Lecture 24: Isolated singularities

I. 3 types of singularities

Let \( f \in \text{Hol}(U), \ U \subset \mathbb{C} \) open. A singularity of \( f \) is really just a point (of \( U^c \)) where \( f \) is not defined.

**Definition** \( p \in U^c \) is an isolated singularity of \( f \) if \( \exists \ R > 0 \text{ s.t. } D^*(p, R) \subset U^c \).

(That is, \( p \) is not an accumulation point of the other singularities of \( f \).)

Consider the case \( p = 0 \). Let \( f \in \text{Hol}(D^*_R) \), then

\[
 f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{there, for some unique \( \{a_n\}_{n \in \mathbb{Z}} \).}
\]

The 3 possibilities for "species" of \( f \)'s isolated singularity at 0 are:
(i) all \( a_n = 0 \) for \( n < 0 \) \( \iff \) 0 "removable" singularity (obvious) 
\( \iff \) can extend \( f \) to \( \Bbb{C} \) (\( D_R \))

(ii) all \( a_n = 0 \) for \( n < -m \) \( \iff \) 0 pole of order \( m \) (but \( a_{-m} \neq 0 \)) (\( \text{ord}_0 (f) = -m \))

(iii) "otherwise" (i.e. \( f \) has only \( n < 0 \) s.t. \( a_n \neq 0 \))

**Proposition (Riemann)**

\[ f \text{ bounded on } D_R^* \implies 0 \text{ removable}. \]

**Proof:** By the formula for Laurent coefficients,

\[
|a_{-m+1}| = \left| \frac{1}{2\pi i} \int_{C_{r_0}} w^m f(w) dw \right| \leq \frac{2\pi r_0}{2\pi} \epsilon^n \|f\|_{C_{r_0}} \\
\leq \epsilon^{n+1} B \to 0 \quad (\epsilon \to 0)
\]

\[
\|f\|_{D_R^*} =: B
\]

for all \( n \geq 0 \).

A different approach to Riemann's theorem (and a slightly stronger statement) is given by the
Proposition \( \lim_{z \to 0} f(z) = 0 \Leftrightarrow 0 \text{ removable} \).

Proof: Using the "can extend } f \text{ to the} (0_R) \text{" form of the det'n. of removable singularity, we can forgo Laurent series entirely: for } z \in D^*_R, \text{ taking } 0 < \varepsilon < |z| < r < R, \text{ we have \( f(z) = \frac{1}{2\pi i} \oint_{C_{r}} \frac{f(w)}{w-z} \, dw - \frac{1}{2\pi i} \oint_{C_{\varepsilon}} \frac{f(w)}{w-z} \, dw \) yielding holomorphic function inside } D^*_R, \text{ hence extending } f \text{ across the origin.}

By means of Laurent series, one readily verifies the following results on poles:

- \( \text{ord}_0(f) = -m \Leftrightarrow 2^m f(z) \in H^0(D^*_R) \) \( \text{as } z \to 0 \)

- \( \text{ord}_0(f) + \text{ord}_0(g) = \text{ord}_0(fg) \),

- \( \text{ord}_0(f) - \text{ord}_0(g) = \text{ord}_0(f/g) \)

- If } f \neq 0 \text{ is holomorphic with } f(0) = 0, \text{ then } \frac{1}{f} \text{ has a pole at } 0.

A much more interesting result is the
**Theorem (Casorati-Weierstrass)**: $0$ essential $\iff$

\[ \text{(*) } f(D^*_e) \text{ is dense in } C \quad (\forall e > 0). \]

**Remark**: This means that $\lim_{z \to z_0} |f(z)|$ is undefined, even if we think of $z_0$ as a limit. If the limit is $0$, then you've got a pole, not an essential singularity. //

**Proof (by contrapositive)**: If (*) is NOT the case, then there exists $\delta \in C$ s.t. $f(D^*_e)$ omits a nhbd. $\delta < \infty$:

\[ |f(z) - \delta| > \delta(>0) \quad \forall 0 < |z| < \varepsilon \]

\[ \implies |\frac{1}{f(z) - \delta}| < \frac{1}{\delta}(<< \infty) \quad \forall z \in D^*_e \]

\[ \implies \frac{1}{f(z) - \delta} \in \text{hol}(D_e) \]

\[ \implies f(z) - \delta \text{ has (at most) a pole at } 0 \]

\[ \implies f(z) \text{ has (at most) a pole at } 0 \]

\[ \implies \text{no essential singularity.} \]
Corollary: The only analytic automorphisms $f : \mathbb{C} \to \mathbb{C}$ are of the form $f(z) = az + b$ (translation-dilation-rotation).

