in (-1, 1) \cap \mathbb{R}$ (by Calculus) $\implies$ for $z \in D(1, 1)$.
So \( \log(e) = -\text{Something} \) at different \( z \)’s:

\[ e^{\log z} = z \]

or \( z \in \text{Something} \) as \( \epsilon > 0 \).
Formal consequence: for \( \theta \in (-\pi, \pi) \),
\[
\int_1^{re^{i\theta}} \frac{dw}{w} =: \log(re^{i\theta}) = \log(e^{\log r + i\theta}) = \log r + i\theta
\]
(or you can just compute this by direct integration along
\[
\begin{array}{c}
1 \\
\uparrow \\
r
\end{array}
\]
).

Example:

Without a moment's hesitation you may write
\[
\int_1^{re^{i\theta}} \frac{dw}{w} = \log(w) \bigg|_{2e^{i\pi/4}}^{2e^{-i\pi/4}}
\]
\[
= \log(2 + i\pi/4) - \log(2 - i\pi/4)
\]
\[
= i\pi/2.
\]

More general situations:

1. Different branches of \( \log(z) \): \( U \subset \mathbb{C} \setminus \{0\} \) simply conn.,

\[
\log(z) := w_0 + \oint_{z_0}^{z} \frac{dw}{w} \quad (\Rightarrow \log(e^{w_0}) = w_0)
\]

where \( a^{w_0} = z_0 \)

The "branch" depends on \( U \) of \( w \) (can add \( 2\pi i \)m here to "change" branch).
(2) Primitive for $f'(z)/f(z)$ on $U \subset \mathbb{C}$:

- $U$ simply connected, $f$ never $0$ on $U$.

\[ \log (f(z)) := w_0 + \int_{z_0}^{z} \frac{df}{f} \]

in $U$, i.e. $\frac{f(w)}{f(w_0)} \, dw$.

**WARNING:** This is NOT $f(z)$ plugged into \( \log \cdot \), or $\int \frac{f(z)}{f(z_0)} \, dw$. Reason: can't necessarily define $\log$ on $f(U)$, because $U$ simply connected $\Rightarrow f(U)$ simply connected!

**Example:** $f(z) = \exp(z)$, $U = \mathbb{C}$. The thing called \( \log (f(z)) \) above is a primitive on all $\mathbb{C}$ for $f'(z)/f(z) = e^z/e^z = 1$, i.e. $\pm$.

$\Rightarrow \log (f(z)) = \pm$, \( \log (f(z_0)) = 2\pi i n \).

Deceptive notation, as there's no single branch of $\log$ itself that takes all these values!!
(3) Other functions:

(a) \( \frac{1}{1+z^2} \) is holomorphic on \( \mathbb{C} \setminus \{ \pm i \} \) but this isn't simply connected. To make it so, remove 2 rays:

Then \( \int_{0}^{2} \frac{dz}{1+z^2} = \arctan(z) \in \text{Hol}(U) \).

(b) \( f(z) = \frac{1}{\sqrt{(z-x)(z-\beta)(z-\gamma)}} \) is holomorphic & well-defined on the complement of ones

But this region isn't simply connected. To fix this, remove a strip from \( \beta \) to \( \alpha \):
Then the *abelian integral*

\[ A(z) := \int_{z_0}^{z} \frac{dw}{\sqrt{(w-a)(w-b)(w-c)}} \]

is well-defined & holomorphic on \( U \).

On the Riemann surface of \( f(z) \),

\[ \hat{C} \quad \hat{C} \quad = \quad \hat{T} \]

this corresponds to cutting open the torus as shown, but one can omit the cuts from \( \hat{a} \) to \( \hat{b} \) without sacrificing simple connectivity. This produces a "parallelogram"
corresponding to the fact that the "existence domain of $A(z)$ over $T$" is actually the complex plane:

The dots form a lattice $\Lambda \cong \mathbb{Z} \langle a_1, a_2 \rangle \subset \mathbb{C}$ and there is a complex analytic isomorphism from $\mathbb{C}/\Lambda$ to $T$.
We'll see this from a different angle (and more clearly) later on in the course.
II. The dilogarithm

\[ \log(z) \in \text{hol}(C\setminus \{0\}) \Rightarrow \log(1-z) \in \text{hol}(C\setminus \{1\}) \]

\[ \Rightarrow -\frac{\log(1-z)}{z} \in \text{hol}(C\setminus \{1\}) \]

Since we divided by \( z \), check analycity (holomorphicity) at 0:

\[ -\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots \]

\[ \Rightarrow -\log(1-z) = 1 + \frac{z^2}{2} + \frac{z^3}{3} + \ldots \]

