Lecture 25: Residue Calculus

I. Residues of functions

Recall the setup for studying isolated singularities:
\[ f \in \mathcal{H} (D^* (z_0, r)) \]
\[ \Rightarrow f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n. \]

Define \[ \text{Res}_{z_0} (f) := a_{-1}. \]

Fact: \[ \int_{C} f(z) \, dz = 2\pi i \text{Res}_{z_0} (f) \]

\[ \int_{C} \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n \, dz = \sum_{n \in \mathbb{Z}} a_n \int_{C} (z - z_0)^n \, dz = 2\pi i a_{-1} \]

Remark: \( f \) has a primitive on \( D^* \) \( \iff \) \( \text{Res}_{z_0} (f) = 0 \).

Rules for computing residues:

- Brute force with Laurent series

Example: \[ \text{Res}_0 \left( \frac{e^z}{\sin^2 z} \right) = 1 \]
\[
\frac{e^z}{\sin^2 z} = \frac{1 + z + \frac{z^2}{2} + \ldots}{(z - \frac{z^3}{3!} + \ldots)^2} = \frac{1 + z + \frac{z^2}{2} + \ldots}{z^2(1 - \frac{z^3}{3} + \ldots)}
\]
\[
= \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right) \left(1 + \frac{z^2}{2} + \ldots\right) = \frac{1}{z} + \frac{1}{4} + \ldots \quad //
\]

Ex:// \text{Res}_{z=2} \left(\frac{1}{z(z-2)^2}\right) = -\frac{1}{4} ;
\[
\frac{1}{z(z-2)^2} = \frac{1}{z-2} \cdot \frac{1}{z-2} = \frac{1}{2} \frac{1}{z-2} \cdot \frac{1}{z-2} = \frac{1}{z-2} \left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \ldots\right)
\]
\[
= \frac{1}{2} \left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \ldots\right) = \frac{1}{2} (z-2)^2 = \frac{1}{4} (z-2)^{-1} + \ldots \quad //
\]

There's a better method for the 2nd example.
In fact, here are six:

1. \( f \) holomorphic at \( z_0 \) \( \Rightarrow \text{Res}_{z_0} \left(\frac{f(z)}{z-z_0}\right) = f(z_0) \)

   Ex:// \text{Res}_{z=\infty} \left(\frac{\log(z+1)}{z-\infty}\right) = \log \pi \quad //

   Proof: \( f(z) = b_0 + b_1(z-z_0) + \ldots \), where \( b_0 = f(z_0) \)
   \[
   \frac{f(z)}{z-z_0} = \frac{b_0}{z-z_0} + b_1 + \ldots \quad \Box
   \]

2. \( f \) holomorphic at \( z_0 \) \( \Rightarrow \text{Res}_{z_0} \left(\frac{f(z)}{(z-z_0)^n}\right) = \frac{f^{(n)}(z_0)}{(n-1)!} \)

   Ex:// \text{Res}_{z=0} \left(\frac{\sin z}{z^{10}}\right) = \frac{[\frac{d^{10}}{dz^{10}} \sin z]}{9!} = \frac{\cos 0}{9!} = \frac{1}{9!} \quad //

   Ex:// \text{Res} \left(\frac{1}{z(z^2-2)^2}\right) = -\frac{1}{2} \frac{1}{1!} = -\frac{1}{4} \quad (\text{using} \frac{d}{dz} \frac{1}{z^2} = -\frac{2}{z^3}) \quad //
Proof: \( f(z) = b_0 + b_1(z-z_0) + \cdots + b_n(z-z_0)^n + \cdots \)

\[ = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(n)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \cdots \]

\[ \frac{f(z)}{(z-z_0)^n} = \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!} \frac{1}{z-z_0} \]

\[ \text{Res}_{z_0} f(z) = \lim_{z \to z_0} (z-z_0)f(z) \]

Remark: f has a simple pole at \( z_0 \) \( \iff \) Laurent series def. starts at \( n=-1 \), and \( a_{-1} \neq 0 \).