Proof: Assume $f : 0 \to 0$, show $f(1) = 1$. If $f$ takes $D_1$ to an open nbhd. of $0$ (since invertible):

\[ f(0) = 0, \quad f(D_1) \ni 0 \]

So $|z| > 1 \implies f(z) \not\in f(D_1) \implies |f(z)| > \epsilon$

$g = \frac{1}{z} \uparrow$

$0 < |g| < 1$

$|f(g)| > \epsilon$

Set $h(s) = f\left(\frac{1}{s}\right)$: then $h(D_1^*)$ omits a nbhd. of $0$

$|h(s)| > \epsilon \quad \forall \, s \in D_1^*$

$\implies h$ has at least a pole at $0$.

But now $f(z) = \sum a_n z^n \implies h(s) = f\left(\frac{1}{s}\right) = \sum a_n s^{-n}$

must terminate for $n \geq m$, so $h$ must have only

$a_0, a_1, \ldots, a_m$ possibly nonzero.
That is, \( f(z) = \text{polynomial} = a_T(z - z_i) \).

If \( z_i \) not all same, then \( f(z_i) = f(z_j) = 0 \) \( \implies \)

\( f \) not injective \( \implies \) \( f \) not automorphism.

Thus \( f(z) = a(z - z_0)^m \implies m = 1 \), \( f(z) = a(z - z_0) \).

But if \( f(z) = 0 \), then \( z_0 = 0 \), i done.

Remark: The Big Picard Theorem is a substantial strengthening of C-W which we will prove later in this course. It says that:

\[
\begin{cases}
\text{If } f \text{ has an essential singularity at } 0, \\
\text{then on any } D_{r,0}^x, f \text{ takes on all possible complex values, with at most a single exception,} \\
\text{infinitely often.}
\end{cases}
\]

Example: \( e^z \) has an essential singularity at \( 0 \), and obviously omits the value \( 0 \).

We conclude this part with an application of Riemann's theorem.
Taylor's theorem  \( f \in \mathcal{H}(U), \ a \in U \Rightarrow \)

\[
f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (z-a)^k + f_n(z) \frac{(z-a)^n}{n!}
\]

where \( f_n \in \mathcal{H}(U) \), \( f^{(n)}(a) = n! f_n(a) \), and

\[
f_n(z) = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f(w)}{(w-z)^{n+1}} \, dw \quad \text{for} \quad |z-a| < r.
\]

**Proof:** Since \( g_1(z) := \frac{f(z)-f(a)}{z-a} \) times \( (z-a) \) goes to 0 as \( z \to a \), Riemann \( \Rightarrow \) \( g_1 = f \big|_{U \setminus \{a\}} \) for \( f \in \mathcal{H}(U) \).

Repeat with \( g_2(z) = \frac{f_1(z)-f_1(a)}{z-a} \quad \text{many times} \quad f_2 \in \mathcal{H}(U) \),

by Riemann etc.

This gives

\[
f(z) = f(a) + (z-a) f_1(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(z)
\]

and

\[
\text{hence} \quad 2\pi i f_n(z) = \int_{\partial D(a,r)} \frac{f_n(w)}{w-z} \, dw
\]

\[
= \int_{\partial D(a,r)} \frac{f(w)}{(w-z)^{n+1}} \, dw - \left\{ \text{terms of the form} \quad f_2(w), \ 0 \leq n \leq n. \right\}
\]
But \((a-x)F_1(a) = W(3D,\overline{a}) - W(3D,\overline{z}) = 0\),
and \(F_k(a) = \text{const.} \times F_{k+1}(a) = 0\).

