Lecture 34: Functions w/ prescribed principal parts

I. Constructing meromorphic functions in C

We begin by recalling the estimate on the Taylor remainder for \( f \in \text{Hol}(U) \) at \( a \in U \) (\( \text{region} \)). In \( U \),

\[
(*) \quad f(a) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k + f_{n+1}(z) (z-a)^{n+1}
\]

Now writing \( D := D(a, r) \subset U \), let \( z \in D \Rightarrow \)

\[
f_{n+1}(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_{n}(z) \, ds}{z - \zeta}
\]

(by \( (*) \))

\[
= \frac{1}{2\pi i} \int_{\partial D} \frac{f(z) \, ds}{(z-a)^{n+1}} - \frac{1}{2\pi i} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \int_{\partial D} \frac{ds}{(z-a)^{k+1}}
\]

The \( k=0 \) term at the end is (omitting a constant)

\[
\frac{1}{z-a} \int_{\partial D} \left( \frac{1}{z-a} - \frac{1}{z-\zeta} \right) \, ds \quad (= 0)
\]

the other terms are (up to const.) derivatives of this in \( a \) and therefore zero as well. So

\[
|f_{n+1}(z)| \leq \frac{M}{r^n (r-1-z)}
\]

and the estimate on the remainder term \( f_{n+1}(z) (z-a)^{n+1} \) in \( (*) \) is
\[ |f_{n+1}(z)| |z-a|^{-n+1} \leq \frac{M |z-a|^{n+1}}{r^n (r-|z-a|)} . \]

In the special case where \( z \) is restricted to \( D_0 := D(a, r/2) \subset D \), with a slight weakening and assuming \( a = 0 \), this becomes
\[ |f_{n+1}(z)| |z|^{-n+1} \leq \frac{M |z|^{n+1}}{(\frac{r}{2})^n (\frac{r}{2})} = M \left( \frac{2|z|}{r} \right)^{n+1} . \]

We want to build meromorphic functions from principal parts. For \( F \in \text{Mer}(U) \) with a pole at \( a \), and Laurent expansion about \( a \)
\[ F(z) = \sum_{-m \leq n < 0} A_n (z-a)^n + \text{holomorphic} , \]
\[ =: P(z) \]

\( P \) is called the \textbf{principal part} of \( F \) at \( a \).

So the idea will be to add a holomorphic function on \( U \) to a sum of principal parts at distinct points \( \{b_i\} \subset U \). When this set is infinite, the convergence question is interesting. It is also clear, by the definition of a meromorphic function (poles must be isolated in \( U \)) that the \( f_{b_i} \)'s may not have an accumulation point in \( U \).
but they will have at least one or \( \cup \cup \{a_0\} \) if they are infinite in number. In what follows we take \( U = \mathbb{C} \).

So let \( \{b_n\}_{n \geq 1} \subset \mathbb{C} \) be a sequence with \( |b_n| \to \infty \), \( \{P_n\} \) a sequence of polynomials with zero constant term; we may assume \( b_n \neq 0 \) \( (\forall n) \) w/o loss of generality.

The goal is to sum \( \sum_n P_n \left( \frac{1}{z-b_n} \right) \), or (failing that) \( \sum_n \left\{ P_n \left( \frac{1}{z-b_n} \right) - \text{holo}_n \right\} \) which will at least have the same principal parts.

Set \( M_n := \left\| \frac{P_n \left( \frac{1}{z-b_n} \right)}{D(b_n)} \right\|_{\mathbb{C}} \), and (for now) let \( n_n \) be an arbitrary increasing sequence of natural numbers. Expanding \( P_n \left( \frac{1}{z-b_n} \right) \) about the origin in \( D(b_n) \), write \( P_n \) for the \( n_n \) partial sum. The remainder is bounded by (using \( r = \left| b_n \right| \))

\[
|P_n \left( \frac{1}{z-b_n} \right) - P_n(z)| \leq M_n \left( \frac{4/\left| z \right|}{\left| b_n \right|} \right)^{n_n} \quad \forall z \in D_{b_n}.
\]

