Lecture 32: More on harmonic functions

I. Fourier series

This is a nice application of the Theorem on Dirichlet's problem from Lecture 31. Moreover, it gives a better idea of what the harmonic functions of that Theorem look like.

Let $f \in C^0([0,2\pi])$, $f(0) = f(2\pi)$. The Theorem just mentioned guarantees the existence of $u \in C^0(\overline{D_1})$ satisfying:

(a) $u|_{D_1}$ is harmonic, hence of the form

$$u(re^{i\theta}) = \Re \left( \sum_{n=0}^{\infty} a_n (re^{i\theta})^n \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \right)$$

Where the series are absolutely and uniformly convergent on $D_r$ for $r < 1$

(b) $u(e^{i\theta}) = f(\theta)$.
Now using basic trigonometric integrals, (a) gives
\[
\frac{1}{\pi} \int_{0}^{2\pi} \cos(n\theta) u(re^{i\theta}) \, d\theta = a_n \, r^n
\]
\[
\frac{1}{\pi} \int_{0}^{2\pi} \sin(n\theta) u(re^{i\theta}) \, d\theta = b_n \, r^n
\]
Taking \( r \to 1 \) and using (uniform) continuity of \( u \) on \( \overline{D_1} \) together with (b),
\[
\begin{align*}
\frac{1}{\pi} \int_{0}^{2\pi} \cos(n\theta) f(\theta) \, d\theta &= a_n \\
\frac{1}{\pi} \int_{0}^{2\pi} \sin(n\theta) f(\theta) \, d\theta &= b_n
\end{align*}
\]

(*) may be regarded as the definition for Fourier coefficients of \( f \). Together with (a), this gives a formula for \( u|_{D_1} \) as a series. What about \( f \)? This is a nontrivial question, since "the series converges to \( u \) on \( D_1 \)" and "\( u \) is continuous on \( \overline{D_1} \)" do NOT imply that the series converges to \( u \) on \( \overline{D_1} \).

Since \( f \) is (uniformly) continuous, \( \forall \varepsilon > 0 \exists \delta > 0 \) s.t. \( |\theta_2 - \theta_1| < \delta \Rightarrow |f(\theta_2) - f(\theta_1)| < \frac{\varepsilon}{2} \). This means that for \( n > \frac{2\varepsilon}{\delta} \) (i.e. sufficiently large),
\[
\left| \frac{1}{\pi} \int_{\frac{2\pi}{n}}^{2\pi} f(\theta) \cos(n\theta) \, d\theta \right| = \left| \frac{1}{\pi} \int_{\frac{2\pi}{n}}^{2\pi} f_{m}(\theta) \cos(n\theta) \, d\theta \right| < \frac{1}{\pi} \cdot \frac{2\pi}{n} \cdot \frac{\epsilon}{2} = \frac{\epsilon}{n}
\]

\[
\Rightarrow \left| a_n \right| = \left| \sum_{m=0}^{n-1} \frac{1}{\pi} \int_{\frac{2\pi}{n}}^{2\pi} f(\theta) \cos(n\theta) \, d\theta \right| < \epsilon. \quad (\#\#)
\]

So \( a_n \to 0 \) as \( n \to \infty \). (Similar for \( b_n \))

But this isn't good enough for convergence, and indeed if \( S \subset [0, 2\pi] \) is any set of measure zero, then \( \exists f \in C^0([0, 2\pi]) \) whose Fourier series diverge (unboundedly!) on \( S \). A famous theorem of Carleson implies that the Fourier series at least converge pointwise almost everywhere, but still this is a bit shocking.

(A) When you first learn Fourier series from physicists who repeat the mantra that \( f \in \mathcal{C}^k \Rightarrow \lim_{n \to \infty} a_n = 0 \).

(B) In light of our theorem on Dirichlet for \( D_1 \).

The problem is that, while \( \lim_{n \to \infty} a_n \) exists, the statement amounts...
to Abel summability of the Fourier series at \( \Theta_0 \), which is weaker than ordinary summability.

To fix this, suppose now that \( f \) is everywhere differentiable, with bounded derivative (weaker than \( C^1 \)). Then if \( \|f'\|_{[0,2\pi]} \leq M \), and so if \( \varepsilon = \frac{2\pi M}{n} \) then we can take \( \delta = \frac{\varepsilon}{M} = \frac{2\pi}{n} \Rightarrow |n c_n| \leq \frac{2\pi M}{\varepsilon} \cdot \varepsilon = 2\pi M \) (same for \( b_n \)).

If \( f \) is \( C^1 \), then by parts

\[
|a_n| = \left| \frac{1}{n} \int_{0}^{2\pi} f(\theta) \cos(n\theta) \, d\theta \right| = \left| -\frac{1}{n} \int_{0}^{2\pi} f'(\theta) \sin(n\theta) \, d\theta \right| \leq \frac{\varepsilon}{n} \left[ \text{same technique appl. to } f' \text{ as in the derivative of } \sin(n\theta) \right].
\]

and we conclude that \( |a_n| \to 0 \) and \( |b_n| \to 0 \).