Example \( \text{Res}_i \left( \frac{e^{\pi z}}{z^2+1} \right) = \lim_{z \to i} \frac{z-i}{(z^2+1)} = \lim_{z \to i} \frac{e^{\pi z}}{2i} = e^{\pi/2} = \frac{e^{\pi i}}{2i} = -\frac{i}{2} \)

and \( \text{Res}_{-1} \left( \frac{e^{\pi z}}{z^2+1} \right) = \lim_{z \to -i} \frac{e^{\pi z}}{z-i} = \lim_{z \to -i} \frac{e^{-\pi i}}{-2i} = -\frac{i}{2} \).

Proof: \( f(z) = a_{-1}(z-z_0)^{-1} + a_0 + \cdots \) \( \text{holo.} \)

\( (z-z_0)f(z) = a_{-1} + (z-z_0) \text{(holo.)} \)

\[ \to 0 \text{ as } z \to z_0. \]

Here is another generalization of (1):
4. \( f \) has a simple pole at \( z_0 \), \( g \) holomorphic at \( z_0 \) \( \Rightarrow \) 
\[ \text{Res}_{z_0} (fg) = g(z_0) \text{Res}_{z_0} (f) \]
(Compute by \( \text{Res} \) or other method)

Proof: HW
Ex 1. see below (\( \text{Res} \) II).

5. \( f \) holomorphic at \( z_0 \), \( f(z_0) = 0 \), \( f'(z_0) \neq 0 \) \( \Rightarrow \)
\[ \text{Res}_{z_0} \left( \frac{1}{f} \right) = \frac{1}{f'(z_0)}. \]

Ex// \( \text{Res}_0 \left( \frac{1}{e^{z} - 1} \right) = \frac{1}{3} \)
\[ \frac{d}{dz} (e^z - 1) \bigg|_{z=0} = 3e^0 = 3 \]

Proof: \( f(z) = b_1 (z - z_0) (1 + h(z)) \), \( h(z_0) = 0 \) (\( h \) holomorphic)
\[ \Rightarrow \frac{1}{f(z)} = \frac{1}{b_1 (z - z_0)} (1 - h + h^2 - ...) \]
\[ \Rightarrow \frac{a_n}{b_1} = \frac{1}{b_1} = \frac{1}{f'(z_0)}. \]

6. \( \text{ord}_{z_0} (f) = m \) \( \Rightarrow \) \( \text{Res}_{z_0} (f'/f) = m. \)
(\( z_0 \) could be a pole or zero)

Ex// \( \text{Res}_1 \left( \frac{10 + 9}{z^{10} - 1} \right) = 1 \), since \( z^{10} - 1 \) has
a "simple zero" at each 5.
(any 10th root of 1)
Proof: \( f(z) = a_m (z-z_0)^m \left( 1 + h(z) \right) \), \( h(z_0) = 0 \) (here \( b.) \)
\[
f'(z) = m a_m (z-z_0)^{m-1} (1 + h(z)) + a_m (z-z_0)^m h'(z)
\]
\[
\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{h'(z)}{1 + h(z)} \Rightarrow \text{Res}_{z_0} \left( \frac{f'}{f} \right) = m.
\]

II. Residue formulas

In each case, \( U \subseteq \mathbb{C} \) is open, \( \gamma \subset \overline{U} \) is a closed path which is homologous to zero (in \( U \)) and avoids points where we are taking residues.

(Residue theorem)

\[
\sum_{j} f(z) \, dz = 2\pi i \sum_{j=1}^{m} W(Y, z_j) \text{Res}_{z_j} (f)
\]

Proof: Use \( Y \equiv \sum_{j} W(Y, z_j) \, \partial D_j \)

\[
\int_{\partial D_j} f(z) \, dz = 2\pi i \, \text{Res}_{z_j} (f).
\]

Ex:

\[
\int_{|z|=2} \frac{e^{i\pi z}}{1 + z^2} \, dz = 2\pi i \left( \text{Res}_{z_1} \left( \frac{e^{i\pi z}}{1 + z^2} \right) + \text{Res}_{z_2} \left( \frac{e^{i\pi z}}{1 + z^2} \right) \right)
\]
\[
= 2\pi i \left( \frac{i}{2} - \frac{i}{2} \right) = 0.
\]

(See Ex. 66-68)
\( b \) \( f \in \text{Mer} (U) \implies \left[ \text{writing } \Omega_{z_1, \ldots, z_m} = U \text{ (f's zeroes)} \right] \)

\[ \int_{\gamma} \frac{f'}{f} \, dz = 2\pi i \sum_{j=1}^{\infty} W(\gamma, z_j) \text{ord}_j (f) \]