Remark: Ahlfors used Taylor's Thm. to

(i) bound the error in the Taylor series partial sums:
\[
|f_n(b)(z-a)^n| \leq \frac{1}{2\pi} \int_{|z-a| = r} |\frac{f(z)}{z-a}| \, dz = \frac{1}{2\pi} \int_{|z-a| = r} \frac{|f(z)|}{|z-a|} \, dz 
\]
\[\Rightarrow \text{uniform convergence on compact sets}\]

(ii) Show that zeros are isolated: a similar estimate demonstrates that if all \(f^{(k)}(a)\) vanish then \(f\) vanishes on an open disk \(\Rightarrow\) the set \(S\) where all \(f^{(k)}\) vanish is open. Obviously \(S^c\) is open. So \(U = \text{region} \Rightarrow \emptyset\) or \(S^c = \emptyset\). If \(f \neq 0\), then \(U \neq \emptyset\) \(\Rightarrow U = S^c\) is locally at a zero, some \(f^{(k)}\) is \(\neq 0\) \(\Rightarrow\) in a punctured nbhd. \(f \neq 0\).
II. Meromorphic functions

We write (for $U \subset \mathbb{C}$ open)

$$f \in \text{Mec}(U) \iff f \text{ restricts to a hol. fn. on the complement of a discrete set } \mathcal{D} \subset U$$

(no limit pts. in $U$), with poles at each point of $\mathcal{D}$.

Example // If $g_1, g_2 \in \text{Hol}(U)$, we have

$$g_1 / g_2 \in \text{Mec}(U).$$

The Riemann sphere: Given $f(z)$ on $\mathbb{C}$ (holo/mero/whatever), consider $f(\frac{1}{s})$, $s = \frac{1}{z} \Rightarrow z = \frac{1}{s}$. Look at both on $\overline{\mathbb{D}}$, and notice they line up along the boundary.
We say \( f(z) \) has a singularity of a certain type at \((z) = 0 \) \( \iff \) \( f \left( \frac{1}{z} \right) \) has this type of singularity at \((z) = 0 \). (Same goes for zero/pole of certain order at \( \infty \).)

Now given a cover of \( S^2 \) by open sets (= open set in \( C \), or \( \{ \infty \} \cup \{ \text{complement of a compact set in } C \} \)), we get finite subcovers on each of \( \overline{D_{1,2}} \) and \( \overline{D_{1,5}} \) \( \Rightarrow \) \( S^2 \) compact.

Remark: \( \infty \) is an accumulation point of \( Z \).

So for example the zeros of \( \cos(z) \) have \( \infty \) as an accumulation point; this guarantees that it has \( \infty \) as an essential singularity. //

**Proposition** \( f \in \text{Mer}(C) \) s.t. \( f \) has finitely many zeros and poles in \( D_{1,2} \) (and more generally any compact set).

**Proof:** If \( \{ \text{zeros of } f \} \cup \overline{D_{1,2}} \) is infinite, then it has an accumulation point \( \Rightarrow \) not discrete \( \Rightarrow \) \( f \) not meromorphic. So this set is finite: the poles of \( f \) in \( \overline{D_{1,2}} \) are
\{x_1, \ldots, x_n\}. Let \( U = U_i = \{x_{i1}, \ldots, x_{in}\} \).

If \( f \) has \( n \) many zeros in \( U \), then they have an accumulation point in \( U \), which cannot be one of the \( \{x_i\} \). (Near \( x_i \), \( |f| \) is big.) Hence, the zeroes have accumulation point in \( U \Rightarrow f \equiv 0 \), contradiction.

