Lecture 18: Liouville's Theorem:

Homology classes

I. Entire functions

An entire function is just an element of $\text{Hol}(\mathbb{C})$.

The classic result here is

**Theorem (Liouville)** $f$ entire and bounded

(i.e. $|f(z)| \leq C \forall z \in \mathbb{C}$) $\Rightarrow$ $f$ constant.

**Corollary 1** If a function is entire and nonconstant, then it is unbounded.

Examples: $\cos, \sin, \exp$.

**Corollary 2** Any nonconstant polynomial $p(z) = a_n z^n + \ldots + a_0$ has a root in $\mathbb{C}$. ($n > 0, a_n \neq 0$)
Proof of Cor 2: Assume \( P \) has no root; then \( \frac{1}{P} \) is entire. Writing
\[
P(z) = a_n z^n \left(1 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n}\right),
\]
we find that for any \( M > 0 \) \( \exists R \in \mathbb{R}_+ \) s.t. \( |z| > R \Rightarrow |P(z)| > M \). Hence
\[
\frac{1}{P(z)} \to 0 \text{ as } |z| \to \infty,
\]
and defining \( \frac{1}{P(a_n)} = 0 \) therefore makes \( P \) continuous on \( \hat{C} \). Now compactness of \( \hat{C} \Rightarrow \frac{1}{P} \) is bounded \( \Rightarrow \) \( \frac{1}{P} \) constant \( \Rightarrow P \) constant.

Proof of Liouville: \( f \) entire \( \Rightarrow \) power series for \( f \) at \( 0 \) converges on all of \( C \) \( (r = \infty) \).
So if all \( a_1, a_2, \ldots = 0 \) in this expansion, then \( f(=a_0) \) is constant.

Let \( \gamma_R = \partial D_R \), \( R > 0 \) arbitrary. Recall that the Cauchy integral formula yields a power series
Expansion about any point in \( D_R \); in particular, about 0 we have

\[
f(z) = \sum_{n \geq 0} \left( \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w) \, dw}{w^{n+1}} \right) z^n.
\]

Then \( |a_n| = \frac{1}{2\pi} \left| \int_{\gamma_R} \frac{f(w) \, dw}{w^{n+1}} \right| \)

\[
\leq \frac{1}{2\pi} \cdot L(\gamma_R) \| \frac{f(w)}{w^n} \|_{\gamma_R}
\]

\[
\leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{C}{R^n}
\]

\[
= \frac{C}{R^n} \to 0 \quad \text{for } n \geq 1.
\]

**Generalization:** If \( f \) entire with polynomial bound, i.e., \( |f(z)| \leq C |z|^m \) for \( |z| \geq R_0 \) sufficiently large, \( \Rightarrow f \) is a polynomial of degree \( \leq m \).

**Remark:** If you want to get rid of "for all \( |z_0| = R_0 \)," then make the bound instead \( |f(z)| \leq C_0 + C_1 |z|^m \) (\( z_0 \in C \)).
This implies the above bound with $C = 1 + C_1$, $R_0 = C_0^{-\frac{1}{m}}$:

$$|z| \geq C_0^{-\frac{1}{m}} \Rightarrow |z|^m \geq C_0 \Rightarrow |z|^m + C_2 |z|^m \geq C_0 + C_2 |z|^m.$$ 

Idea of proof ($m = 1$; $mW$ = general case):

- $|f(x)| \leq C |x|$, for $|x| \geq R_0$
- So $|x_k| \leq \frac{2\pi R}{2\pi} \left| \frac{f(x)}{v_{x_k}} \right| \leq R \cdot \frac{C \cdot R}{R^{k+1}} = \frac{C}{R^{k-1}} \to 0$ if $k \geq 2$

II. Homotopy classes of paths

The point of Cauchy’s integral formula is to compute integrals, and we want a stronger version than the “homotopy form”. Let’s recall what we know: given

- $U$ open
- $x \in D \subset U$
- $f \in \text{Hol} (U)$
- $x \sim D$ in $U \setminus \{x\}$ homotopic to
More generally if \( Y \) \( \sim \) \( \eta \subset D \) with \( W(\eta, x) = n \) then \( n f(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-x} \, dz \), but this is the problem — having to be homotopic to something in the disk. If this is not true for our given \( \delta \), might something like (\( t \)) hold anyway?