\text{(Proof: use (a) + (b).)}

**Example**

\( \int_{|z|=1} \frac{10 + 9}{z^{10} - 1} \, dz = 2\pi i \sum_{j=1}^{10} = 20\pi i. \)

\( c \) \( f \in \text{Mer} (U) \implies \)

\[ \int_{\gamma} \frac{f'}{g} \, dz = 2\pi i \sum_{j=1}^{\infty} W(\gamma, z_j) \text{ord}_j (f) g(z_j) \]

\text{(Proof: use (a), (b), and (c).)}

**Example**

\( \int_{|z|=1} \frac{10 + 9}{z^{10} - 1} g(z) \, dz = 2\pi i \sum_{j=1}^{10} g(z_j). \)

These are amazing exercises, but why do we really care about this?

Because we can use it to compute real integrals:
Example

Set $I := \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)^2} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{x}{(x^2 + 1)^2} \, dx$.

Let $f(z) := \frac{2}{(z^2 + 1)^2} \in \mathcal{M}(C \setminus \{i\})$,

and notice that

$$| \int_{C_R^+} f(z) \, dz | \leq \pi R \| f \|_{C_{R}^+} \leq \pi R \frac{R^2}{(R^2 - 1)^2} \xrightarrow{R \to \infty} 0 \text{ as } R \to \infty.$$

For $|z| = R$, $|z^2 + 1| \geq |z|^2 - 1 = R^2 - 1$.

So

$$I = \lim_{R \to \infty} \left( \int_{C_{R}^+} f(z) \, dz - \int_{C_{R}^-} f(z) \, dz \right) = \lim_{R \to \infty} \int_{C_{R}^+} f(z) \, dz$$

$$= 2\pi i \text{ Res}_1 (f).$$

To evaluate this residue we use (2) with $n = 2$:

writing $f(z) = \frac{2}{(z^2 + 1)^2} = \frac{F(z)}{(z-i)^2}$, we have

$$F'(z) = \frac{2i z^2 - 2z}{(z+i)^4} \Rightarrow F'(i) = -\frac{i}{4} \Rightarrow \text{Res}_1 (f) = \frac{F'(i)}{(z-i)!} = -\frac{i}{4}$$

$\Rightarrow I = 2\pi i \left( -\frac{i}{4} \right) = \frac{\pi}{2}$.

\[ \text{Slightly more gen' condition than simple + closed} \]

Definition

$Y$ has an interior if $W(y, x) = 0$ or $1$ for every $x \in C \setminus \{y\}$, and $\text{Int}(Y) := \{ x \in C \mid W(y, x) = 1 \}$. 

\[ \text{for every } x \in C \]

\[ \text{and } \text{Int}(Y) := \{ x \in C \mid W(y, x) = 1 \}. \]
Let $f, g \in \mathcal{M}(U)$ for some open set $U$ containing both $Y$ and $\text{Int}(Y)$.

**Corollary of (b):** If $Y$ has an interior, then
\[
\frac{1}{2\pi i} \int_Y \frac{f'}{f} \, dz = \sum_{p \in \text{Int}(Y)} \text{ord}_p(f) = (\# \text{ of zeroes of } f \text{ inside } Y, \text{ counted }) - (\# \text{ of poles of } f \text{ inside } Y, \text{ counted }).
\]

**Corollary of (c):** If $Y$ has an interior, then
\[
\frac{1}{2\pi i} \int_Y \frac{f^*}{f} \, dz = \sum_{p \in \text{Int}(Y)} \text{ord}_p(f) g(p). \quad [\text{Note that }\text{Cauchy's integral formula is a special case, with } f(g) = \varepsilon - \alpha.]
\]

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**III. Residues of differentials**

It sometimes makes more sense to write “$d\log f$” instead of $\frac{f'(z)}{f(z)} \, dz$. (Note that this is not necessarily a derivative of some function $\log f$.) We have the obvious properties:

- $\frac{dfg}{f g} = \frac{f'}{f} + \frac{g'}{g} \Rightarrow d\log (fg) = d\log (f) + d\log (g)$.

- $d\log (1/f) = -d\log (f)$

- $d\log (f/g) = d\log (f) - d\log (g)$.