Since \( C\setminus \{1\} \) is simply connected, \( -\frac{\log(1-z)}{z} \)

has a primitive there:

\[ \text{li}_2(z) := -\int_0^z \log(1-w) \frac{dw}{w} \]

Put \( \rho(z) := \text{li}_2(z) - \text{li}_2(1-z) \in \text{hol}(C\setminus[1,\infty) \cup (-\infty,0]) \)

\[ \rho(z) := 0 \]

(or \( R(z) := \text{li}_2(z) + \frac{1}{2} \log(z) \log(1-z) \)

\( \uparrow \text{very similar function} \)
Then
\[ d (\rho(z)) = -d \int_0^1 \frac{\log (1-w)}{w} \, dw + d \int_0^{1-t} \frac{\log (1-w)}{w} \, dw \]
\[ = -\frac{\log (1-t)}{t} \, dt - \frac{\log (1-t)}{1-t} \, dt \]
\[ = -\log (1-t) \, d\log t + \log (1-t) \, d\log (1-t). \]

Consider now 5 points \( z_1, \ldots, z_5 \in \mathbb{C} \) such that the "cyclically permuted cross ratios"

\[
\begin{align*}
\alpha_1 &= CR (z_1, z_2, z_3, z_4) \quad \left\{ \begin{array}{l}
\alpha_2 = CR (z_2, z_3, z_4, z_5) \\
\alpha_3 = CR (z_3, z_4, z_5, z_1) \\
\alpha_4 = CR (z_4, z_5, z_1, z_2) \\
\alpha_5 = CR (z_5, z_1, z_2, z_3)
\end{array} \right. \\
\text{belong to \( U \).}
\end{align*}
\]

(This is true for a "generic" choice of \( z_i 's \).)

**Theorem (the "5-term relation")**
\[ \sum_{i=1}^5 \rho(z_i) = \frac{\pi^2}{6}, \]

independently of the \( z_i 's \). (Alternatively: \( \sum \text{Re}(z_i) \sim \frac{\pi^2}{6} \).)
Proof: \((\text{Recall } CR(a, b, c, d) := \frac{a-c}{a-d} \cdot \frac{b-d}{b-c})\)

\[\text{Step 1:} \quad \text{With indices mod 5,} \]

\((\star) \quad 1 - d_i = d_{i-1} d_{i+1} \quad (\forall i) : \]

\[1 - d_i = 1 - \frac{2i - 2i+2}{2i - 2i+2} \cdot \frac{2i+1 - 2i+2}{2i+1 - 2i+2} \]

\[= \left( \frac{2i+1 - 2i - 2i+1 - 2i+2 + 2i+1 - 2i+2}{2i+1 + 2i+2 + 2i+1 - 2i+2} \right) \]

\[= \left( \frac{2i+1 - 2i+2}{2i+1 - 2i+2} \right) \cdot \left( \frac{2i+1 - 2i+2}{2i+1 - 2i+2} \right) \]

\[= d_{i-1} d_{i+1}. \]

\[\text{Step 2:} \quad \text{I like to use the notation} \]

\[\text{to mean } (\star). \quad \text{So one has} \]

\[\begin{align*}
\rho(1/2) &= 0 \\
\rho(1) &= \frac{\pi}{6} \\
\rho(3) &= -\frac{\pi}{6}
\end{align*} \]

\(2\rho(1/2) + 2\rho(0) = \frac{\pi}{3} + \rho(0) = \frac{\pi}{2} \]

\(\text{so works for this configuration!} \)
Step 3: Left to check that as a function of the $x_i$'s, $\sum \varphi(x_i)$ is constant:

\[
\frac{\partial}{\partial x_i} \sum \varphi(x_i) = \sum \frac{\partial \varphi(x_i)}{\partial x_i}
\]

\[
= \sum \log(x_i) \frac{d \log(1-x_i)}{d \log(x_i)} - \sum \log(1-x_i) \frac{d \log(x_i)}{d \log(1-x_i)}
\]

\[
= \left\{ \sum \log(x_i) \frac{d \log(x_{i-1})}{d \log(x_i)} - \sum \log(x_{i+1}) \frac{d \log(x_i)}{d \log(x_{i+1})} \right\}
\]

\[
+ \left\{ \sum \log(x_i) \frac{d \log(x_{i+1})}{d \log(x_i)} - \sum \log(x_{i-1}) \frac{d \log(x_i)}{d \log(x_{i-1})} \right\}
\]

\[
= 0 + 0 = 0.
\]