Lecture 35: Functions with prescribed zeroes

Now that we know how to construct functions in $Mer(U)$ with prescribed principal parts (hence poles), what about functions in $Hol(U)$ with prescribed zeroes? Of course, summing “doesn’t preserve zeroes” of functions in the sum, so we’ll take products instead. Also, note that the two problems are “independent” in the sense that taking the reciprocal of the function solving one of the problems will not solve the other. In particular, you can get “prescribed poles” by taking the reciprocal of the function with prescribed zeroes, but this won’t give the principal parts.

I. Products of numbers

An infinite product $\prod_{n=1}^{\infty} a_n$ of complex numbers is said to converge if (a) only a finite number of the $\{a_n\}$ are 0, and (b) given $N$ s.t. $n \geq N \Rightarrow a_n \neq 0$, $\lim_{M \to \infty} \prod_{n=N}^{M} a_n$ exists and is nonzero. Later we’ll want to be able to control where products of functions are 0.
In this case, we say the product has value

\[ \prod_{n=1}^{N} a_n = \left( \prod_{n=1}^{N} a_n \right) \cdot \lim_{N \to M} \prod_{n=N}^{M} a_n. \]

Now assume all \( a_n \neq 0 \).

Lemma 1: \( \prod a_n \) converges \( \iff \sum \log a_n \) converges

Proof: \((\Leftarrow\Rightarrow)\) Set \( Q_N = \prod_{n=1}^{N} a_n \), \( \delta_n = \sum_{n=1}^{N} \log a_n \). Clearly

\[ Q_N = \exp \delta_N, \quad \text{so} \quad \delta_N \to \delta \Rightarrow Q_N \to \exp \delta. \]

\((\Rightarrow)\) \( Q_N \to Q \neq 0 \Rightarrow \frac{Q_N}{Q} \to 1 \Rightarrow \frac{a_n}{Q_N} = \frac{Q_{N+1}}{Q_N} \to \frac{Q(N+1)}{Q(N)} \to 1.\]

Moreover, \( \log \left( \frac{Q_N}{Q} \right) \to 0 \)

\[ \delta_N = \log \delta + 2\pi i n_N \quad \text{for some} \quad n_N \in \mathbb{Z}, \]

and

\[ 2\pi i (n_{N+1} - n_N) = \log \left( \frac{Q_{N+1}}{Q_N} \right) - \log \left( \frac{Q_N}{Q} \right) - \log a_N \]

\[ = i \log \left( \frac{Q_{N+1}}{Q} \right) - i \log \left( \frac{Q_N}{Q} \right) - i \log a_N \to 0 \]

\( \Rightarrow \) for \( N \geq N_0 \), \( n_N = n \) (fixed) \( \Rightarrow \delta_N \to \log \delta - 2\pi i n \). \( \square \)

Next, we say that \( \prod a_n \) converges absolutely if some tail of \( \sum \lvert \log a_n \rvert \) converges. (It would not do to say \( \text{"if } \prod a_n \text{ converges"} \); for instance, consider the product \( \prod (1 - \frac{1}{n}) \).)

\( \dagger \) otherwise we'd have to assume that all \( a_n \neq 0 \).
**Lemma 2:** $\prod a_n$ converges absolutely $\iff \sum |1 - a_n|$ converges.

**Proof:** In either case, we must have $a_n \to 1$, so that
$$\lim_{n \to \infty} \frac{\log \frac{1}{1 - a_n}}{1 - a_n} = 1 \iff \lim_{n \to \infty} \frac{\log a_n}{1 - a_n} = 1.$$ Thus for any $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \implies 1 - \varepsilon < \frac{|\log a_n|}{1 - a_n} < 1 + \varepsilon$ $\implies (1 - \varepsilon)|1 - a_n| < |\log a_n| < (1 + \varepsilon)|1 - a_n|$$\implies \sum_{n \geq N} |1 - a_n|$ and $\sum_{n \geq N} |\log a_n|$ dominate each other. \hfill \Box

**Example:** $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}$ (will prove later). \hfill //

**II. Products of functions**

Let $U$ be a subset of $\mathbb{C}$ and $\{f_n\} \subset C^0(U)$. Then $\prod f_n$ converges uniformly on $E \subset U$ $\iff$

- $\prod f_n(z_0) (z_0) =: F(z_0)$ converges $(\forall z_0 \in E)$
- $F_N(z) := \prod_{n=1}^{N} f_n(z)$ converges uniformly on $E$, to $F(z)$.

