Lecture 2: Complex functions

I. Some examples

We'll consider complex-valued functions

\[ f : \mathbb{C} \to \mathbb{C} \]

defined on a subset \( \mathbb{C} \subset \mathbb{C} \).

The interplay between "\( \mathbb{C} \to \mathbb{C} \)" and "\( \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \)" is especially important here. To be able to discuss this clearly we shall fix the notation

\[ f(x+iy) = f(z) = u(z) + iv(z) = u(x,y) + iv(x,y) \]

for this lecture.
Example 1

(Exponential map)

Define $f(z) = e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y$.

If $S = \{x + iy \mid x > c, 0 \leq y \leq 2\pi\} \subseteq \mathbb{C}$,

then $f(S) = \{z \mid |z| \geq e^c\}$, as $e^x \geq e^c$ and

$e^{iy}$ takes all angles when $y \in [0, 2\pi]$.

Example 2

$f(z) = \frac{1}{z} = \frac{1}{x - iy} = \frac{x + iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i \frac{y}{x^2 + y^2} = \frac{1}{r} (\cos \Theta + i \sin \Theta) = \frac{1}{r} e^{i \Theta}$

("preserves angle, but inverts length")
Example 3

\[ f(z) = z^5 = r^5 e^{i\theta} \]

\[ u(z) = r^5 \cos(5\theta), \quad v(z) = r^5 \sin(5\theta) \]

(less easy to express in terms of \(\alpha\) and \(y\) ...)

II. Complex differentiability

For a real function of a real variable

\[ g : \mathbb{R} \to \mathbb{R} \]

\(\mathbb{R}\) open

recall the properties of being

(a) differentiable

(b) infinitely differentiable ("smooth" or \(C^\infty\))

(c) analytic (locally representable by power series)

are all distinct.
Example 4

\[ q(x) = \begin{cases} 
0 & x \leq 0 \\
-\frac{1}{x^2} & x > 0
\end{cases} \text{ is smooth but non-analytic (all derivatives are zero at } x=0). \]

For a complex function 

\[ f : U \rightarrow \mathbb{C}, \]

the properties (to be defined) of being

(a) complex differentiable ("holomorphic")

(b) infinitely complex differentiable (not called smooth)

(c) complex analytic (power series in complex variable)

will turn out to, in contrast, be equivalent.

Let's start from a multivariable (real) calculus perspective: suppose we have given
\[ F : U \rightarrow \mathbb{R}^2, \]
and write \( F(\mathbf{x}) = (\mathbf{u}) \). This is differentiable (in the real 2-variable \( \rightarrow \) 2-variable sense) at \( \mathbf{x}_0 \in U \), iff \( \exists A \in M_2(\mathbb{R}) \) such that

\[ (\star) \quad \lim_{\mathbf{h} \to 0} \frac{\| F(\mathbf{x}_0 + \mathbf{h}) - F(\mathbf{x}_0) - A \mathbf{h} \|}{\| \mathbf{h} \|} = 0 \]

Now \((\star) \Rightarrow \) existence of the 4 partials
\( u_x, u_y, v_x, v_y \) at \( \mathbf{x}_0 \),
and in fact
\[ A = \begin{pmatrix} u_x(\mathbf{x}_0) & u_y(\mathbf{x}_0) \\ v_x(\mathbf{x}_0) & v_y(\mathbf{x}_0) \end{pmatrix} = J_{\mathbf{F}}(\mathbf{x}_0) \text{\ \ (Jacobian matrix)} \]

A partial converse is given by the
\[ \textbf{Proposition} \quad \text{The 4 partials exist and are continuous on } U \Rightarrow (\star) \text{ holds (with } A = J_{\mathbf{F}}(\mathbf{x}_0) \text{) for all } \mathbf{x}_0 \in U. \]
Proof: For simplicity assume \( \theta_0 = 0 = F'(\theta_0) \).

\[
\left\| \hat{F}(\theta') - \hat{J}_F(\theta) \cdot \hat{\theta} \right\| \quad \leq \quad \text{(i)} \quad \left\| \hat{F}(\theta') - \hat{F}(0) - \hat{J}_F(0) \cdot \theta \right\| \\
\quad \quad \uparrow \quad \text{Anginity} \quad \left\| \hat{F}(\theta') - \hat{F}(0) - \hat{J}_F(0) \cdot \theta \right\| \\
\quad \quad \uparrow \quad \text{(ii)} \quad \left\| \hat{F}(0) - \hat{J}_F(0) \cdot \theta \right\| \\
\quad \quad \uparrow \quad \text{each} \quad \rightarrow 0 \\
\quad \text{by definition} \quad \text{of the partial derivative} \quad u_y(\theta'), v_y(\theta') \\
\text{of} \quad \frac{\partial y}{\partial } \\
\text{since} \quad \frac{\partial y}{\partial } \leq 1, \quad \frac{\hat{\theta}}{\| \hat{\theta} \|} \rightarrow 0 \quad \text{with} \quad \| \hat{\theta} \|. \\
\text{A similar argument yields} \quad \text{(ii)} \quad \rightarrow 0. \quad \text{Finally,} \\
\left\| \left( u_y(0) \theta - u_x(0) \theta \right) \right\| \\
\left\| \left( v_y(0) \theta - v_x(0) \theta \right) \right\| \\
\rightarrow 0 \\
\frac{\| y(\theta) - y(0) \|}{\| \theta \|} \rightarrow 0 \quad \text{by the continuity} \quad \text{of} \quad u_y \quad \text{and} \quad v_y. \\
\]
Now call \( f : U \to \mathbb{C} \) “complex differentiable” at \( z_0 \in U \), if \( \exists \lambda \in \mathbb{C} \) such that
\[
\lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - \lambda h|}{|h|} = 0.
\]

Denote \( \lambda := f'(z_0) = \frac{df}{dz} \bigg|_{z_0} \).