† this is locally true, but need not be globally true on $U$. 
This goes beyond a mnemonic for logarithmic differentials rules: first, for the above "Corollary of (6)", it expresses the fact that you are recording how much \( \log(f) \) (or \( \arg(f) \)) changes around \( y \). But it also is because of the following:

**FACT:** Residues of \( \left\{ \text{functions ARE NOT inverted under \( \text{deformation} \)} \right\} \) \( \text{ARE} \)

local analytic isomorphism.

Here \( \text{Res}_{w_0}(F(z) \, dz) := \text{Res}_{w_0}(F(z)) \), so they appear to be exactly the same.

But now substitute \( z = z(w) = z_0 + b_1 (w-w_0)^1 \), \( h(w_0) = 0 \),

and take \( \text{Res}_{w_0} \):

\[
\begin{align*}
F(z) & \to F(z(w)) \quad \text{vs.} \quad F(z) \, dz \to F(z(w)) \, \frac{dz}{dw} \, dw \\
\text{Res}_{w_0} \left( F(z(w)) \right) & \quad \text{vs.} \quad \text{Res}_{w_0} \left( F(z(w)) \, \frac{dz}{dw} \right) = \\
& = \text{Res}_{w_0} \left( F(z(w)) \right).
\end{align*}
\]

Obviously, these can't be the same in general.

**Example**

\[
\begin{align*}
F(z) & = a_1 (z-z_0)^{-1}, \quad z(w) = z_0 + b_1 (w-w_0), \quad w = b_1 \\
F(z(w)) & = a_1 \, b_1^1 (w-w_0)^{-1}, \quad \text{but} \\
F(z(w)) \, \frac{dz}{dw} & = a_1 \, b_1^1 (w-w_0)^{-1} \frac{dz}{dw} = a_1 \, (w-w_0)^{-1} \quad \text{"right" \quad \text{side}.}
\end{align*}
\]
This makes sense, if we write \[ \text{Res}_{z_0} \left( \frac{F(z)}{z - w} \right) = \lim_{z \to z_0} \frac{1}{2\pi i} \int_{C(z_0)} \frac{F(z)}{z - w} \, dz \]

where \( C(z_0) \) is some contour around \( z_0 \) or "\( z_0 \)". So the differential works because it transforms like the differential in the integral under \( \Delta \) of variable.

Residue at 0: Here we transform the differential into the local coordinate at 0, \( s = \frac{1}{z} \) (\( \to z = \frac{1}{s} \)):

\[ f(z) \, dz = f \left( \frac{1}{s} \right) \, \frac{dz}{s^2} = -f \left( \frac{1}{s} \right) \frac{ds}{s^2}. \]

Now assume \( f \) has finitely many poles on \( C \), so that \( \exists \varepsilon > 0 \) s.t. the only pole enclosed by \( C_E \) (in \( s \)) is at \( s = 0 \) (if there even is one). Then

\[ \text{Res}_{z_0} \left( f(z) \, dz \right) = \text{Res}_{s=0} \left( -f \left( \frac{1}{s} \right) \frac{ds}{s^2} \right) = \text{Res}_{s=0} \left( -f \left( \frac{1}{s} \right) \frac{ds}{s^2} \right). \]

by above defn. of Res (differential)

\[ = \frac{1}{2\pi i} \left( \frac{-f \left( \frac{1}{0} \right)}{0^2} \right) \left( \frac{ds}{s^2} \right) \text{ on } C_E \]

\[ = \frac{1}{2\pi i} \left( \frac{-f \left( \frac{1}{0} \right)}{0^2} \right) \left( \frac{ds}{s^2} \right) \text{ on } C_E \]

\[ = \frac{-1}{2\pi i} \int_{C} f(z) \, dz \quad \text{where } R > 0 \]

(sufficiently large to enclose all poles \( q \) on \( C \)).
But then

\[ \sum_{p \in S^1} \text{Res}_p(f(z)dz) = \sum_{p \in \mathbb{C}} \text{Res}_p(f(z)) + \text{Res}_{\infty}(f(z)dz) \]

\[ = \frac{1}{2\pi i} \int_{C_R} f(z)dz + \left( \frac{-1}{2\pi i} \! \! \int_{C_R} f(z)dz \right) \]

\[ = 0, \]

and we conclude the

**Theorem** \( \) The sum of the residues of a differential on \( S^1 \) is 0.