**Lemma 3:** Let $K \subset \mathbb{C}$ be compact, and let $\{g_n\} \subset C^0(K)$ be such that $\sum |1 - g_n|$ converges uniformly on $K$. Then

1. $(F(z) := \prod_{n=1}^{\infty} g_n(z))$ converges absolutely and uniformly on $K$,
2. $\exists N \in \mathbb{N}$ s.t. $(\forall z \in K) \quad F(z_0) = 0 \iff g_n(z_0) = 0$ for some $n \leq N$.\hfill \Box
Proof: Let $\varepsilon > 0$. Then
\[ \|g_n\|_K \to 0 \Rightarrow \left\| \frac{\log g_n}{g_n} - 1 \right\|_K \to 0 \]
\[ \Rightarrow \left| \log g_n(x) - (1 + \varepsilon) \right| \leq (1 + \varepsilon) \|g_n(x) - 1\| \quad \forall x \in K \quad \forall n \geq N \]
\[ \Rightarrow G_M := \sum_{n=N}^{M} \log g_n \text{ converges (M-to-\infty)} \text{ absolutely and uniformly on K,} \]
with limit function $G \in C^0(K)$.

Let $A := \|G\|_K$; then $\|e^A\|_K \leq e^A$, and we pick

- $\delta > 0$ s.t. $|e^A - 1| < \varepsilon / e^A$

- $M_0$ s.t. $M \geq M_0 \Rightarrow \|G_m - G\|_K < \delta$.

Then $\varepsilon > \|e^A\|_K \|e^{G_m - G} - 1\|_K \geq \|e^{G_m} - e^G\|_K$, and so

\[ \lim_{m \to \infty} \prod_{n=N}^{M} g_n = \lim_{m \to \infty} e^{G_m} \text{ converges uniformly on K, to the nowhere zero function } e^G, \text{ and} \]
\[ F = \left\{ \sum_{n=1}^{N-1} g_n \right\} e^G \text{ clearly implies (ii).} \]

Proposition 1: Let $U \subset \mathbb{C}$ be a region, and
\[ \{f_n\} \subset H(\mathbb{D})^{(0)} \] a sequence with $\sum |1 - f_n|$ converging uniformly on compact subsets. Then

(a) $F := \prod f_n$ converges uniformly on compact subsets ($\Rightarrow F \in H(U)$),

and

(b) For each $z_0 \in U$, $F(z_0) = 0 \Rightarrow f_n(z_0) = 0$ for some $n$ (with $0 \neq F(z_0) = \sum_{n}^\infty d_n(z_0)$).
Proof: (a) is a trivial consequence of Lemma 3(a).

(b) Suppose \( f(z_0) = 0 \), and consider \( \overline{D(z_0, r)} \subset U \).

Since \( \Sigma |1-f_n| \) converges uniformly on \( \overline{D} \), Lemma 3(b) implies:

(1) \( \exists N \in \mathbb{N} : F = f_1 \cdots f_N, \) where \( g \) doesn't vanish on \( \overline{D} \) (and being a pole-free quotient of holomorphic functions, is clearly holomorphic).

III. Canonical products (informal motivation)

Suppose \( \Sigma \frac{1}{|a_n|} \) converges. Then

\[ \frac{1}{|a_n|} \] converges absolutely and uniformly on all \( \overline{D}_R \subset \mathbb{C} \)

(hence on all compact \( K \))

\[ \Rightarrow \prod (1 - \frac{z}{a_n}) \] converges uniformly on compact subsets,

(Prop 1 is valid)

and clearly has zeroes exactly at the \( a_n \)

(with multiplicity equal to the \# of times \( a_n \) occurs).

This is called a canonical product of genus zero.

