Lecture 26: Rouche's theorem

I. Counting zeroes

Recall that a closed path $\gamma \in C$ is said to "have an interior" if $W(\gamma, x) = 0$ or $1$ $(\forall x \in C \setminus \gamma)$; and that the $\text{Int}(\gamma) := \{x \in C \setminus \gamma \mid W(\gamma, x) = 1\}$.

For such a $\gamma$, let $U$ be an open set containing $\text{Int}(\gamma) = \text{Int}(\gamma) \cup \gamma$, and if $f \in \text{Hol}(U)$ have no poles or zeroes on $\gamma$ itself. Then we have

**Argument principle (AP):** \[ \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} \, dz = \sum_{p \in \text{Int}(\gamma)} \text{ord}_{p}(f) \]

**Generalized argument principle (GAP):** given also $g \in \text{Hol}(U)$,

\[ \frac{1}{2\pi i} \int_{\gamma} g \frac{f'}{f} \, dz = \sum_{p \in \text{Int}(\gamma)} g(p) \text{ord}_{p}(f). \]

Of course, $\sum \text{ord}_{p}(f)$ has the standard interpretation as $(\# of zeroes w/ mult.) - (\# of poles w/ mult.)$. 
The following application of the AP is very useful:

Rouché's Theorem: If $Y$ has an interior,
\[ \text{Int}(Y) \subset U \quad (\Rightarrow Y \equiv 0 \mod U), \]
and $f, g \in \text{Hol}(U)$ satisfy
\[ |f - g| < |f| \quad \text{on} \quad Y, \]
then $f$ & $g$ have the same number of zeroes (counted w/multiplicity) in $\text{Int}(Y)$.

Proof: It follows at once from (6) that on $Y$
$f$ & $g$ are nowhere zero and $|\frac{g}{f} - 1| < 1$. Thus, the values of $g/f$ lie in $D(1, 1)$. It follows that $W(\frac{g}{f}, 0) = 0$, hence
\[
0 = \frac{1}{2\pi i} \int_Y d\log (t) = \frac{1}{2\pi i} \int_Y d\log (\frac{g}{f})
= \frac{1}{2\pi i} \int_Y d\log (t) - \frac{1}{2\pi i} \int_Y d\log (g)
= (\# \text{ of zeroes of } f) - (\# \text{ of zeroes of } g). \]
Example

Consider a polynomial with one BIG-coefficient and the others small:

\[ P(z) = a_1 z^k + \ldots + a_i z + a_0 + A z^m, \quad |A| > \ell(z, \epsilon), \]

where \( 0 \leq m \leq d \). How many zeros does \( P \) have inside the unit circle?

Set \( g(z) = P(z) \), \( f(z) = A z^m \), so that \( g(z) - f(z) = \ell a z^j \). On \( C_1 \),

\[ |g(z) - f(z)| = \ell |a| z^j < |A| = |f| \]

Hence \( f \) and \( g \) have same \( \epsilon \) \( 0 \)s w/mult. in \( D_1 \).

Since \( f \) has one zero of multiplicity \( m \) at \( z = 0 \), conclude that

\[ P(z) \] has \( m \) zeros (counted w/mult.) inside the unit circle.
Example

How many solutions does
\[ 2^{2n+1} + e^{-z} = \lambda, \quad \lambda \in \mathbb{R}_{>0}, \]
have in the right half-plane?

Set
\[ f(z) = 2^{2n+1} - \lambda \]
\[ g(z) = e^{-z} + 2^{2n+1} - \lambda \]
\[ \Rightarrow f - g = -e^{-z} \]
and consider their behavior
on \( \mathbb{R} \): we have
\[ \int |f - g| = |e^{-z}| = \frac{1}{e^z} \leq 1 \]
and
\[ |f| \leq \lambda \quad (\lambda > 1) \quad \text{there}. \]

To see this:
\[ \begin{align*}
2^{2n+1} & \leq \sqrt{\lambda^2 + 2^{2n+1}^2} \\
& \geq \lambda \\
\end{align*} \]

Taking \( R \to \infty \),
Rosch \( \Rightarrow \)
\( f, g \)
have the same th of zeroes (w/lost)
in the right half-plane.
\[ f = 0 \Leftrightarrow z = \sqrt[2n+1]{\lambda} \leq 2n+1 \]
\( \frac{n}{2n+1} \) (n odd) of which
lie in the right half-plane.
So \( g \) has \( n \) resp. \( n+1 \) zeroes there too.
II. Inverse functions

Rouché's theorem provides a neat proof of the IMT:

\[ F \in Hol(U), F'(z_0) \neq 0 \Rightarrow F \text{ is a local analytic isomorphism at } z_0. \]

• Assume \( z_0 = 0 \), write \( F(z) = z^n h(z) \), with \( h \in Hol(U) \) bounded on some closed disc: \( |h(z)| \leq K \) for \( z \in \overline{D}_R \).

• Take \( \epsilon < \frac{1}{2n} \), let \( |w| < \epsilon^2 / \epsilon \); and set

\[ g(z) := F(z) - w, \quad f(z) := z - w. \]

• For \( z \in C_{\epsilon} \), \( |f| = |z - w| \geq |z| - |w| > \epsilon - \frac{\epsilon^2}{\epsilon^2} = \frac{\epsilon}{\epsilon^2} \),

and so \( |g - f| = |g - f| = K |z - w| = K e^2 < \frac{\epsilon}{\epsilon^2} < |f| \).