That is, the series we want to sum is dominated by the power series

\[
\sum_n \frac{M_n}{\left| b_n \right|^{n_n+1}} z^{n_n} = \sum_n a_n z^n.
\]
which converges absolutely and uniformly on \( \overline{D_R} \) iff
\[
\lim_{v \to \infty} \frac{m^n}{|b_v|} < \frac{1}{k}.
\]
In particular, if
\[
\lim_{v \to \infty} \frac{M^n_v}{|b_v|} = 0
\]
then taking \( N(R) \) s.t. \( v \geq N(R) \Rightarrow |b_v| > 4R \),

\[
\sum_{v \geq N(R)} \left( P_v \left( \frac{1}{z-b_v} \right) - P_v(z) \right)
\]
converges unif. on \( \overline{D_R} \).

Consequently, the tails of the series converge to holomorphic functions on arbitrarily large balls, and so the entire series converges to a meromorphic function on \( \mathbb{C} \) with the desired principal parts.

To arrange condition (**) we simply demand that
\[
n_v > \log M_n^v;
\]

since then
\[
\log \frac{M_v^n}{|b_v|} = \frac{\log M^n_v}{n_v} - \log |b_v| \to -\infty.
\]

(Note that since the condition that \( n_v \gg n_n \) is already in place, we are really assuming something more like \( n_v > n + \log M_n \).)
II. Runge's Theorem

On a disk, an arbitrary holomorphic function is the limit of a sequence of polynomials. This does not work on an arbitrary region — already \( \frac{1}{z} \) on \( \mathbb{D}^* \) furnishes a counterexample, since there is no polynomial with \( \int_{|e| = \frac{1}{2}} P(z) \, dz = 2\pi i \) (or even nonzero). But if we replace "polynomial" by "rational function", we get a nice result:

**Theorem**

Given \( K \subset U \subset \mathbb{C} \), \( E \subset \mathbb{C} \setminus K \), and \( f \in \text{hol}(U) \). Then \( \forall \varepsilon > 0 \) there exists a rational function \( R(z) \) with polar set \( CE \) and

\[
|f(z) - R(z)| < \varepsilon \quad \text{on} \quad K.
\]

With this in hand, we'll be able to prove a more general result on "constructing meromorphic functions from principal parts". We begin with the following easy

**Lemma 1**

Given a path \( \gamma : [0,1] \to \mathbb{C} \setminus K \), \( f \in C^0(\gamma) \),

\[
F(z) := \int_{\gamma} \frac{f(w)}{w-z} \, dw.
\]

Then \( \forall \varepsilon > 0 \), \( \exists \varepsilon \)-rat'l func. \( R(z) \) with polar set \( CE \), and \( |F(z) - R(z)| < \varepsilon \) on \( K \).
Proof: Let $r \in (0, d(K, V))$, $c > \max \{ \|x\|_{K}, \|y\|_{1, \infty}, \|z\|_{1, \infty} \}$, $\alpha, \beta \in \mathbb{R}$. Then for $\varepsilon \in K$,\[
\left| \frac{f(x)}{\alpha - x} - \frac{f(p)}{\beta - p} \right| \leq \frac{\varepsilon}{\alpha - x} \left| f(x) - \alpha f(p) - \beta (f(x) - f(p)) \right| \leq \frac{\varepsilon}{\alpha - x} \left[ (\beta - p) + (x - \varepsilon) (f(x) - f(p)) \right] \leq \frac{\varepsilon}{\alpha - x} \left[ (\beta - p) + (x - \varepsilon) \|f(x) - f(p)\| \right]. \]