Next, suppose \( \Sigma \frac{1}{|a_n|} |g_n| \) converges, and that \( g \) is the smallest integer for which this holds. Consider the product

\[ \prod \left( 1 - \frac{z}{a_n} \right)^{p_n(g)}, \quad P_n = \text{polynomials} \]

Log of this product

is

\[ \text{non-zero correction term} \]

(similar idea to Mittag-Leffler)
\[
\sum_{n} \left\{ -\frac{\epsilon}{a_n} - \frac{1}{2} \left( \frac{\epsilon}{a_n} \right)^2 - \frac{1}{3} \left( \frac{\epsilon}{a_n} \right)^3 - \ldots \right\} + P_{n}(\epsilon)
\]

so let this be (the first $g$ terms of this series)

\[
\sum_{n} \left\{ -\frac{1}{g+1} \left( \frac{\epsilon}{a_n} \right)^{g+1} - \frac{1}{g+2} \left( \frac{\epsilon}{a_n} \right)^{g+2} - \ldots \right\}
\]

with absolute value

\[
\left| \sum \frac{1}{g+1} \left( \frac{R}{|a_n|} \right)^{g+1} \right| \leq \sum_{n} \frac{1}{g+1} \left( \frac{R}{|a_n|} \right)^{g+1}
\]

on \( \overline{D_{R}} \).

Now, (\( \star \)) converges if \( \sum_{n} \frac{1}{g+1} \left( \frac{R}{|a_n|} \right)^{g+1} \) does (here the point is that \( a_n \to \infty \), so \( 1 - \frac{R}{a_n} \to 1 \Rightarrow \frac{1}{1 - \frac{R}{a_n}} \to 1 \)), and

in this way we get uniform convergence of the original product on compact sets. The result is called a canonical product of genus \( g \).

A function which is entire and of the (unique) form

\[
\sum_{n} c_{n} G(\epsilon) \prod \left( -\frac{\epsilon}{a_n} \right)^{P_{n}(\epsilon)}
\]

(\( \star \star \))

where

- \( G \) is a polynomial of minimal degree
- (\( \star \star \)) is a canonical product of genus \( g_0 \) (also minimal)

is said to have genus

\[
g := \max \left( g_0, \deg (G) \right).
\]
IV. Weierstrass Factorization Theorem

We want to generalize this approach to something which both works for an arbitrary sequence \( \{a_n\} \) we accumulate points, and reproduces an arbitrary entire function. Also we want to know if the idea generalizes (like Mittag-Leffler) beyond \( C \) to arbitrary regions.

By an elementary factor, we shall mean one of the entire functions

\[
\begin{cases}
  E_0(z) = 1 - z \\
  E_p(z) = (1 - z) e^{\frac{z^2}{2} + \frac{z^5}{5} + \ldots + \frac{z^p}{p}} \quad (p \geq 1)
\end{cases}
\]

Lemma 4: \( |1 - E_p(z)| \leq |z|^{p+1} \) on \( D_1 \).

Proof: \( E_p(z) = 1 + \sum_{k=1}^{p} \varepsilon_k z^k \)

\[
E'_p(z) = \sum_{k=1}^{p} k \varepsilon_k z^{k-1} = \frac{1 - \varepsilon_0}{1 - z} (1 + \varepsilon_1 z + \ldots) - z
\]

\( = -z \varepsilon(z) \) has power series expansion with \( \varepsilon(z) \) all \( > 0 \)

\[
E_1 = \ldots = E_p = 0, \quad \varepsilon_k = 0 \text{ for } k \geq p + 1
\]

\( 0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} \varepsilon_k \)

\[
|E_p(z) - 1| = \left| \sum_{k=p+1}^{\infty} \varepsilon_k z^k \right| \leq |z|^{p+1} \sum_{k=p+1}^{\infty} |\varepsilon_k| = |z|^{p+1}.
\]
Proposition 2. Given sequences

- \( \{a_n\} \subset \mathbb{C}^+ \) with \( |a_n| \to \infty \)
- \( \{p_n\} \subset \mathbb{Z} \) satisfying \( \sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty \) \( \forall r \in \mathbb{R}_{>0} \),

we have

(i) \( f(z) = \prod_{n=1}^{\infty} E_{p_n} \left( \frac{e^{a_n}}{a_n} \right) \) converges (uniformly on compact sets) to an entire function (\# of a_n's tending to \( \infty \))

(ii) \( \text{ord}_{z_0} (f) = \text{multiplicity of } z_0 \) in the sequence \( \{a_n\} \).