We are heading toward a homology version of Cauchy, which is stronger because it takes far less for closed paths to be homotopic than for them to be homotopic. So let’s review these competing “global topological” notions.

Define the “homology” or “fundamental” group of \( U \) — first by just specifying the underlying set:

\[
\pi_1(U, x) := \left\{ \text{closed C}^0 \text{ paths starting ending at } x \right\} / \text{homotopy equivalence} \sim
\]

i.e. \( \delta \) and \( \eta \) are the “same element” of \( \pi_1 \sim \delta \sim \eta \).

A path \( Y \) is the “trivial element” \( \{x\} \) in \( \pi_1 \subseteq \exists C^0 \Psi : [0,1] \times [0,1] \to U \) with \( \Psi_0(t) = \gamma(t) \), \( \Psi_1(t) = \{x\} \) for \( \Psi \).
Intuitively, this means (viewing $I$ as a "rope") you can stand at "$x$", grip 2 ends of the rope and pull to yourself without the rope passing through the hole:

Given $Y, y : [0,1] \to \Omega$, define $y \cdot Y : [0,1] \to \Omega$:

This makes $\Pi_2$ into a group, with identity $\{x\}$ (constant path) and "$y^{-1}$" just $y$ traversed backwards. (Why does this work?) However, in general we'll have $y \cdot y' \neq y' \cdot y$, or equivalently $y^{-1} \cdot y \cdot y^{-1} \cdot y \neq \{x\}$. The crucial here is that we aren't allowed to subdivide the path and cancel pieces — more like tying the
ends of 4 strings together and trying to pull the whole thing towards you (and you have beginning of $y$ and end of $y^{-1}$ in your hands). This

![Diagram of 4 strings]

is an example of a nontrivial commutator, i.e. one you cannot pull to $[x]$. Accordingly, homotopy Cauchy can't tell us anything about the integers over $q^7 \cdot q^3 \cdot q \cdot y$.

III. Homology classes of chains

... which is, of course, completely ridiculous: if you integrate over $q^7 \cdot q^3 \cdot q \cdot y$ (regardless of the integrand $f \in \mathbb{H}^1(U)$), you get 0 by cancellation!!

Define for a closed path $\Gamma \subset U$
\[ \Gamma \equiv 0 \iff 0 = W(\Gamma, \alpha) := \sum_{\gamma} \frac{d\gamma}{\gamma - \alpha} \quad \forall \alpha \in \mathbb{C} \setminus \text{U}. \]

So: \[ \eta \cdot \gamma \cdot \gamma \cdot \eta \equiv 0. \]

The reason I wrote "chains", is that I want you to think of homology in terms of subdivisible objects, in contrast to homotopy. Why? Because integrals can be subdivided by definition (Riemann), and homology is defined in terms of winding numbers (which are integrals).

**Definition**

On \( \text{U} \subset \mathbb{C} \), we define

- \( 0 \)-chain := formal sum of points \( \text{in} \text{U} \) = complex points with \( \mathbb{Z} \)-coefficients
- \( 1 \)-chain := formal sum of paths with \( \mathbb{Z} \)-coefficients

\[ \sum_{\gamma} \gamma_i \]

- can define "boundary", e.g.: \[ \partial \delta_2 = \delta_1 \]

- each "piece" is parameterized, but (except for the direction) we forget this.

\[ \uparrow \text{In our ad hoc definition above, there is a better, more general one, to be given in a moment.} \]
- 2-chain := formal sum of 2-chains w/ $\mathbb{Z}$-coeff.

\[ \sum_{i} n_{i} \cdot 2_{i} \]

- boundary:

\[ \partial 2_{i} = 8_{i} + 2_{i} + 3_{i} \]

- can get any shape region by subdividing into

countless triangles (triangulation)

- We write sums of chains with $+$, not $\cdot$, whether or not they are end-to-end; they commute by definition.