In the first case (\( n a_n \) bounded) we can use Littlewood's theorem, in the second case (\( n a_n \to 0 \)) Titchmarsh's theorem (proved in Lecture 8), to assert that \( \Phi(n\Theta) \)

\[
f(\Theta_0) = \lim_{r \to 1^-} n(\Re e^{i\Theta_0})
\]
\[
\lim_{n \to 1} \left( \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\theta) + b_n \sin(n\theta) \right] r^n \right)
\]
\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\theta) + b_n \sin(n\theta) \right]
\]

(and that this last expression converges).

Conversely, by Abel's theorem, whenever the last expression converges, the \( \lim_{r \to 1} \) must equal \( f(\theta) \) and since the \( \lim_{r \to 1} \) gives \( u(r e^{i\theta}) = f(\theta_0) \), we have the

**Theorem**

Let \( f \in C^0([0, 2\pi]) \) and \( a_n, b_n \) be its Fourier coefficients. Then:

(i) Whenever \( \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta) \) (the Fourier series of \( f \)) converges, it converges to \( f(\theta) \).

(ii) The Fourier series converges everywhere if \( f \) is everywhere differentiable, with bounded derivative.
II. Harmonic differentials

Though we only want to deal with 1-forms on $\mathbb{C}$, I'll start by putting this in a broader context:

If $M$ is a Riemannian $n$-manifold, one can take an oriented orthonormal basis $\{\Theta_i\}$ for local coframes and define

$$\Theta_{i_1} \wedge \ldots \wedge \Theta_{i_k} \wedge \ast (\Theta_{i_1} \wedge \ldots \wedge \Theta_{i_k}) := \Theta_{i_1} \wedge \ldots \wedge \Theta_{i_n} \wedge \ast \Theta_{i_k} \wedge \ldots \wedge \ast \Theta_{i_1}.$$ 

This is called the Hodge star operator.

Presently this yields

$$dx \wedge \ast dx = dx \wedge dy \quad \Rightarrow \quad \ast dx = dy \quad \text{note} \quad \ast x = -1$$
$$dy \wedge \ast dy = dx \wedge dy \quad \Rightarrow \quad \ast dy = -dx$$

So

$$\ast (A \, dx + B \, dy) = -B \, dx + A \, dy$$
$$\ast dx = \ast (dx + i \, dy) = -i \, dx + i \, dy = -i(dx + idy) = -i \, d\bar{z}$$
$$\ast d\bar{z} = \ast (dx - i \, dy) = i \, dx + i \, dy = i(dx - idy) = i \, dz.$$ 

\[\ast\] that is, a basis (over $C^\infty$ functions) for differential 1-forms in a neighborhood, which is orthonormal in the metric on the cotangent space at each point.
Hence \( i* (A dt + B d\bar{z}) = A dz - B d\bar{z} \), and in particular
\[ i* A dz = A dz. \]

Now let \( \omega \) be a real 1-form:
\[ \omega = c dt + b dy = A dt + B d\bar{z}, \quad c, b \text{ real-valued} \text{ functions}. \]
Suppose \( d \omega = 0 = d(\omega) \) (\( \leftrightarrow \omega \) is a harmonic differential)
Then \( \phi := \omega + i* \omega = 2A dt \) satisfies
\[ d\phi = 0 = -2 \frac{\partial A}{\partial \bar{z}} d\bar{z} dt + \text{differential}, \text{ i.e. } \frac{\partial A}{\partial \bar{z}} = 0 \] (A holo. 1-form).
Since \( \omega = \bar{\omega} \Rightarrow B = \bar{A} \), we have \( \omega = \text{Re}(2A dt) \).

Conversely, taking \( \phi = 2A dt \) with \( \frac{\partial A}{\partial \bar{z}} = 0 \), write
\[ \phi = \frac{1}{2} (\phi + \bar{\phi}) + i \cdot \frac{1}{2i} (\phi - \bar{\phi}) = \omega + i\eta. \text{ We have} \]
\[ \text{(real 1-forms)} \]
\[ 0 = d\phi = d\omega + i d\eta \Rightarrow d\omega = 0 = d\eta. \]
But \( i* \phi = \phi \Rightarrow \omega + i\eta = i(\ast \omega + i\ast \eta) = -\eta + i\omega \]
\[ \Rightarrow \eta = \ast \omega \]
\[ \Rightarrow d(\ast \omega) = 0 \] (\(- \) \( \omega \) harmonic).

