Lecture 27: An algebra-geometric detour

In this lecture I will give two important applications of Rouché + GAP to complex algebraic curves.

I. Connectedness of irreducible algebraic curves

An affine algebraic curve $C \subseteq \mathbb{C}^2$ is the solution set of a polynomial equation

(1) \[ 0 = f(x, y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x) \quad \text{(deg}(a_i) \leq i) \]

where the $\{a_i(x)\}$ are polynomials (in $x$). Here I am using $x$ & $y$ to denote two complex variables.

Projecting $\pi : C \to \mathbb{C}$

$(x,y) \mapsto x$

we find that on the complement of a finite point set $\Delta \subseteq \mathbb{C}$, the preimage consists of exactly $n$ points:

(2) \[ \pi^{-1}(x) = \{ y_1(x), \ldots, y_n(x) \} \]

More precisely, one verifies (2) on a small disk.

\[ \text{or rather, the equation can always be put in this form by a change of variable of the form } (x_0, y_0) \mapsto (x, y) = (x_0 + \lambda y_0, y_0). \]
$D \subset C \setminus \Delta$ and analytically continues the resulting $\{g_i(x)\}$ to the simply connected complement of a set $\Gamma$ (as shown): even if continued through $\Gamma \setminus \Delta$, these analytically continued $\{g_i(x)\}$ still satisfy

$$f(x, y, g_i(x)) = 0.$$  

But: they may swap! (Think back to Riemann surfaces)

**Proposition** Let $E \subset \{1, \ldots, n\}$ be a proper subset which is closed under this swapping (under continuation over all of $C(\Delta)$, which we may assume for simplicity to be of the form $\{1, \ldots, m\}$ ($m < n$). Then $\prod_{i=1}^{m} (y - g_i(x))$ is a polynomial in $x$ and $y$, and so (1) factors nontrivially in $C[x, y]$.

**Theorem** If $C$ is irreducible (i.e. $f$ is irreducible in $C[x, y]$), then $C \setminus \pi^{-1}(\Delta)$ is connected (and therefore so is $C$).
**Prop. ⇒ Thm.** If \( f \) is irreducible, then by the Proposition, the only subset of \( \{1, \ldots, n\} \) closed under the "swapping action" \( \pi_1(\mathcal{C} \setminus \Delta) \to S_n \) is \( \{1, \ldots, n\} \) itself. So the complete set of "branches" \( \{y_i(x)\} \) is acted on transitively by "monodromy" about \( \Delta \), and one can therefore draw a path on \( C \setminus \pi^{-1}(\Delta) \) connecting any two points.

**Proof of Prop.:** \( \prod_{l=1}^{m} (y - y_l(x)) \) is clearly well-defined on \( C \setminus \Delta \), since continuation through \( \pi \setminus \Delta \) ("monodromy about \( \Delta \)) merely swaps its factors. Thus it is in \( \text{Hol}(C \setminus \Delta) \) for each fixed \( y \), and we write

\[
\prod_{l=1}^{m} (y - y_l(x)) = \sum_{j=0} e_{m-j}(\{y_l(x)\}_{l=1}^{m}) \cdot y^j
\]

again, monodromy invariant hence in \( \text{Hol}(C \setminus \Delta) \). But why polynomials ?

Let \( d \in \Delta \), \( N_\alpha \) a small disk about \( x \).

We have

\[
x \in N_\alpha \implies |a_j(x)| \leq M (N_j).
\]

Fix \( x_0 \in N_\alpha^* \), put \( a_j := a_j(x_0) \); then we shall
apply Rouché to

\[ \begin{align*}
\delta(y) &= y^n \\
\gamma(y) &= y^n + a_1 y^{n-1} + \cdots + a_n \\
Y &= \{ |y| = M+1 \} \subset \mathbb{C}
\end{align*} \]

\[ |\delta - \gamma| = |a_1 y^{n-1} + \cdots + a_n| \leq M ((M+1)^{n-1} + \cdots + 1) \]

\[ = (M+1)^n - 1 < (M+1)^n = |z| \]

\(\Rightarrow\) \(\delta\) \& \(\gamma\) have the same \# of zeroes

\(= n\) for \(\delta\), hence for \(\gamma\) inside \(Y\)

\(\Rightarrow\) \(|y_j(x_0)| < M+1\) \(\forall j = 1, \ldots, n\ \& \ x_0 \in \mathbb{N}_x^k\)

\(\Rightarrow\) the \(\epsilon_k(\{y_1(x), \ldots, y_n(x)\})\) are bounded on \(\mathbb{N}_x^k\)

\(\Rightarrow\) extend to \(\mathbb{N}_x^L\).

Riemann

Conclude that the \(\epsilon_k(\{y_1(x), \ldots, y_n(x)\}) \in \text{hol}(\mathbb{C})\)

(are entire).