Proof: By Lemma 4, \( \|1 - E_{p_n}(e^{a_n})\|_{\overline{D}_r} \leq \|\frac{e^{a_n}}{a_n}\|_{\overline{D}_r} \leq \left( \frac{r}{|a_n|} \right)^{p_n+1} \) provided \( r \leq |a_n| \). If we fix \( \overline{D}_r \), then \( \exists N \) s.t. \( n \geq N \Rightarrow |a_n| \geq r \), so \( \sum |1 - E_{p_n}(e^{a_n})| \) is dominated by the sequence which converges by assumption \( \Rightarrow \) done. [Prop. 1]

Remark: If we take \( p_n = n-1 \), then taking \( N \) s.t. \( n \geq N \Rightarrow |a_n| \geq 2r \), \( \sum_{n=1}^{\infty} \left( \frac{r}{a_n} \right)^{n+1} \leq \sum_{n=1}^{\infty} \left( \frac{1}{r} \right)^n \) on \( \overline{D}_r \), so that we get the second condition of Prop. 2 for every sequence \( \{a_n\} \).

Theorem 1. Given \( f \in H(\mathbb{C}) \setminus \{0\} \), let \( \{a_n\} \) be the list of its nonzero terms, repeated according to multiplicity, and set \( n = \text{ord}_0 (f) \). Then \( \exists g \in H(\mathbb{C}) \) and \( \{p_n\} \subset \mathbb{Z} \) s.t.

\[ f(z) = z^n e^{g(z)} \prod_{n=1}^{\infty} E_{p_n} \left( \frac{e^{a_n}}{a_n} \right). \]
Proof: Taking $p_n = n-1$, Prop. 2 implies that
\[
\frac{f(z)}{2^{n-1} \pi \prod_{n \geq 1} E_p_n \left( \frac{\pi}{n} \right)} \text{ has removable singularities and extends to a normal vanishing holomorphic function } G(z) \text{ on } C. \text{ Since } C \text{ is simply-connected, } F \text{ is holomorphic on } C \text{ with } F' = G / G'.
\]
\[
\Rightarrow \frac{\partial}{\partial z} G(z) e^{-F(z)} = G(z) e^{-F(z)} - G(z) \frac{G'(z)}{G(z)} e^{-F(z)} = 0
\]
\[
\Rightarrow G(z) = \text{constant} \times e^{F(z)} = e^{f(z)}.
\]

Example: \( \sin(z) \) has zeroes at all \( k \in \mathbb{Z} \), but \( \sum \frac{1}{k^2} \) diverges. On the other hand, \( \sum \frac{1}{k^2} \) converges.
\Rightarrow \text{can construct genus 1 product (all } p_n = 1). \text{ Noting that } E_1(z) = (1-z)e^z, \text{ we have that}
\[
\left( e^{g(z)} := \frac{\sin(\pi z)}{z \prod_{n \in \mathbb{Z} \setminus \{0\}} (1 - \frac{z}{n})} \right)
\]
\[
d \log \downarrow
\]
\[
g'(z) = \pi \cot(\pi z) - \frac{1}{z} - \sum_{n \neq 0} \left( \frac{1}{\pi z - n} + \frac{1}{n} \right) = 0
\]
\Rightarrow \text{g(z) constant. So } e^{g(z)} = \lim_{z \to 0} \frac{\sin(\pi z)}{z} = \pi.
\Rightarrow \sin(\pi z) = \pi z \prod_{n \neq 0} \left( 1 - \frac{z}{n} \right) e^{\gamma/n} = \pi z \prod_{n \geq 1} \left( 1 - \frac{z^2}{n^2} \right).
\]
\[
\text{Taking } \lim_{n \to \infty} \frac{\sin(\pi x)}{\pi x (1 - \frac{x^2}{n^2})} = \lim_{z \to 1} \frac{\sin(x)}{x (1 - \frac{x^2}{n^2})} = \lim_{x \to 1} \frac{\sin(x)}{x - \frac{x^3}{3!}} = -\frac{\pi}{2} \text{ or } z = -\frac{1}{2},
\]
\text{we find (using uniform convergence on compact sets) that }
\[
\prod_{n \geq 2} \left( 1 - \frac{1}{n^2} \right) = \frac{1}{2}.
\]
II. Arbitrary regions

**Theorem 2** Given $U \subseteq \mathbb{C}$, \( \{a_n\} \subseteq U \) w/ limit point in $U$, \( \{m_n\} \subseteq \mathbb{Z} \). Then $\exists f \in \mathcal{H}(U)$ whose only zeroes are at the points $a_n$, with multiplicity $m_n$.