By (uniform) continuity of $y$ and $f$, there exists a partition $0 = t_0 < \varepsilon < \ldots < t_{n-1} = \varepsilon$ such that $\left| \frac{f(y(t_i)) - f(y(t_{i+1}))}{y(t_i + \varepsilon) - y(t_i)} \right| < \frac{\varepsilon}{L(y)} \forall \varepsilon \in [t_i, t_{i+1}]$. Setting $R_j := \sum_{j=1}^{\infty} \frac{f(y(t_{j-1}))}{y(t_j) - y(t_{j-1})}$, we have

\[
\left| F(x) - R(x) \right| = \left| \sum_{j=1}^{\infty} \int_{t_{j-1}}^{t_j} \left[ \frac{f(y(t))}{y(t) - x} - \frac{f(y(t_{j-1}))}{y(t_j) - y(t_{j-1})} \right] y'(t) \, dt \right| < \frac{\varepsilon}{L(y)} L(y) = \varepsilon. \]

Define a distance function $\rho$ on $C^0(K)$ by

$\rho(\phi, \psi) = \|\phi - \psi\|_{K, \infty}$. \]

Since the uniform limit of a sequence of functions on $K$ is $C^0$, this makes $C^0(K)$ into a complete metric space. Define $B(E) := \{ \text{rational functions with poles } C \leq E \}$ where the bar denotes closure in $C^0(K)$ w.r.t. $\rho$ i.e., uniform limits of the restrictions of such rational functions to $K$. $B(E)$ is obviously closed under $+, \cdot$, scalar mult.
Lemma 2: \( \frac{1}{x-a} \in B(E) \) for every \( x \notin K \).

**Proof:** Assume first that \( x \notin E \); set \( V := \{ a | \frac{1}{x-a} \in B(E) \} \).

Clearly \( E = V \subseteq C \backslash K \).

Claim: If \( a \in V \) and \( |x-a| < d(a, K) \), then \( b \in V \) (\( \Rightarrow \) \( V \) open).

\[ \text{Pf: } B(E) \ni \frac{1}{x-a} \Rightarrow B(E) \ni \frac{1}{x-a} \cdot \sum_{n=0}^{\infty} \left( \frac{b-a}{x-a} \right)^n \quad \text{which conv. uniformly on } K \]

\[ \Rightarrow B(E) \ni \frac{1}{x-a} \cdot \sum_{n=0}^{\infty} \left( \frac{b-a}{x-a} \right)^n = \frac{1}{x-b}. \]

Given \( b \in dV \), let \( V \ni a_n \rightarrow b \). Since \( V \) is open, \( b \notin V \) \( \Rightarrow \) \( b \in K \) compact.

Let \( H \) be any component of \( C \backslash K \). By assumption, \( H \cap E \neq \emptyset \), so \( H \cap V \neq \emptyset \). But \( V \subseteq K \Rightarrow H \cap dV = \emptyset \).

If \( H \) has points both not in \( V \) and in \( V \), path connectedness shows \( H \cap dV \neq \emptyset \), a contradiction; so in fact \( H \subseteq V \).

Since \( H \) was arbitrary, \( C \backslash K \subseteq V \) and so \( V = C \backslash K \), and we're done in this case.

Next suppose \( a_0 \in E \), and set \( E_0 := E \backslash \{ a_0 \} \) for \( a_0 \in (C \backslash K)_{xy} \) with \( |a_0| > 2 \| x \| K \). (This is in \( C \) and still meets each component of \( C \backslash K \).) By the above, \( B(E_0) \ni \frac{1}{x-a} \forall x \in C \backslash K \). Moreover, \( \| \frac{x}{x-a_0} \| K \leq \frac{1}{2} \Rightarrow \frac{1}{x-a_0} = \frac{-x}{x-a_0} \sum_{n=0}^{\infty} \left( \frac{x}{x-a_0} \right)^n \) is a uniform limit on \( K \) of rational functions \( 1/pole \) at \( a_0 \), hence \( \in B(E) \). Thus \( B(E) \supset B(E_0) \) and we're done.
Proof of Range's Thm.: We want to show that 

if $f$ is analytic in a nbhd. of $K$, then $f|_K \in \mathcal{B}(E)$. 