• By Rouché, \( g \equiv f \) have the same \# zeros (w/mult.) in \( D_{\epsilon} \subset \text{Int}(C_{\epsilon}) \); \( f \) has one \( z = w \),

so \( g \) has exactly one (w/mult. = 1).

Conclusion: For \( |w| < \epsilon / \epsilon^2 \) (i.e. sufficiently small),

\( \exists z \in D_{\epsilon} \text{ s.t. } F(z) = w \), giving an inverse

\[ F^{-1} : D_{\epsilon^2} \to D_{\epsilon}, \quad w \mapsto F^{-1}(w) = z. \]
Is this $F^{-1}$ continuous? holomorphic? — need this to conclude that $F$ is a local analytic isomorphism.

Use the GAP (now with $f = F(z) - w$ and $g = z$)

$$\frac{1}{2\pi i} \oint_{C_E} \frac{F'(z)}{F(z) - w} \, dz = \sum_{p \in D_E \cap \mathbb{C} \setminus \{w\}} \text{ord}_p (F(z) - w) \cdot z(p) = F^{-1}(w),$$

such representations are holomorphic as a rule.

III. Weierstrass factorizations

Similarly, the GAP gives us the analyticity of symmetric polynomials in the values of "multivalued inverses" — more important than you might think, as we'll see in the next lecture.

Assume

$$f(z) = w, \quad |w - w_0| < \delta$$

has a solutions in $D(z_0, \epsilon)$. By the GAP with $g(z) = z^m$,
\[
\frac{1}{2\pi i} \int_{C} \frac{f'(z)}{f(z) - w} \, dz = \sum_{j=1}^{n} \frac{\tau_j(w)}{\tau_j^{(n)}(z)}
\]

\[\text{Hol}(D(w_0, \delta))\]

[in particular, single-valued]

(As in II we basically saw the \( m = 1 = n \) case.)

Formally writing (on \([z, w) \cap D(w_0, \delta) \cap D(w_0, \delta)\])

\[
f(z) - w = \bigcup_{j=1}^{n} \frac{1}{\tau_j(z)} \prod_{j=1}^{n} (z - \tau_j(w))
\]

\[(\text{cf.})\]

defines the elementary symmetric polynomials in the \( \{\tau_j(z)\} \). A well-known fact from algebra is that the \( \{\tau_j\} \) are linear combinations of the \( \{p_k\} \). Hence they are also single-valued and holomorphic, and we may replace \( "f(z) = w" \) by

\[
0 = z^n - E_1(w) z^{n-1} + E_2(w) z^{n-2} + \cdots + (-1)^n E_n(w).
\]

The factorization (\textit{cf.}) is called a \textit{Weierstrass factorization}.\]
IV. Bergman kernel on the disk

As a concluding application of residue theory, we will derive the first case of a "reproducing kernel" (in the setting of several complex variables) on a bounded symmetric domain (which in this case will just be $D_1$).

Writing we $D_1$, $\mathbf{f} \in \mathbf{Hol}(D_1)$ (bounded on $D_1$), and

$$K(z,w) := \frac{i}{2\pi} \cdot \frac{1}{(1-z\overline{w})^2}$$

we will show

(1) $$\mathbf{f}(w) = \iint_{D_1} K(z,w) \mathbf{f}(z) \, dz \, d\overline{w}.$$

First note that $dz \, d\overline{w} = (dx + idy) \, (d\overline{x} - id\overline{y}) = -2i 
\ dx \, d\overline{w} = -2i \ dx \, d\overline{w} = -2i \ \partial_x \, d\partial w$ , and so

$$\text{RHS} (1) = \frac{1}{\pi} \iint_{D_1} \frac{\mathbf{f}(z) \, d\partial w \cdot dz}{(1-z\overline{w})^2}.$$

\[ \theta = \theta e^{i\theta} \Rightarrow \]

\[ dB = \frac{1}{i} \theta d\overline{\theta} \]

\[ = \frac{1}{i\pi} \int_{\partial D_1} \left( \frac{f(z) \, dz}{z (1-z\overline{w})^2} \right) \, d\theta \]
In general the Bergman kernel gives rise, by taking \( \overline{\partial} \partial \log K(z, \bar{z}) \), to a Hermitian metric. What is the metric in this case?

\[
\frac{\partial}{\partial z} \log K(z, \bar{z}) = -2 \frac{\partial}{\partial z} \log (1 - z \bar{z}) = \frac{2z}{1 - z \bar{z}}
\]

\[
\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log K(z, \bar{z}) = \frac{\partial}{\partial z} \frac{2z}{1 - z \bar{z}} = 2 \left( \frac{1 - z \bar{z}}{(1 - z \bar{z})^2} \right) = \frac{2}{(1 - z \bar{z})^2}
\]

so \( \overline{\partial} \partial \log K(z, \bar{z}) = \frac{2 d\bar{z} \wedge dz}{(1 - |z|^2)^2} \), and replacing the antiholomorphic (vector) product by symmetric product \( dz \wedge d\bar{z} \) gives

\[
\frac{d\bar{z} \wedge dz + dz \wedge d\bar{z}}{(1 - |z|^2)^2},
\]

the Poincaré metric! (We'll find out why later.)