\[ \textbf{Corollary} \quad f \in \text{Mer}(S^2) \Rightarrow f \text{ has finitely many zeroes and poles.} \]

\[ \textbf{Proof:} \quad \text{Writing } s = \frac{1}{z}, S^2 = \{ |z| \leq 1 \} \cup \{ |z| > 1 \} \text{ and } f(z) \in \text{Mer}(\mathbb{C}), f(s) \in \text{Mer}(\mathbb{C}) \text{ (by restriction). Now apply the Proposition.} \]

\[ \textbf{Theorem} \quad f \in \text{Mer}(S^2) \Rightarrow f \text{ rational (quotient of polynomials).} \]

\[ \textbf{Proof:} \quad \text{ord}_x(f(z)) = \text{ord}_x(f(s)) = m, \text{ and } \]
\[ \text{ord}_x(f) = m; \in \mathbb{Z} \setminus \{0\} \text{ for finitely many } x; \in \mathbb{C}. \]

Hence \( G(z) := \frac{f(z)}{\prod (z - x_i)^{m_i}} \text{ has NO zeroes or poles on } \mathbb{C}. \]
\[ \Rightarrow G \setminus \frac{1}{G} \text{ entire. At } \infty, \text{ ord}_\infty(G) = m + e_m. \]

\[ \Rightarrow \lim_{z \to \infty} |G(z)| = \begin{cases} \infty & \text{if } \delta > \frac{1}{G} \text{ bounded} \\ \text{finite} & \delta \text{ largest} \\ 0 & \delta \text{ bounded} \end{cases} \quad \text{Liouville} \]

So \[ C = \frac{f(z)}{\Gamma(z-d):^m} \Rightarrow f(z) = C T_1(z-d):^m = \frac{P(z)}{Q(z)}, \]

where \[ P(z) = C T_1(z-d):^m, \quad Q(z) = \Gamma(z-d):^{-m}. \]

Corollary \(f \in \text{Mer}(\mathbb{C}), \lim_{|z| \to \infty} |f(z)| = \infty \Rightarrow f \text{ unbounded}\)

Proof: For \(|z| > \frac{1}{2}\), \[ |f(z)| > \frac{1}{e} \Rightarrow i.e. \]

If \(|z| < \frac{1}{2}\) gains \[ |f(z)| > \frac{1}{e}. \] New Cauchy-A.

Weierstrass \(f(z) \) doesn't have an essential
csingularity \(\Rightarrow \) it has a pole. Apply last theorem.

Corollary \(a) f \in \text{hol}(\mathbb{C}) \text{ with pole at } \infty \Rightarrow \text{polynomial}\)

\(b) f \in \text{hol}(S^2) \Rightarrow f \text{ constant.} \)
Proof: (a) \( f = \frac{P}{Q} \) by Theorem; \( Q \) can't have zeros on \( C \)
(b) by (a), \( f = \frac{P}{Q} = C \mathrm{Ti} (z - a_i) \) (\( a_i \geq 0 \))
\( \Rightarrow \) \( \text{ord}_{a_i}(f) = -\Sigma k_i \). But \( \text{ord}_{a_0}(f) \geq 0 \) \( \Rightarrow \) all \( k_i = 0 \).

It will turn out that all simply connected Riemann surfaces (complex 1-manifolds) are bi-holomorphic to \( S^2 \), \( D_1 \), or \( C \). If we knew the automorphism groups of the latter, we knew them for the RS. With this in mind:
- \( \text{Aut}(D_1) \cong \text{PSL}_2(\mathbb{R}) \)
- \( \text{Aut}(C) \cong \mathbb{R}^* \times \mathbb{R} \) (via transformations \( z \rightarrow az+b \))
- \( \text{Aut}(S^2) \cong \text{PSL}_2(\mathbb{C}) \).

Proof: Since \( \text{FLT} \cong \text{PSL}_2(\mathbb{C}) \), suff. to show \( \text{Aut}(S^2) = \text{FLT} \).

The \( \geq \) inclusion is clear. Now let \( f \in \text{Aut}(S^2) \), i.e. \( f \) is a 1-1 homeo. map \( S^2 \rightarrow S^2 \), hence a meromorphic function with a single, simple, pole. By the Corollary above, \( f \) is a rational function \( P/Q \). Written in lowest terms, \( Q \) must have degree \( \leq 1 \). In order to be 1-1, also, \( \deg(P) \leq 1 \). So \( f \) is a FLT.