Also, and

are identified.

- A chain $\Gamma$ is closed $\iff \partial \Gamma = 0$. Define the first homology group of $U$:

\[ H_{1}(U) := \frac{\{ \text{closed 1-chains on } U \}}{\partial \{ \text{2-chains on } U \}} \]

Element in here denoted by $Y$ is denoted $[Y]$ ("homology class")
Cauchy’s theorem will just say that integrals of holomorphic functions on \( U \) are well-defined as homology classes; that is, \( \int_{[y]} \psi \, dz \) makes sense.

The assertion in the background is that the two definitions of “homologous” are the same:

\[
\gamma \equiv 0 \iff W(\gamma, \mu) = 0 \iff \gamma = \partial K, \ K \text{ a 2-chain in } U
\]

\[\text{hom} \quad (\forall \mu \in U^*) \quad \text{good for drawing picture}\]

**Examples**

1. \( \partial \sigma \) \( \Gamma \) homologous to 0

\[
\begin{align*}
\gamma &\equiv 0 \iff W(\gamma, \mu) = 0 \iff \gamma = \partial K, \ K \text{ a 2-chain in } U \\
\forall \mu \in U^* &\quad \text{good for drawing picture}
\end{align*}
\]

2. (similar to commutator example)

\[
\gamma \equiv 0 \iff W(\gamma, \mu) = 0 \iff \gamma = \partial K, \ K \text{ a 2-chain in } U
\]
Now, \( Y \approx \{z\} \iff \text{path "contractible" to } x \) (in \( U \))

\( \Rightarrow W(y, x) = 0 \quad \forall x \in U \).

\[ Y \approx z \Rightarrow \int_y \frac{dz}{z - a} = 0 \text{ by homotopy Cauchy,} \]

\[ \text{since } d \notin U = 0 \quad \frac{1}{z - a} \in H_0(U) \]

So: \( Y \text{ "homotopic to } 0 \" \Rightarrow Y \text{ homologous to } 0 \)

but NOT vice versa.

Hurewicz homomorphism: \( \pi_1(U) \to H_1(U) \)

\( Y \mapsto [Y] \)

\( \eta \cdot Y \mapsto [\eta] + [\delta] \)

\( \eta \cdot Y \mapsto 0 \)

In fact, \( H_1 \cong \pi_1 / [\pi_1, \pi_1] \)

commutator subgroup, generated by all commuting

Example:

\[ U = \]

\[ \pi_1(U, x) = \langle \eta, \eta \rangle \text{ (free group): elements are the words } \eta^{n_1} \eta^{n_2} \eta^{n_3} \ldots \eta^{n_k} = 1. \]
$H_1(U) = \mathbb{Z} \oplus \mathbb{Z}$, the free abelian group on $[x] \oplus [y]$.

The Hurwitz map sends $\Gamma \to \left( \sum_i a_i [x] + \sum_i b_i [y] \right) = [\Gamma]$.

Note that $a = W(\Gamma, a)$, $b = W(\Gamma, b)$.

Before explaining \((\star)\) above, I recall properties of the winding \#.

(i) \underline{it's an integral}. For all \(S^1\) s, \(S^1_{x+y} = S^1_x + S^1_y\).

(This is why homology works so well with integration.)
So \(W(y, x) + W(y, x) = W(x+y, x)\) for closed \(x, y\).

(ii) \underline{it's an integer}, and \(it\ is\ constant\ as\ \alpha\ varies\ in\ a\ connected\ component\ of\ \partial \mathbb{C}\).