To show that they are polynomials, we've got to analyze their behavior about \(x_0\). Change coordinates to \(\tilde{x} = \frac{1}{x}, \tilde{y} = \frac{y}{x^n}\); \(1\) becomes
\[ 0 = \tilde{\chi} f\left( \frac{1}{x}, \frac{\tilde{\alpha}}{x} \right) = \tilde{y}^n + \left( 2 \tilde{\alpha} \eta \left( \frac{1}{x} \right) \right) \tilde{y}^{n-1} + \cdots + \tilde{x}^n a_n \left( \frac{1}{x} \right), \]

with roots \( \tilde{y}_i \left( \tilde{\alpha} \right) = \tilde{x} \cdot y_i \left( \frac{1}{x} \right) \).

From this formula, clearly the \( \{ \tilde{y}_i \}_{i=1}^m \) are permuted amongst themselves by monodromy. So the above argument applies to \( e_k(\{ \tilde{y}_1 \left( \tilde{\alpha} \right), \ldots, \tilde{y}_m \left( \tilde{\alpha} \right) \}) \) on a neighborhood \( \mathcal{N}_0 \) of \( \tilde{\alpha} \); that is,

\[ \text{Hol}(\mathcal{N}_0) \ni e_k(\{ \tilde{y}_1 \left( \tilde{\alpha} \right) \}_{i=1}^m) = \prod_{\alpha=1}^k e_k(\{ y_{i_1} \left( \frac{1}{\alpha} \right) \}_{i=1}^m) = \prod_{\alpha=1}^k \left( \frac{x}{\alpha} \right)^{k-x} \]

and so \( e_k(\{ y_{i_1} \left( \frac{1}{\alpha} \right) \}_{i=1}^m) \) has at least a pole of order \( k \) at \( \alpha \) (and is holomorphic \( \alpha \)). Therefore it is a polynomial of degree at most \( k \) (in \( \alpha \)).

\[ \square \]

II. Meromorphic functions on algebraic curves are algebraic (rational)

Starting with the same projection (taking \( C \) now to be irreducible)

\[ \pi : C \to \tilde{C}, \]

we have inclusions of fields
\[ \pi^* C(x) \subset C(C) \subset \text{Mer}(\hat{C}) \]

Claim: (a) \([C(C): \pi^* C(x)] \geq n\)
(b) \([\text{Mer}(\hat{C}): \pi^* C(x)] \leq n\)

**Theorem** \(\text{Mer}(\hat{C}) = C(C)\).

(a) is trivial: by irreducibility of \(C\),

\[ 0 = f(x, y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x) \]

is the minimal polynomial of \(y\) over \(\pi^* C(x)\).

More interesting is the

**Proof of (b):** Let \(g \in \text{Mer}(\hat{C})^k\), and denote by \(P\) the image under \(\pi\) of its polar set and

\[ \Delta = \Delta \cup P. \]

Write

\[ e^g_i(x) := e_i(\{g(x, y, 0, \cdots, \ iterated integral \}

The same argument as above involving Riemann - Rouché
g\( \in \text{Mer}(\hat{C}\backslash \Delta)\).

About a point \(p \in P\), we can look at \(e^g_i\) times a
Sufficiently high power of \((x-\beta)\); this cancels out the pole of \(\varphi\) at \((\beta, \cdots)\) and allows the same argument to be applied, with the consequence that \(e_{\beta}^\varphi\) has meromorphic extension to \(\mathbb{C}\). Again, analyzing about \(x_0 \Rightarrow e_{\beta}^\varphi \in \mathbb{M}\mathbb{C}_2(\mathbb{P}^1) \cong \mathbb{C}(\mathbb{P}^1)\).

Now for any \(x \in \mathbb{C} \setminus \mathbb{P}^1\) and \(j \in \{1, \ldots, n\}\),

\[
\Omega = \prod_{i=1}^{n} (\varphi(x, y_j(x)) - \varphi(x, y_i(x)))
\]

\[= \sum_{k=0}^{n} \varphi(x, y_j(x))^{n-k} (-1)^k e_{\beta}^\varphi(x) \]

\(\Rightarrow\) the meromorphic function \(\varphi\) itself satisfies

\[
\Omega = \sum_{k=0}^{n} (-1)^k (\prod_{i=1}^{k} \varphi(x)) \cdot \varphi^{n-k} \]

\(\Rightarrow\) any element of \(\mathbb{M}\mathbb{C}(\mathbb{C})^*\) has degree \(\leq n\) over \(\pi^* \mathbb{C}(\mathbb{C})\)

\(\Rightarrow\) the degree of the extension is no more than \(n\).

(Otherwise, there is a primitive element, of degree \(> n\).)