Let's first look at the (striking) consequence:

**Corollary 1** For any $F \in \mathcal{M}_\mathbb{C}(U)$, $\exists g, f \in \mathcal{H}(U)$ s.t. $F = g/f$.

**Proof**: We know the poles \( \{a_n\} \) (w/multiplicities \( \{m_n\} \)) of $f$ must not have accumulation points in $U$. Let $f$ be the holomorphic function provided by Theorem 2. Then $fF$ has removable singularities at each $a_n$ (as $\text{ord}_{a_n} f = \text{ord}_{a_n} f + \text{ord}_{a_n} F = 0$), so extends to a hol. fn. $g$ on $U$.

Now assume $U \subseteq \mathbb{C}$ is a region, $f \in \mathcal{H}(U)$, $P \in \partial U$. We shall say that $P$ is regular for $f$ if $f \in \mathcal{D}(P, r) = \mathbb{D}$ and $\tilde{f} \in \mathcal{H}(D)$ s.t. $f|_{\partial U} = \tilde{f}|_{\partial U}$.
**Corollary 2** For any region $U \subseteq \mathbb{C}$, \( \exists f \in \mathcal{H}(U) \) s.t. no $p \in \partial U$ is regular for $f$.

**Proof (idea):** Construct a sequence \( \{a_n\} \) which has every point of $\partial U$ as a limit point, using smaller and smaller grids (cf. Greene-Krantz, pp. 268-70).

**Proof of Theorem 2:** Suppose first that

- \( \{z \mid |z| > R\} \subseteq U \)
- \( |a_n| \leq R \) \((\forall n)\).

Let \( \{w_n\} \) consist of the $a_n$'s w/mults. \( \{\tilde{a}_n\} \),

and \( \{w_n\} \subseteq U \) satisfy \( |w_n - a_n| = d(w_n, \partial U) \rightarrow 0 \) \( (n \rightarrow \infty) \).

The functions \( \mathcal{E}_n(\frac{z-w_n}{z-a_n}) \) have (simple) zeros at $z = a_n$ (where the argument $= 1$) and none else.

Let $K \subseteq U$ be compact, and set $\delta_0 := d(C \subseteq U, K) > 0$.

We have \( \frac{|2z-w_n|}{|z-a_n|} \leq \frac{1}{\delta_0} \), so that for any $\delta \in (0, 1)$

\( \exists N \) s.t. \( n \geq N \Rightarrow \frac{|2z-w_n|}{|z-a_n|} < \delta \) \( \rightarrow \mathcal{L} \) \( \mathcal{L} \)

Therefore \( \sum \mathcal{E}_n(\frac{z-w_n}{z-a_n}) \) is absolutely & uniformly convergent on $K$.

Then \( f(z) := \prod_{n \geq 1} \mathcal{E}_n(\frac{z-w_n}{z-a_n}) \in \mathcal{H}(U) \), with the right zeros.
Now taking $|z| \geq 1$ sufficiently large that (for any given $\varepsilon \in (0, \frac{1}{2})$)

$$\left| \frac{z_n - w_n}{z - w_n} \right| < \varepsilon \quad (\forall n),$$

we have (Lemma 4)

$$\left| E_n \left( \frac{z_n - w_n}{z - w_n} \right) - 1 \right| < \varepsilon^{n+1} \quad (\forall n).$$

But then

$$\sum_{n \geq 1} \log E_n (...) \leq \sum_{n \geq 1} \log E_n (...)$$

is dominated by (a constant multiple of)

$$\sum_{n \geq 1} \left| E \left( \frac{z_n - w_n}{z - w_n} \right) - 1 \right| \leq \sum_{n \geq 1} \varepsilon^{n+1} < 2\varepsilon$$

and so $\to 0$ as $|z| \to \infty$. Hence, $f(\alpha) \to 1$

as $|z| \to \infty$.

Finally, let $U \subseteq \mathbb{C}$ be arbitrary and $\{e_n\} \subseteq U$ be w/o limit point in $U$, etc.; and consider

$\Omega = \Omega(e, r) \subseteq U$ a ball not meeting $\{e_n\}$. Applying

the FLT $T(z) := \frac{1}{z - e}$ recovers the above situation, and the fact that $\lim_{z \to \alpha} f(z) = 1$ in the above mean that the pullback via $T$ (call this $g$) will have

$$\lim_{z \to \alpha} g(z) = 1$$

hence a removable singularity at $\alpha$. \[\square\]