**Definition** \( f \) is holomorphic on \( U \iff \)
\( f \) is complex differentiable at every point of \( U \).

**Comparing \( \hat{f} \) & \( f \):** Writing \( \hat{F} = (u(x,y)), f = u(x,y) + iv(x,y), \)
- \( f \) is continuous (at \( z_0 \)) \( \iff \lim_{h \to 0} f(z_0 + h) = f(z_0) \) for some \( h \in \mathbb{C} \).
- \( \hat{F} \) is continuous (at \( z_0 \)) \( \iff \lim_{h \to 0} \hat{F}(z_0 + h) = \hat{F}(z_0) \).

**WARNING** I don’t consider this to be a correct definition of (complex) “analytic”, even though they will turn out to be the same. This is my only sources terminological difference with Ahlfors.
(Moreover, it's clear that in both cases differentiability $\Rightarrow$ continuity.)

Let's write out complex differentiability in matrix/vector form: $z = a + ib$, $h = \Delta x + i\Delta y$, $f = u + iv$.

\[
\begin{align*}
\lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - v h|}{|h|} &= \lim_{h \to 0} \frac{|F(z_0 + h) - \bar{F}(z_0) - (\Re(z_0h), \Im(z_0h))|}{|h|} \\
&= \lim_{h \to 0} \frac{|F(z_0 + h) - \bar{F}(z_0) - (\Delta x - i\Delta y)h|}{|h|} \\
&= \lim_{h \to 0} \frac{|F(z_0 + h) - \bar{F}(z_0) - (\Delta x - i\Delta y)h|}{|h|}
\end{align*}
\]

**KEY COMPUTATION:**

$\Delta h = (a + ib)(\Delta x + i\Delta y)$

$= (a\Delta x - b\Delta y) + i(a\Delta y + b\Delta x)$

$\Rightarrow \begin{align*}
(\Re(z_0)) &= (a - b)(\Delta x) \\
(\Im(z_0)) &= (b \ a)(\Delta y)
\end{align*}$

The form of $A$ suggests that complex differentiability (of $f$) is more "rigidifying" than real differentiability (of $F$). The next result makes this more precise.
Theorem (a) If \( u_x, u_y, v_x, v_y \) exist and are continuous on \( U \) with
\[
 u_x = v_y, \quad u_y = -v_x
\]
then \( f \) is holomorphic on \( U \).

(b) If \( f \) is holomorphic on \( U \), then the \( 4 \) partials exist everywhere and satisfy the Cauchy-Riemann equations.

Proof: (b) \( f \) C-differentiable \( \Rightarrow \) LHS \((\frac{\partial f}{\partial x}) = 0 \)
\[ \Rightarrow \text{RHS } (\frac{\partial f}{\partial x}) = 0 \Rightarrow f \text{ differentiable with } \left( \begin{array}{c} u_x \\ u_y \end{array} \right) = \left( \begin{array}{c} v_x \\ v_y \end{array} \right) \text{ everywhere of the form } \left( \begin{array}{cc} a & -b \\ b & a \end{array} \right) \]
\( \Rightarrow \) Cauchy-Riemann equations

(a) \( f \) & continuity of partials \( \Rightarrow \) \( f \) differentiable. Proposition

Cauchy-Riemann equations \( \Rightarrow \) \( \text{LHS } (\frac{\partial f}{\partial x}) = 0 \)
\[ \Rightarrow \text{LHS } (\frac{\partial f}{\partial x}) = 0 \Rightarrow f \text{ holomorphic}. \]
Alternate proof of (b): The definition of complex differentiability says that
\[ \lim_{h \to 0} \frac{f(z+ih) - f(z)}{h} = f'(z) \]
c exists.

In particular, it is independent of the direction by which \( h \to 0 \). So we have
\[
\lim_{\Delta y \to 0} \frac{f(x+i(y+\Delta y)) - f(x+i y)}{i \Delta y} = \lim_{\Delta x \to 0} \frac{f(x+i(y+\Delta x)) - f(x+i y)}{\Delta x}
\]
\[
\lim_{\Delta y \to 0} \left( \frac{u(x,y+\Delta y) - u(x,y)}{i \Delta y} \right) + \frac{v(x,y+\Delta y) - v(x,y)}{i \Delta y} = \lim_{\Delta x \to 0} \left( \frac{u(x+\Delta x,y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x,y) - v(x,y)}{\Delta x} \right)
\]
\[
-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial y}
\]

So
\[
\left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = 0
\]
real part
0

real part
0, done.
Remark on $J^f$: using $|x| = \sqrt{a^2 + b^2}$, $\theta = \arg(x)$, the polar form $x = |x| e^{i\theta}$ has a related matrix factorization

\[
\begin{pmatrix}
a & -b \\
b & a
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 \\
0 & |x|
\end{pmatrix} \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

\(\text{dilation} \times \text{rotation} \rightarrow \text{preserve angles!}\)

\[J^f\] should be viewed as the infinitesimal linear transformation on vectors given by \((a, b)\); clearly this doesn't change the angles between vectors. The “integrated” form of this statement is that $F$ (or $f$) doesn't change the angle between the tangent vectors at curves where they meet; i.e., $f$ is CONFORMAL.

\[\text{Note that the polar form expresses any complex number as a rotation } e^{i\theta} \text{ times a dilation } r. \text{ In the HW you'll make the correspondence between matrices and complex numbers a bit more formal.}\]

\[\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin \theta = \frac{b}{\sqrt{a^2 + b^2}}\]