We have proved

**Proposition 1** The harmonic differentials \( \omega \) are precisely the real parts of holo. differentials, with \( d\omega \) equal to the imag. part.
Now again let $\omega$ be harmonic. Using Prop. 1, $\omega = \text{Re}(f \, dz)$, if holomorphic. But if $f$ is locally the derivative of something else analytic, say $F$, (This is OK "globally" in any simply-connected region in which $f$ is defined & holomorphic.) So (locally) $\omega = \text{Re}(dF) = d(\text{Re}(F))$.

By differentiating the Cauchy-Riemann equations once, $\text{Re}(F)$ is a harmonic function. [\text{Re}(F)_x = \text{Im}(F)_y \implies \text{Re}(F)_{xx} = \text{Im}(F)_{xy}, \\
\text{Re}(F)_y = \text{Im}(F)_x \implies \text{Re}(F)_{yy} = \text{Im}(F)_{yx}; \text{ now subtract.}]

Conversely, suppose that $u$ is harmonic and $w := du$ (locally). Using $\Delta = \partial_x \partial_y \, [\text{Helmholtz}], \; d(du) = d^2 du = 0$; and $d\omega = ddw = 0$. So $\omega$ is harmonic.

**Proposition 2**  The harmonic differentials are (locally) precisely the differentials of harmonic functions.

If $u$ is a harmonic function in a region then we may define the harmonic conjugate of $u$ as the integral of $\bar{u} du$. (Easy exercise: this agrees with the previous definition.) This both explains why harmonic conjugates aren't well-defined in general in multiply-connected regions, and illustrates...
Why the differential form point of view is better!

If \( u_1, u_2 \in \mathcal{H}(U) \) (\( U \) any region in \( \mathbb{C} \)) then on any simply-connected subregion \( U_0 \), \( u_j \) has a conjugate function \( v_j \), with \( u_j + iv_j \in \mathcal{H}(U_0) \).

\[ \mathcal{L}^1(U_0) \ni \tau := (u_1 + iv_1) d(u_2 + iv_2) = i f v_1 du_2 + u_1 dv_2 \] + Re(\( \eta \))

\[ \Rightarrow \quad \mathcal{O} = \int_{\partial \mathcal{K}} v_1 du_2 + u_1 dv_2 \quad \forall \mathcal{K} \subset U_0 \text{ 2-chain} \]

\[ \Rightarrow \quad \mathcal{O} = \int_{\partial \mathcal{K}} u_1 dv_2 - u_2 dv_1 = \int_{\partial \mathcal{K}} u_1 x du_2 - u_2 x dv_1, \]

subtract \( u_2 dx + v_1 dy = d(u_2 v_1) \) (exact 1-form)

\[ \Rightarrow \quad \mathcal{O} = \int_Y u_1 x du_2 - u_2 x dv_1 \quad \forall \gamma \subset U. \quad (\#) \]

An application:

\[ \{ \text{HW}\}: \{ \begin{align*}
& \ast dx = r \, d\theta \\
& \ast d\theta = -\frac{1}{r} \, dr
\end{align*} \]

If \( u \in \mathcal{H}(D_\mathbb{R}^k) \), then taking \( U = D_\mathbb{R}^k \), \( \gamma = C_{r_2} - C_{r_1} \) \( (0 < r_1, r_2 < \infty) \)

we want to apply \((\#)\) to

\[ u_1 = u \quad \rightarrow \quad \ast du_1 = \ast \left( \frac{\partial u}{\partial r} \, dr + \frac{\partial u}{\partial \theta} \, d\theta \right) = r \frac{\partial u}{\partial r} \, d\theta - \frac{1}{r} \frac{\partial u}{\partial \theta} \, dr \]

\[ u_2 = \log(r) \quad \rightarrow \quad \ast du_2 = \frac{1}{r} \, r \, d\theta = d\theta \]

\( (\in \mathcal{H}(D_\mathbb{R}^k)) \quad \text{and} \quad u_3 = 1 \quad \rightarrow \quad \ast du_3 = 0. \)
\[ \Rightarrow 0 = \int_y u_1 \, dx_2 - u_2 \, dx_1, \]

\[ = \int_y u \, d\theta - \log(r)\int_y x \, dx_1, \]

and

\[ 0 = \int_y u_3 \, dx_4 - u_2 \, dx_3, \]

\[ = -\int_y x \, dx_1 \]

\[ \Rightarrow \begin{cases} \int_{C_r} u \, d\theta - \log(r)\int_{C_r} x \, dx_1 & (=: \beta) \\ \int_{C_r} x \, dx_4 & (=: \lambda) \end{cases} \]

are both constant in \( r \in (0, R) \).

\[ \Rightarrow \int_{C_r} u \, d\theta = \beta + \lambda \log(r) \quad (\forall r \in (0, R)) \]

which generalizes the MVT.  

\[ \uparrow \]  

Recall MVT says that if \( u \in \mathcal{D}_R \), then \( \lambda = 0 \) and \( \beta = u(0) \).