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**Proof of \((\star)\):**

\[ \Gamma \quad \Rightarrow \quad W(x, \alpha) = 0 \quad \text{for any } \alpha \not\in U \]

\(\Gamma = \sum_{i=1}^m D_i \Gamma_i, \Gamma_i \subset U\). By triangulating, make \(\{\Gamma_i\}\) as small as needed: make them fit in a circle of radius \(d(x, U^c) \Rightarrow D \Gamma_i\) homotopic to \(0\).

\((\text{disk is convex} \Rightarrow \text{simply connected})\)
Hence,
\[ \oint_{\partial \Omega} f \, ds = \sum_{i=1}^{n} \oint_{\partial R_i} f \, ds = 0 \quad \text{for } f \in \mathcal{H}(U), \]
and \( \frac{1}{z-2} \in \mathcal{H}(U) \) for \( z \not\in U \).

(2) \( W(\Omega, z) = 0 \quad \forall z \not\in U \Rightarrow z = \partial \Omega : \) (Th)

is the harder direction, as we must construct \( \mathcal{K} \).

Partition \( \Gamma \) so that points are closer together than \( \delta(\Omega, U) \).

Then "rectangularize" the path to obtain \( \Gamma' \). Clearly \( \Gamma' - \delta' \approx \sum \mathcal{T}_i \) and \( \Gamma', \delta \) close together \( \Rightarrow \) \( \Gamma', \delta \) homotopic \( \Rightarrow \Gamma', \delta \) have same winding \( \Rightarrow \) can replace \( \delta \) by \( \Gamma' \).

Now subdivide the plane into rectangles \( \partial R_i \), using vertical & horizontal extensions of \( \mathcal{K} \), let \( \beta_i \in \partial R_i \) be arbitrary.

Lemma: Let \( \eta \) be a rectangular path, with segment between \( R_i \) & \( \partial R_i \):

Then \( W(\eta, \beta_1) \neq W(\eta, \beta_2) \).
Pf: "divert" the path by adding $\partial R_1$; then $\beta$, $\beta_2$ is some connected component of $C \setminus \eta + \partial R_1$. So

$$0 = W(\eta + \partial R_1, \beta) - W(\eta + \partial R_1, \beta_2)$$

$$= (W(\eta, \beta) - W(\eta, \beta_2)) + (W(\partial R_1, \beta) - W(\partial R_1, \beta_2))$$

as desired. \hfill \Box

Now set $m_i := W(\Gamma, \beta_i)$, $\beta_i \in R_i$ arbitrary.

$$m_i \neq 0 \implies \beta_i \in U \implies R_i \subseteq U.$$

(Contrapositive of our hypothesis: $\alpha \notin U \implies W(\Gamma, \alpha) = 0$)

So $K := \sum m_j \overline{R_j}$ is a 2-chain in $U$. Notice that for $\beta_i \in R_i$,

$$(k) \quad W(\partial K, \beta_i) = \sum m_j W(\partial R_j, \beta_i) = m_i = W(\Gamma, \beta_i).$$

Claim: $\Gamma = \partial K$.

Suppose otherwise: then the closed rectangular 1-chain $\Gamma - \partial K$ is nonzero, hence has a segment. That segment divides 2 rectangles, but (say, $R_1$ & $R_2$)
\[ W(\Gamma - \delta K, \beta_1) = W(\Gamma, \beta_1) - W(\delta K, \beta_1) \]
\[ = 0 = W(\Gamma - \delta K, \beta_2) \]  
(4) (4)

contradicting the Lemma.

We have proved

**Proposition** \( y \equiv 0 \iff W(\chi, y) = 0 \iff y = \chi K \)  
\( \forall \alpha \notin U \)  
\( \forall \alpha \notin U \)  
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\( \forall \alpha \notin U \)

(You should think of \( K \) as a finitely triangulable compact subset of \( U \).) This will lead pretty directly to "homology versions" of Courant's Thm. of Courant's Thm. Integral formula in the next lecture.