Let $\gamma$ be a path in $U \setminus K$ with winding number 1 about every point of $K$. (That this is feasible follows from the construction of a homology basis for the connected open set $U \setminus K$ earlier in this course.) Then $f|_K = \int_\gamma \frac{f(z)}{z-w} \, dw$, and Lemma 1 $\Rightarrow f$ of $R$ with points in $(\gamma \circ) C \setminus K$ s.t. $||f-R||_K \leq \varepsilon$. 

But the fact that $\mathcal{B}(E)$ is an algebra and $\frac{1}{z-w} \in \mathcal{B}(E)$ $(\forall z \in C \setminus K)$ by lemma 2 $\Rightarrow R \in \mathcal{B}(E)$. By taking $\varepsilon$ smaller and smaller we get a sequence $R_n \to f$ unit on $K$, so that $f \in \mathcal{B}(E)$ too.

III. Mittag-Leffler's Theorem

We are now ready for the generalization of 3.1.

**Theorem 2** Let $G \subseteq \mathbb{C}$ be an open set, $\{a_k\}_{k \in \mathbb{N}}$ a sequence of distinct points with no point of accumulation in $G$, and $S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z-a_{jk})}$ the sequence of divided principal parts. Then $\exists f \in \operatorname{Mer}(G)$ with polar set $\{a_k\}_{k \in \mathbb{N}}$ and principal parts $S_k$ (on $a_k$).
Proof: By taking \( K_n := \{ z : d(z, \partial \Omega) \geq \frac{1}{n} \} \cap D_n \), we get a sequence of compact subsets \( K_n \subset G \) with \( G = \bigcup_{n=1}^{\infty} K_n \), \( K_n \to K_n^{+} \), and

\((\dagger)\) each component of \( \hat{\Omega} \setminus K_n \) containing a component of \( \hat{\Omega} \setminus G \).

In each \( K_n \) there are finitely many \( \{a_k\} \), and we define

\[ I_1 := \{ k \in \mathbb{N} \mid a_k \in K_1 \}, \quad I_n := \{ k \in \mathbb{N} \mid a_k \in K_n \setminus K_{n-1} \}, \]

\[ f_n(z) := \sum_{k \in I_n} S_k(z) \quad (n \geq 1). \]

Clearly \( f_n \) is rational & holomorphic in a neighborhood of \( K_{n-1} \). Because of \((\dagger)\), Runge's Thm. \( \Rightarrow \) \( f \) rational for \( R_n(z) \) with poles in \( \hat{\Omega} \setminus G \), s.t. \( \|f - R_n\|_{K_{n-1}} < \left(\frac{1}{n}\right)^2 \). Hence

\[ f_n(z) + \sum_{k \geq 2} (f_k(z) - R_k(z)) \]

has tails converging uniformly on each \( K_k \) (by Weierstrass M-test \( w/M_k = \left(\frac{1}{k}\right)^2 \)), and therefore on every compact subset of \( G \) it thus defines a meromorphic function with the desired principal parts.

IV. Examples

These results are nice, but the main applications are on \( \Omega \) and don't necessarily use them (just the method of subtracting off a polynomial from each principal part).
\[ f(\zeta) = \frac{\pi^2}{\sin^2 \pi \zeta} \xrightarrow{\zeta \to 0} \frac{\pi^2}{(\pi \zeta)^2 - (\pi \zeta)^3 + \ldots} = \frac{1}{\zeta^2} + \frac{1}{\zeta} + \text{horo.} \]

But if even \( \Rightarrow \) \( \alpha = 0 \Rightarrow PP_0(f) = \frac{1}{\zeta^2} \xrightarrow{\text{periodicity}} \)

\[ PP_n(f) = \frac{1}{(\pi \zeta)^n}. \]

Moreover, \( \sum_{n \in \mathbb{Z}} \frac{1}{(\pi \zeta)^n} \) is convergent by comparison to \( \sum \frac{1}{n^2} \) (essentially), and so

\[ g(\zeta) := f(\zeta) - \sum_{n \in \mathbb{Z}} \frac{1}{(\pi \zeta)^n} \in \mathcal{H}(\mathbb{C}). \]

Now \( g \) is periodic with period 1, and since

\[ |\sin \pi \zeta|^2 = \frac{e^{i \pi \zeta} - e^{-i \pi \zeta}}{2i} \cdot \frac{e^{-i \pi \zeta} - e^{i \pi \zeta}}{-2i} = \frac{e^{-i \pi \zeta} + e^{i \pi \zeta}}{4} = \cosh^2 \pi \zeta - \cos^2 \pi \zeta \to \infty \text{ as } |\zeta| \to \infty, \]

\( g \) is bounded on strips \( \xrightarrow{\text{periodically}} g \) bounded on \( \mathbb{C} \)

\( \text{Liouville} \)

But since the limit as \( |\zeta| \to \infty \) of both functions \( f(\zeta) \) and \( \sum \frac{1}{(\pi \zeta)^n} \) is zero, \( g \) must then be zero.

Therefore

\[ \frac{\pi^2}{\sin^2 (\pi \zeta)} = \sum_{n \in \mathbb{Z}} \frac{1}{(\pi \zeta)^n} \]
\[ f(t) = \pi \cot(\pi t) \]  
Consider

\[
\sum_{n \in \mathbb{Z}} \left( \frac{1}{t-n} + \frac{1}{n} \right) = \sum_{n \in \mathbb{Z}} \frac{2}{n(t-n)} ,
\]

which is uniformly convergent on compact sets hence may be differentiated termwise:

\[
-\sum_{n \in \mathbb{Z}} \frac{1}{(t-n)^2} = -\frac{\pi^2}{\sin^2 \pi t} + \frac{1}{t^2}.
\]

So

\[
\frac{d}{dt} \left( \pi \cot(\pi t) \right) = -\frac{\pi^2}{\sin^2 \pi t} = \frac{\pi^2}{\sin^2 \pi t}
\]

\[
= \frac{d}{dt} \left\{ \frac{1}{t} + \sum_{n \geq 1} \left( \frac{1}{t-n} + \frac{1}{n} \right) \right\}
\]

\[
= \frac{d}{dt} \left\{ \frac{1}{t} + \sum_{n \geq 1} \frac{2}{n(t-n)} \right\}
\]

when both starred functions are odd, hence have 0 constant term and

\[
\pi \cot(\pi t) = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2\pi}{n^2 - t^2}
\]

\[
\text{on } \mathbb{C}
\]

\[
\text{on } \mathbb{D}_1,
\]

\[
(e) \quad g_{2m-1} = \frac{1}{2m} \int_{-1}^{1} \frac{\pi \cot(\pi t)}{t^{2m}} \, dt = \sum_{n=1}^{\infty} \frac{1}{2m} \int_{-1}^{1} \frac{2t^{1-2m}}{t^{2m} - n^2} \, dt
\]

\[
= \sum_{n=1}^{\infty} \operatorname{Res} \left( \frac{2/(t^{2m} - n^2)}{t^{2m}} \right) = \sum_{n=1}^{\infty} \frac{-2/n^2 - 1}{1 - 2/n^2 - 2/n^2}
\]

\[
\frac{2/(t^{2m} - n^2)}{1 - 2/n^2 - 2/n^2} = -\frac{2/n^2 - 1}{1 - 2/n^2 - 2/n^2} \sum_{k \geq 0} n^{2k}
\]

\[
\text{(Res picks out the term } k = m - 1)\]

\[
= -2 \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = -2 \zeta(2m).
\]
But we computed before that \( a_{2m-1} = \frac{-i1^{2m} \beta_{2m} (-4)^m}{(2m)!} \).

So \( S(2m) = 2^{2m-1} \frac{1}{B_{2m}} \pi^{2m} \), a rational multiple of \( \pi^{2m} \).